Stable model categories and triangulated categories

We have just seen that the homotopy category of a pointed model category \( \mathcal{C} \) is naturally a pre-triangulated category. In this chapter, we examine what happens when the suspension functor is an equivalence on \( \text{Ho} \mathcal{C} \). We refer to a pre-triangulated category where the suspension functor is an equivalence as a triangulated category, and we refer to a pointed model category whose homotopy category is triangulated as a stable model category. Of course, there is already a well-known definition of a triangulated category, and the definition we give does not coincide with the classical definition. We justify this in Section 7.1 by showing that every triangulated category is a classical triangulated category, and that we can recover most of the structure of a triangulated category from a classical triangulated category. Our position is that every classical triangulated category that arises in nature is the homotopy category of a stable model category, so is triangulated in our sense.

For the rest of the chapter, we examine generators in the homotopy category of a stable model category. These generators are very important in [HPS97], and we try to uncover their precursors in the model category world. In Section 7.2, we remind the reader of the definition of an algebraic stable homotopy category, the only kind we treat in this book. This section provides some of the motivation for the next two sections. In Section 7.3, we construct weak generators in the homotopy category of a pointed cofibrantly generated model category. In Section 7.4 we determine what we need to know in order for these generators to be small in an appropriate sense.

The material in this chapter is all new, so far as the author knows. We do demand a little more of the reader than in previous chapters as well. In particular, we use the theory of homotopy limits of diagrams of simplicial sets from [BK72].

7.1. Triangulated categories

In this section we define triangulated categories and study some of their properties. Triangulated categories were first introduced by Verdier in [Ver77], and have been very useful since then. A good introduction to triangulated categories can be found in [Mar83, Appendix 2]. The definition we give is new, and is stronger than the usual one. Perhaps we should call our triangulated categories simplicially triangulated, but we do not, since every triangulated category with the standard definition that we know of is also a triangulated category with our stronger definition.

**Definition 7.1.1.** A triangulated category is a pre-triangulated category in which the suspension functor \( \Sigma \) is an equivalence of categories. A pointed model category is stable if its homotopy category is triangulated.
We then have an obvious 2-category of triangulated categories, namely the full sub-2-category of the 2-category of pre-triangulated categories whose objects consist of triangulated categories. This full sub-2-category is closed under the duality 2-functor, since $\Sigma$ is an equivalence if and only if its adjoint $\Omega$ is an equivalence.

Similarly, we have a 2-category of stable model categories. If $R$ is a ring, the model category $\text{Ch}(R)$ is stable, with any of the model structures in Section 2.2. Similarly, if $B$ is a commutative Hopf algebra over a field, then $\text{Ch}(B)$ is a stable model category. On the other hand $\text{CGTop}$ and $\text{SSet}_*$ are definitely not stable model categories. The model categories of [EKMM97] and [HSS96] are stable model categories whose homotopy categories are isomorphic to the standard stable homotopy category of spectra. In the model categories $\text{Ch}(R)$ and $\text{Ch}(B)$, the suspension functor is already an equivalence before passing to the homotopy category. The reader may think it preferable to require this of any stable model category. This is not reasonable, however, because changing the functorial factorization changes the definition of the suspension. The suspension may be an equivalence with one functorial factorization and not with another. Furthermore, the suspension is not an equivalence in the model category of symmetric spectra studied in [HSS96], though it is an equivalence in the homotopy category.

We have analogous definitions of a closed triangulated category and of a closed (pre-)triangulated module over a closed triangulated category. Such a closed module is in fact automatically triangulated, as the reader can easily check. The homotopy category of a stable Quillen ring is a closed triangulated category, and will be a central closed triangulated category if Conjecture 5.7.5 holds. Similarly, the homotopy category of a stable symmetric Quillen ring is both a symmetric closed category and a closed triangulated category, but need not be a symmetric closed triangulated category unless Conjecture 5.7.5 holds.

We must of course relate our definition of a triangulated category to the standard one. We begin this process with the following lemma.

**Lemma 7.1.2.** Triangulated categories are additive.

**Proof.** Suppose $S$ is a triangulated category. Since $\Sigma$ is an equivalence, so is $\Sigma^2$. Thus we have a natural isomorphism $\Sigma^2 \Omega^2 X \to X$. Since $\Sigma^2 Z$ is an abelian cogroup object for any $Z$, and $\Sigma^2 f$ is an abelian cogroup map for any $f$, this proves that every object of $S$ is an abelian cogroup object and that every map is an abelian cogroup map. To complete the proof that $S$ is additive, we only have to show that the canonical map $X \amalg Y \to X \times Y$ is an isomorphism. But since both coproducts and products exist in $S$, this is purely formal, and we leave it to the reader. \qed

**Remark 7.1.3.** Because of this lemma, a cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in a triangulated category is completely determined by $f$, $g$, and the map $Z \xrightarrow{\partial} \Sigma X$. Indeed, the coaction of $\Sigma X$ on $Z$ is a map $Z \to Z \amalg \Sigma X \cong Z \times \Sigma X$. The unit axiom forces the first component of this coaction to be $1_Z$, and the second component is $\partial$. For this reason, in a triangulated category $S$, we will refer to a cofiber sequence, or triangle, as a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$. Note that if we have a commutative
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Diagram

\[
\begin{array}{cccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} Z & \xrightarrow{h} \Sigma X \\
\downarrow{a} & & \downarrow{b} & & \downarrow{\Sigma a} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} Z' & \xrightarrow{h'} \Sigma X'
\end{array}
\]

A fill-in map \( c: Z \rightarrow Z' \) is \( \Sigma a \)-equivariant if and only if \( \Sigma a \circ h = h' \circ c \), so our notion of a map of cofiber sequence also translates correctly to the triangulated situation. Dually, we will refer to a fiber sequence in a triangulated category as a diagram \( \Omega Z \xrightarrow{v} X \xrightarrow{u} Y \xrightarrow{h} Z \).

We now show that a triangulated category in the sense of Definition 7.1.1 is also a triangulated category in the classical sense. First we recall the triangulated version of Verdier’s octahedral axiom.

**Definition 7.1.4.** Suppose \( S \) is an additive category equipped with an additive endofunctor \( \Sigma: S \rightarrow S \) and a collection of diagrams of the form \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \), called triangles. We abbreviate such a triangle by \( (X, Y, Z) \). We say that *Verdier’s octahedral axiom holds* if, for every pair of maps \( X \xrightarrow{v} Y \xrightarrow{u} Z \), and triangles \( (X, Y, U) \), \( (X, Z, V) \) and \( (Y, Z, W) \) as shown in the diagram (where a circled arrow \( U \rightarrow \Phi X \) means a map \( U \rightarrow \Sigma X \)), there are maps \( r \) and \( s \) as shown, making \( (U, V, W) \) into a triangle, such that the following commutativities hold:

\[
\begin{align*}
au &= rd \\
es &= (\Sigma v)b \\
sa &= f \\
br &= c
\end{align*}
\]

This is the form of the octahedral axiom given in [HPS97], and is equivalent to the original definition given by Verdier in the presence of the other axioms for a classical triangulated category. The reader should compare this to Proposition 6.3.6.

We can now give the classical definition of a triangulated category.

**Definition 7.1.5.** Suppose \( S \) is an additive category. A classical triangulation on \( S \) is an additive self-equivalence \( \Sigma: S \rightarrow S \) together with a collection of diagrams of the form \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \), called triangles, satisfying the following properties.

(a) Triangles are replete. That is, any diagram isomorphic to a triangle is a triangle.
(b) For any \(X\), the diagram \(* \to X \to X \to \Sigma X\) is a triangle.

(c) Given any map \(f: X \to Y\), there is a triangle \(X \to Y \to \Sigma X\).

(d) If \(X \to Y \to Z \to \Sigma X\) is a triangle, so is \(Y \to Z \to \Sigma Y \to \Sigma X\).

(e) Given a map \(f: X \to Y\), there is a triangle \(X \to Y \to \Sigma X\).

(f) Verdier’s octahedral axiom holds.

A classical triangulated category is an additive category together with a classical triangulation on it.

We then have the following proposition, whose proof is just a matter of rewriting the pre-triangulated axioms in the additive case.

**Proposition 7.1.6.** Suppose \(\mathcal{S}\) is a triangulated category. Then the suspension functor and cofiber sequences in \(\mathcal{S}\) make \(\mathcal{S}\) into a classical triangulated category.

The converse to Proposition 7.1.6 is extremely unlikely to be true, though we do not know of a counterexample. A classical triangulated category is not a closed \(\text{Ho} \mathbb{S}\text{Set}_*\)-module, and there doesn’t seem to be any reason it should be. It also does not have fiber sequences, only cofiber sequences. However, that problem turns out not be a problem at all. Indeed, we will show that, in a triangulated category, the fiber sequences are completely determined by the cofiber sequences.

Before doing this, we show that, in a triangulated category, a cofiber sequence can be shifted to the left as well as to the right. The following lemma is [Mar83, Lemma A2.8].

**Lemma 7.1.7.** Suppose \(\mathcal{S}\) is a triangulated category, and suppose \(\Sigma X \to \Sigma g \to \Sigma Z \to \Sigma^2 X\) is a cofiber sequence. Then so is \(X \to Y \to Z \to \Sigma X\).

Note that the converse to this lemma is immediate from the axioms.

**Proof.** There is some cofiber sequence \(X \to Y \to Z \to \Sigma X\). We then get a commutative diagram

\[
\begin{array}{cccccc}
\Sigma X & \to & \Sigma Y & \to & \Sigma Z & \to & \Sigma^2 X \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Sigma X & \to & \Sigma Y & \to & \Sigma Z' & \to & \Sigma^2 X.
\end{array}
\]

where the rows are cofiber sequences. There is then a fill-in map \(\Sigma Z \to \Sigma Z'\), which is an isomorphism by part (b) of Proposition 6.5.3. Since \(\Sigma\) is an equivalence of categories, we can write this map as \(\Sigma k\) for some map \(k: Z \to Z'\). We then get an isomorphism of sequences from the desired sequence to the cofiber sequence \(X \to Y \to Z \to \Sigma X\).
This lemma allows us to shift cofiber sequences to the left, as we prove in the following proposition. We need some notation to do so. Let $\varepsilon_X : \Sigma \Omega X \to X$ and $\eta_X : X \to \Omega \Sigma X$ denote the counit and unit of the adjunction between $\Sigma$ and $\Omega$ in a closed $\mathrm{Ho} \ SS\mathrm{et}$-module. Then we have $(\Omega \varepsilon_X) \circ \eta_X = 1$ and $\varepsilon_{\Sigma X} \circ (\Sigma \eta_X) = 1$.

**Proposition 7.1.8.** Suppose $S$ is a triangulated category. Then $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a cofiber sequence if and only if $\Omega Z \xrightarrow{-\eta_X^{-1} \circ (\Omega h)} X \xrightarrow{f} Y \xrightarrow{\varepsilon^{-1} \circ g} \Sigma \Omega Z$ is a cofiber sequence.

**Proof.** Suppose first that $(\Omega, X, Y)$ is a cofiber sequence. Then we find by shifting to the right that the top row in the following diagram is a cofiber sequence.

$$
\begin{array}{cccccc}
X & \xrightarrow{f} & Y & \xrightarrow{\varepsilon^{-1} \circ g} & \Sigma \Omega Z & \xrightarrow{(\Sigma \eta_X)^{-1} \circ (\Sigma \Omega h)} & \Sigma X \\
\end{array}
$$

The right-most square of this diagram commutes because $(\Sigma \eta_X)^{-1} = \varepsilon_X$ and because $\varepsilon$ is natural. Thus the bottom row must also be a cofiber sequence.

Conversely, suppose $(X, Y, Z)$ is a cofiber sequence. The commutative diagram

$$
\begin{array}{cccccc}
\Sigma \Omega X & \xrightarrow{\Sigma \Omega f} & \Sigma \Omega Y & \xrightarrow{\Sigma \Omega g} & \Sigma \Omega Z & \xrightarrow{(\Sigma \eta_X)^{-1} \circ (\Sigma \Omega h)} & \Sigma^2 \Omega X \\
\varepsilon_X & \downarrow & \varepsilon_Y & \downarrow & \varepsilon_Z & \downarrow & \varepsilon_X \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
\end{array}
$$

shows that the top row is also a cofiber sequence. Lemma 7.1.7 then shows that the sequence $\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\varepsilon_X^{-1} \circ \eta_X^{-1} \circ \Omega h} \Sigma \Omega X$ is a cofiber sequence. Shifting this cofiber sequence to the right two places, we find that the top row of the following commutative diagram is a cofiber sequence.

$$
\begin{array}{cccccccc}
\Omega Z & \xrightarrow{-\varepsilon_X^{-1} \circ \eta_X^{-1}} & \Sigma \Omega X & \xrightarrow{\Sigma \Omega f} & \Sigma \Omega Y & \xrightarrow{\Sigma \Omega g} & \Sigma \Omega Z \\
\varepsilon_X & \downarrow & \varepsilon_Y & \downarrow & \varepsilon_Z & \downarrow & \varepsilon_X \\
\Omega Z & \xrightarrow{-\eta_X^{-1} \circ \Omega h} & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
\end{array}
$$

is a cofiber sequence. Hence the bottom row is as well, completing the proof.

**Remark 7.1.9.** The dual of Proposition 7.1.8 says that we can shift fiber sequences in a triangulated category to the right. That is, $\Omega Z \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ is a fiber sequence if and only if $\Sigma \Omega X \xrightarrow{-\varepsilon_X^{-1} \circ \eta_X^{-1} \circ \Omega h} Y \xrightarrow{h} Z \xrightarrow{-\varepsilon_X^{-1} \circ \eta_X^{-1} \circ \Omega h} \Sigma X$ is a fiber sequence.

We also need to know that mapping into a cofiber sequence in a triangulated category gives an exact sequence, just as mapping out of one does.

**Lemma 7.1.10.** Suppose $S$ is a classical triangulated category, and suppose $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle in $S$. Then, for any $W \in S$, the sequence

$$
[W, X] \xrightarrow{f_*} [W, Y] \xrightarrow{g_*} [W, Z] \xrightarrow{h_*} [W, \Sigma X]
$$

is exact.
Of course, the dual statement also holds, and tells us that mapping out of a
fiber sequence in a triangulated category gives an exact sequence.

**Proof.** We first show that \( gf \) and \( hg \) are both 0. Indeed, consider the com-
mutative diagram

\[
\begin{array}{cccccc}
\ast & \to & Y & \to & Y & \to \ast \\
0 & \downarrow & \downarrow & \downarrow & 0 \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} \Sigma X
\end{array}
\]

where the rows are triangles (by axiom (b) of Definition 7.1.5). There is a fill-in
map \( c : Y \to Z \) making the diagram commute. It follows that we must have
\( c = g \), and therefore that \( hg = 0 \). Similarly, by applying axiom (d) to axiom (b), we get a
commutative diagram

\[
\begin{array}{cccccc}
X & \to & X & \to \ast & \to 0 & \to \Sigma X \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} \Sigma X
\end{array}
\]

Thus there is a fill-in map \( * \to Z \) making the diagram commutative. This fill-in
map must of course be the zero map, and so we have \( gf = 0 \).

Now suppose we have a map \( j : W \to Y \) such that \( gj = 0 \). Then we get a
commutative diagram

\[
\begin{array}{cccccc}
W & \to & * & \to 0 & \to \Sigma W & \xrightarrow{-\Sigma j} \Sigma W \\
\downarrow & \downarrow & 0 & \downarrow & \downarrow & \downarrow \\
Y & \xrightarrow{g} & Z & \xrightarrow{h} \Sigma X & \xrightarrow{-\Sigma f} \Sigma Y
\end{array}
\]

Thus there is a fill-map \( \Sigma W \to \Sigma X \) making the diagram commute. Since \( \Sigma \) is an
equivalence of categories, we can write this map as \( \Sigma k \) for some map \( k : X \to W \).
Then \( \Sigma (f \circ k) = \Sigma j \), so \( f \circ k = j \), as required.

Similarly, suppose we have a map \( j : W \to Z \) such that \( hj = 0 \). Then the same
argument, shifted over to the right one spot, yields a map \( k \) such that \( j = gk \).

We can now show that the fiber sequences in a triangulated category are com-
pletely determined by the cofiber sequences, as promised.

**Theorem 7.1.11.** Suppose \( \mathcal{S} \) is a triangulated category. Then the sequence
\( \Omega Z \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z \) is a fiber sequence if and only if the sequence \( \Omega Z \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{-\varepsilon^{-1} \circ h} \Sigma \Omega Z \) is a cofiber sequence.

**Proof.** Suppose \( \Omega Z \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{-\varepsilon^{-1} \circ h} \Sigma \Omega Z \) is a cofiber sequence. There is
some fiber sequence \( \Omega Z \xrightarrow{f'} X' \xrightarrow{g'} Y \xrightarrow{h} Z \). Consider the commutative diagram

\[
\begin{array}{cccccc}
\Omega Z & \xrightarrow{f} & X & \xrightarrow{g} & Y & \xrightarrow{-\varepsilon^{-1} \circ h} \Sigma \Omega Z \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\Omega Z & \xrightarrow{f'} & X' & \xrightarrow{g'} & Y & \xrightarrow{h} Z
\end{array}
\]

\[
\begin{array}{cccccc}
\Omega Z & \xrightarrow{f} & X & \xrightarrow{g} & Y & \xrightarrow{-\varepsilon^{-1} \circ h} \Sigma \Omega Z \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\Omega Z & \xrightarrow{f'} & X' & \xrightarrow{g'} & Y & \xrightarrow{h} Z
\end{array}
\]
Since the top row is a cofiber sequence and the bottom row is a fiber sequence, the compatibility between cofiber and fiber sequences guarantees that there is a map $k: X \to X'$ making the diagram commute. We claim that $k$ is an isomorphism. To see this, we use the five-lemma. Suppose $W$ is an arbitrary object of $S$. Then Proposition 7.1.8 and Lemma 7.1.10 imply that we have a commutative diagram where the rows are exact sequences:

\[
\begin{array}{ccccccc}
[W, \Omega Y] & \xrightarrow{(\Omega h)_*} & [W, \Omega Z] & \xrightarrow{f_*} & [W, X] & \xrightarrow{g_*} & [W, Y] & \xrightarrow{(-\varepsilon Z^{-1} \circ h)_*} & [W, \Sigma \Omega Z] \\
\| & & \| & & k_* & & \| & & \| \\
[W, \Omega Y] & \xrightarrow{(\Omega h)_*} & [W, \Omega Z] & \xrightarrow{f_*} & [W, X] & \xrightarrow{g_*} & [W, Y] & \xrightarrow{h_*} & [W, Z] \\
\end{array}
\]

The five-lemma then implies that $k_*$ is an isomorphism, so, since $W$ was arbitrary, $k$ is an isomorphism. We then have a commutative diagram

\[
\begin{array}{ccccccc}
\Omega Z & \xrightarrow{f} & X & \xrightarrow{g} & Y & \xrightarrow{h} & Z \\
\| & & k & & \| & & \| \\
\Omega Z & \xrightarrow{f'} & X' & \xrightarrow{g'} & Y' & \xrightarrow{h} & Z \\
\end{array}
\]

Since the bottom row is a fiber sequence, so is the top row. The proof of the converse is dual.

Theorem 7.1.11 implies that a classical triangulated category is not so far away from a triangulated category. Indeed, given a classical triangulated category, we can recover the loop functor $\Omega$ up to natural isomorphism by taking the right (and also left) adjoint of $\Sigma$. Such an adjoint always exists for any equivalence of categories. We can then define fiber sequences as in Theorem 7.1.11. The interested reader can check that these fiber sequences satisfy all the properties of fiber sequences in a pre-triangulated category, except of course the compatibility with the (non-existent) closed $\text{Ho SSet}_*$-module structure. It is most instructive to check the compatibility between the cofiber and fiber sequences.

Since the fiber sequences in a triangulated category are determined by the cofiber sequences, we would expect morphisms of triangulated categories also to depend only on the cofiber sequences. The following proposition is based on [Mar83, Proposition A2.11].

**Proposition 7.1.12.** Suppose $S$ and $T$ are triangulated categories. Suppose $(F, U, \varphi): S \to T$ is an adjunction of closed $\text{Ho SSet}_*$-modules. Then $(F, U, \varphi)$ is an exact adjunction if and only if $F$ preserves cofiber sequences.

**Proof.** If $(F, U, \varphi)$ is an exact adjunction, then by definition $F$ preserves cofiber sequences and $U$ preserves fiber sequences. Conversely, suppose $F$ preserves cofiber sequences. We must show that $U$ preserves fiber sequences. Suppose $\Omega Z \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ is a fiber sequence. We must show that the sequence $\Omega UZ \xrightarrow{Uf \circ Dm} UX \xrightarrow{Ug} UY \xrightarrow{Uh} UZ$ is a fiber sequence. Here $m$ is the natural isomorphism $\Sigma FW \to F\Sigma W$ and $Dm: \Omega UZ \to U\Omega Z$ is its dual, as in Section 1.4. Note first that adjointness implies that mapping into this latter sequence produces an exact sequence. There is some fiber sequence $\Omega UZ \xrightarrow{f} X' \xrightarrow{g'} UY \xrightarrow{Uh} UZ$, and mapping into it also produces an exact sequence. The five-lemma then implies
that it suffices to construct a map $X' \to UX$ making the diagram

$$
\begin{array}{ccccccc}
\Omega UZ & \xrightarrow{f'} & X' & \xrightarrow{g'} & UY & \xrightarrow{Uh} & UZ \\
\downarrow & & \downarrow & & \downarrow & & \\
\Omega UZ & \xrightarrow{Uf \circ Dm} & UX & \xrightarrow{Ug} & UY & \xrightarrow{Uh} & UZ
\end{array}
$$

commute. We will construct this map by constructing its adjoint $FX' \to X$. Let $\varepsilon'$ and $\eta'$ denote the counit and unit of the adjunction $(F, U, \varphi)$. Let $j : F \Omega UZ \to \Omega Z$ denote the composite $\varepsilon' \circ F(Dm)$. Consider the commutative diagram below:

$$
\begin{array}{ccccccc}
F \Omega UZ & \xrightarrow{Ff'} & FX' & \xrightarrow{Fg'} & FUY & \xrightarrow{m \circ (-F \varepsilon_U^{-1} \circ Fu h)} & \Sigma F \Omega UZ \\
\downarrow j & & \downarrow \varepsilon'_Y & & \downarrow \Sigma j & & \\
\Omega Z & \xrightarrow{f} & X & \xrightarrow{g} & Y & \xrightarrow{- \varepsilon_Z^{-1} h} & \Sigma \Omega Z
\end{array}
$$

Here the bottom row is a cofiber sequence by Theorem 7.1.11, and the top row is a cofiber sequence by Theorem 7.1.11 and the fact that $F$ preserves cofiber sequences. It takes some work to verify that this diagram commutes, but it does. Since we can shift cofiber sequences over to the left in a triangulated category, there is a fill-in map $FX' \to X$. Its adjoint is the desired map $X' \to UX$.

Another useful fact about triangulated categories is the following.

**Lemma 7.1.13.** Suppose $S$ is a symmetric closed triangulated category. Let $S^{-n} = \Omega^n S$ for $n > 0$. The the following diagram is commutative for arbitrary integers $m$ and $n$.

$$
\begin{array}{ccccccc}
S^m \wedge S^n & \xrightarrow{a} & S^{m+n} \\
T \downarrow & & \downarrow (-1)^{mn} & & \\
S^n \wedge S^m & \xrightarrow{a} & S^{m+n}
\end{array}
$$

Here $a$ is the associativity isomorphism, combined if necessary with the unit and counit of the adjoint equivalence $(\Sigma, \Omega, \varphi)$.

**Proof.** The proof of this lemma is a long diagram chase. We outline the argument but leave the details to the reader. We know the lemma already for nonnegative $m$ and $n$, by Lemma 6.6.2. Suppose that one of $m$ and $n$ is negative. Without loss of generality, let us suppose $n$ is negative. Then, since $T$ is a $\text{HoSSet}_*$-module natural transformation, we have a commutative diagram

$$
\begin{array}{ccccccc}
(S^n \wedge S^m) \wedge S^{-n} & \xrightarrow{m'} & S^n \wedge (S^m \wedge S^{-n}) \\
T \wedge 1 \downarrow & & T \downarrow & & \\
(S^n \wedge S^m) \wedge S^{-n} & \xrightarrow{m} & (S^n \wedge S^{-n}) \wedge S^m
\end{array}
$$

Here we have used the same notation as in Theorem 5.6.5. It follows from the coherence diagrams that $m'$ is determined by $m$ and $T$, and we know how $T$ behaves on $S^m \wedge S^{-n}$. Since we also know how $T$ behaves on $S^0 \wedge S^m$, a long diagram chase tells us that $T$ must behave as claimed on $S^m \wedge S^n$. A similar argument allows us to go from one negative integer to two negative integers, completing the proof. \qed
7.2. Stable homotopy categories

A stable homotopy category, as defined in \[\text{HPS97}\], is a certain kind of closed triangulated category. The goal of the rest of this chapter will be to determine what conditions we need to put on a model category so its homotopy category is a stable homotopy category. We do not entirely succeed in this goal, but we come reasonably close.

In this section, we will recall the definition of an algebraic stable homotopy category and describe the theorems we will prove in the rest of this chapter.

We begin with some definitions.

**Definition 7.2.1.** Suppose \( S \) is a pre-triangulated category, and \( G \) is a set of objects of \( S \). We say that \( G \) is a set of weak generators for \( S \) if \( \Sigma^n G, X \) = 0 for all \( G \in G \) and all \( n \geq 0 \) implies that \( X \cong * \). If \( S \) is triangulated, we usually allow \( \Sigma^n G = \Omega^{-n} G \) for \( n < 0 \) as well, without changing notation.

So, for example, \( S^0 \) is a weak generator of \( \text{HoSSet}_* \), and \( R \) is a weak generator of the triangulated category \( \text{HoCh}(R) \), though we would have to include \( \Sigma^{-n} R \) for all \( n \geq 0 \) if we were thinking of \( \text{HoCh}(R) \) as only a pre-triangulated category.

The goal of the next section is to construct a set of weak generators for any pointed cofibrantly generated model category. The weak generators are simply the cofibers of the generating cofibrations.

However, a set of weak generators by itself is not tremendously useful. Just as in the definition of a cofibrantly generated model category, one also needs an appropriate definition of smallness. The one we adopt is the following.

**Definition 7.2.2.** Suppose \( S \) is a pre-triangulated category. An object \( X \in S \) is called small if, for every set \( Y_\alpha, \alpha \in K \) of objects of \( S \), the induced map

\[
\operatorname{colim}_{S \subseteq K, S \text{ finite}} [X, \coprod_{\alpha \in S} Y_\alpha] \to [X, \coprod_{\alpha \in K} Y_\alpha]
\]

is an isomorphism.

Note that \( X \) is small if every map into a coproduct factors through a finite subcoproduct. If \( S \) is triangulated, then \( X \in S \) is small if and only if for every set \( Y_\alpha, \alpha \in K \) of objects of \( S \), the induced map

\[
\bigoplus_{\alpha \in K} [X, Y_\alpha] \to [X, \coprod_{\alpha \in K} Y_\alpha]
\]

is an isomorphism. This is the definition of smallness given in \[\text{HPS97}\]. Note also that Definition 7.2.2 is the logical definition of smallness in any category where coproducts are the only colimits one can expect to have, such as the homotopy category of a (not necessarily pointed) model category.

We will give sufficient conditions for an object in a pointed model category to be small in the homotopy category in Section 7.4.

Another useful property of an object in any symmetric closed category is the following.

**Definition 7.2.3.** Suppose \( S \) is a symmetric closed category, and \( X \in S \). We say that \( X \) is strongly dualizable if the natural map \( \text{Hom}(X, S) \otimes Y \to \text{Hom}(X, Y) \) is an isomorphism for all \( Y \).

We can now define an algebraic stable homotopy category.
Definition 7.2.4. An algebraic stable homotopy category is a symmetric closed triangulated category $S$ together with a set $\mathcal{G}$ of small strongly dualizable weak generators of $S$.

These algebraic stable homotopy categories are the principal object of study in [HPS97]. The definition given in [HPS97, Definition 1.1.4] is not the same as the one given above, as it involves localizing subcategories and representability of cohomology functors, but it is proven in [HPS97, Theorem 2.3.2] that the definition above is equivalent to that one. Perhaps I should say almost equivalent, since we are certainly using a stronger definition of triangulated category than was used in [HPS97]. Also, we assumed in [HPS97] that the commutativity isomorphism behaved correctly on spheres, as in Lemma 7.1.13, and now we are assuming it also behaves correctly on $S \wedge K$ for any simplicial set $K$.

Another point is that, if the author were writing [HPS97] today, he would not insist that the generators be strongly dualizable. Peter May suggested this at the time, but the authors of [HPS97] were convinced by the importance of strong dualizability in the examples. However, there are too many examples, such as the $G$-equivariant stable homotopy category based on the trivial $G$-universe of [EKMM97], and the homotopy category of sheaves of spectra of [BL96], where the generators are not strongly dualizable. Furthermore, this condition is not amenable to understanding from the model category point of view, as far as the author can tell. We will therefore say no more about it.

We then get a 2-category of stable homotopy categories as the evident full sub-2-category of symmetric closed triangulated categories. One could also make a requirement that the morphisms preserve the generators in an appropriate sense: see [HPS97, Section 3.4]. We do not do this, though.

Combining the results of the next two sections with the results already proven in this book, we get the following theorem. This theorem is close to the author’s original goal when he began thinking about the material in this book. We will define compactly generated model categories in Section 7.4.

Theorem 7.2.5. The homotopy pseudo-2-functor lifts to a pseudo-2-functor from compactly generated simplicial stable symmetric Quillen algebras to algebraic stable homotopy categories. If Conjecture 5.7.5 holds, then we get a pseudo-2-functor from compactly generated stable symmetric Quillen rings to algebraic stable homotopy categories.

7.3. Weak generators

The goal of this section is to construct weak generators in the homotopy category of a cofibrantly generated pointed model category. We will prove the following theorem.

Theorem 7.3.1. Suppose $\mathcal{C}$ is a cofibrantly generated pointed model category, with generating cofibrations $I$. Let $\mathcal{G}$ be the set of cofibers of maps of $I$. Then $\mathcal{G}$ is a set of weak generators for $\mathrm{Ho}\mathcal{C}$.

The proof of Theorem 7.3.1 requires the notion of homotopy limits of diagrams of simplicial sets, for which we rely on [BK72]. The definitive treatment of homotopy colimits and homotopy limits for any model category will be in [DHK]; see also [Hir97].

We begin by studying homotopy classes of maps out of a colimit.
Proposition 7.3.2. Suppose we have a sequence of cofibrations

\[ * \to X_0 \xrightarrow{f_0} X_1 \to \ldots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \ldots \]

in a pointed model category \( \mathcal{C} \), with colimit \( X \). Suppose also that \( Y \) is fibrant. Then we have an exact sequence of pointed sets

\[ * \to \lim^1[\Sigma X_n, Y] \to [X, Y] \to \lim [X_n, Y] \to * \]

When \( \mathcal{C} \) is the category of pointed simplicial sets, this is proved in [BK72, Corollary IX.3.3].

Proof. Recall that the functor \( \text{Map}_{r*}(-, Y) \) of Section 5.2 preserves limits, as a functor from \( \mathcal{C}^{op} \) to \( \text{SSet}_* \). Thus \( \text{Map}_{r*}(X, Y) \cong \lim \text{Map}_{r*}(X_n, Y) \). Furthermore, since each map \( X_n \to X_{n+1} \) is a cofibration of cofibrant objects, each map \( \text{Map}_{r*}(X_{n+1}, Y) \to \text{Map}_{r*}(X_n, Y) \) is a fibration of fibrant pointed simplicial sets, by Corollary 5.4.3. By [BK72, Theorem IX.3.1] we have a short exact sequence

\[ * \to \lim^1 \pi_1 \text{Map}_{r*}(X_n, Y) \to \pi_0 \text{Map}_{r*}(X, Y) \to \lim \text{Map}_{r*}(X_n, Y) \to * \]

But from Lemma 6.1.2, we have \( \pi_0 \text{Map}_{r*}(X, Y) \cong [X, Y] \), \( \pi_0 \text{Map}_{r*}(X_n, Y) \cong [X_n, Y] \), and \( \pi_1 \text{Map}_{r*}(X_n, Y) \cong [\Sigma X_n, Y] \), so we get the required short exact sequence.

Note that the colimit \( X \) in the sequence above is the coequalizer of the identity map of \( \prod X_n \) and the map \( g = \prod f_n \). In general, there is no way to take this coequalizer in \( \text{Ho} \mathcal{C} \) instead of in \( \mathcal{C} \). However, if \( \mathcal{C} \) is stable, we can find a cofiber sequence

\[ \prod X_n \xrightarrow{1-g} \prod X_n \to X' \to \Sigma \prod X_n \]

in \( \text{Ho} \mathcal{C} \). Then \( X' \) is called the sequential colimit, as in [HPS97, Section 2.2]. We can actually form \( X' \) in the homotopy category of any pointed model category, as long as each \( X_n \) is a suspension. Then we have an exact sequence of pointed sets

\[ * \to \lim^1[\Sigma X_n, Y] \to [X', Y] \to \lim [X_n, Y] \to * \]

just as we do for \( X \). This gives us maps \( X' \to X \) and \( X \to X' \) in \( \text{Ho} \mathcal{C} \), but we are not able to prove that these maps are isomorphisms in general.

Corollary 7.3.3. Suppose \( \mathcal{C} \) is a pointed model category,

\[ 0 \to X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \ldots \]

is a sequence of cofibrations with colimit \( X \), and \( Y \) is a fibrant object. If \( [X_n, Y]_* = 0 \) for all \( n \), then \( [X, Y]_* = 0 \).

Proof. Apply Proposition 7.3.2 to \( \Omega^k Y = \text{Hom}_*(S^k, Y) \) for all \( k \).

We also need a transfinite version of Corollary 7.3.3.

Proposition 7.3.4. Suppose \( \mathcal{C} \) is a pointed model category, \( X: \lambda \to \mathcal{C} \) is a \( \lambda \)-sequence of cofibrations of cofibrant objects with colimit also denoted by \( X \), and \( Y \) is a fibrant object. If \( [X_\beta, Y]_* = 0 \) for all \( \beta < \lambda \), then \( [X, Y]_* = 0 \).
Let $\beta$ be the map $Z_\beta \to \lim_{<\beta} Z_{\gamma}$. That is, the map $Z_\beta \to \lim_{<\beta} Z_{\gamma}$ is an isomorphism for all limit ordinals $\beta$. Furthermore, $\Map_{\gamma}(X, Y)$ is the inverse limit of this sequence.

The diagram $Z_\beta$ defines a functor $Z$ from the inverse category $\gamma^{op}$ to pointed simplicial sets $\SSet_*$. Recall from Corollary 5.1.5 that the inverse limit functor on an inverse category is a right Quillen functor, right adjoint to the diagonal functor. Furthermore, we claim that an inverse $\lambda$-sequence $W$ of fibrations, such as $Z$, is fibrant in the model structure given by Theorem 5.1.3. Indeed, given a successor ordinal $\beta$, the map $W_\beta \to M_\beta W$ is the map $W_\beta \to \lim_{<\beta} W_{\gamma}$, which is an isomorphism, and hence a fibration, for an inverse $\lambda$-sequence. Hence we have an isomorphism $\Map_{\gamma}(X, Y) \cong (R\lim) Z$ in the homotopy category $\operatorname{Ho}\SSet^{\lambda^{op}}$, where $R\lim$ denotes the total right derived functor of the inverse limit.

However, there is another approach to this right derived functor, called the homotopy limit. Homotopy limits are developed for diagrams of simplicial sets such as $Z$ in [BK72, Chapter XI]. The homotopy limit $R\lim$ is also a right Quillen functor $\SSet^{\lambda^{op}} \to \SSet$, but with respect to a different model structure on diagrams. The weak equivalences are still defined objectwise, but now the fibrations are also defined objectwise. Since the weak equivalences are the same in the two model structures, they have the same homotopy categories. Furthermore, it is shown in [BK72, Section XI.8] that the total right derived functor $R\lim$ is an inverse category is a right Quillen functor, right adjoint to the diagonal functor. Since $R\lim$ is also right adjoint to the diagonal functor, we have an isomorphism

$$\Map_{\gamma}(X, Y) \cong (R\lim) Z \cong (R\lim \lim) Z \cong \lim Z$$

in the homotopy category of pointed simplicial sets. The last isomorphism comes from the fact that $Z$ is obviously fibrant in the model structure on which $\lim$ is a right Quillen functor.

The advantage of this is that we can calculate $\lim Z$. Indeed, it is proved in [BK72, Section XI.7] that there is a spectral sequence associated to the homotopy inverse limit of any diagram $W$ of fibrant simplicial sets. The $E_2$ term is $E_2^{\ast, t} = \lim^s \pi_t W$, where $\lim^s$ indicates the $s$th derived functor of the inverse limit. In our case, $\pi_t Z_\beta = [\Sigma^t X_\beta, Y] = 0$, using Lemma 6.1.2. Hence the $E_2$ term is identically 0. As pointed out in [BK72], the only obstructions to the convergence of this spectral sequence arise from terms of the form $\lim^0 E_{r+1}^{s, t}$, which are certainly all 0 in our case. We conclude that $\lim Z$ is contractible, and hence that $\Map_{\gamma}(X, Y)$ is contractible. Another application of Lemma 6.1.2 then shows that $[\Sigma^t X, Y] = 0$ for all $t$, as required.

We can now prove Theorem 7.3.1.

**Proof of Theorem 7.3.1.** We must show that if $[G, Y]_* = 0$ for all $G \in \mathcal{S}$, then $Y \cong *$ in $\operatorname{Ho} \mathcal{C}$. We can use the small object argument to factor $* \to Y$ into a cofibration $* \to QY$ followed by a fibration $QY \to Y$. We use $Q'$ instead of $Q$ since this may be a different factorization from the one canonically associated to $\mathcal{C}$. It suffices to show that $QY \cong *$ in $\operatorname{Ho} \mathcal{C}$. To do so, we show that the weak equivalence $QY \to RQY$ is trivial, where $R$ is the fibrant replacement functor canonically associated to $\mathcal{C}$. Note that $[G, RQY]_* \cong [G, Y]_* = 0$ for all $G \in \mathcal{S}$. 


By construction, $QY'$ is the colimit of a $\lambda$-sequence $X : \lambda \to \mathcal{C}$, where each map $X_\beta \to X_{\beta+1}$ fits into a pushout square of the form

$$
\begin{array}{ccc}
A & \longrightarrow & X_\beta \\
\downarrow f & & \downarrow \\
B & \longrightarrow & X_{\beta+1}
\end{array}
$$

where $f$ is a map in $I$ with cofiber $C$. Furthermore, $X_0 = 0$. Thus each $X_\beta$ is cofibrant and each map $X_\beta \to X_{\beta+1}$ is a cofibration.

We show by transfinite induction that $[X_\beta, RQ'Y]'_\ast = 0$ for all $\beta \leq \lambda$, where $X_\lambda = Q'Y$. Since $X_0 = 0$, we can certainly get started. We have a cofiber sequence $X_\beta \to X_{\beta+1} \to C$, and so also a cofiber sequence $\Sigma^nX_\beta \to \Sigma^nX_{\beta+1} \to \Sigma^nC$. Thus, since $C \in \mathcal{G}$, if $[X_\beta, RQ'Y]'_\ast = 0$, then $[X_{\beta+1}, RQ'Y]'_\ast = 0$. Now suppose $\beta$ is a limit ordinal, and $[X_\alpha, RQ'Y]'_\ast = 0$ for all $\alpha < \beta$. Then Proposition 7.3.4 shows that $[X_\beta, RQ'Y]'_\ast = 0$, as required.

7.4. Smallness

This section is devoted to the study of smallness in the homotopy category of a pointed model category. See Definition 7.2.2 for the definition of smallness.

Given a pointed model category $\mathcal{C}$, we would like to know what conditions on an object $X \in \mathcal{C}$ will guarantee that $X$ is small in $\text{Ho} \mathcal{C}$. The most obvious fact along these lines is the following proposition.

**Proposition 7.4.1.** Let $\mathcal{C}$ be a model category in which every object is fibrant. Suppose that $A$ is cofibrant, and that, for every set $\{X_\alpha \mid \alpha \in K\}$ of cofibrant objects of $\mathcal{C}$, the natural map

$$
\text{colim}_{S \subseteq K, S \text{ finite}} \mathcal{C}(A, \coprod_{\alpha \in S} X_\alpha) \to \mathcal{C}(A, \coprod_{\alpha \in K} X_\alpha)
$$

is surjective (it is automatically injective). Then $A$ is small in $\text{Ho} \mathcal{C}$. If $\mathcal{C}$ is cofibrantly generated with generating cofibrations $I$, we can assume each map $0 \to X_\alpha$ is a regular $I$-cofibration.

**Proof.** Given a set $\{Y_\alpha \mid \alpha \in K\}$ of objects of $\text{Ho} \mathcal{C}$, replace each $Y_\alpha$ by the isomorphic object $QY_\alpha$, where $Q$ is the cofibrant replacement functor. Since every object is fibrant, any map $A \to \coprod_{\alpha \in K} Y_\alpha$ in $\text{Ho} \mathcal{C}$ is represented by a map $A \to \coprod_{\alpha \in K} QY_\alpha$ in $\mathcal{C}$. By hypothesis any such map factors through a finite subcoproduct. Thus the map

$$
\text{colim}_{S \subseteq K, S \text{ finite}} [A, \coprod_{\alpha \in S} Y_\alpha] \to [A, \coprod_{\alpha \in K} Y_\alpha]
$$

is surjective. This map is always injective, since any finite coproduct splits off the entire coproduct. In case $\mathcal{C}$ is cofibrantly generated, repeat the same argument using $Q'Y_\alpha$ instead, where $Q'$ is the cofibrant replacement functor derived from the small object argument.

Proposition 7.4.1 applies in particular to $\text{Top}$, by virtue of the following lemma.

**Lemma 7.4.2.** Suppose $A$ is a compact topological space, and $\{X_i\}$ is an arbitrary family of spaces. Then any map $A \to \coprod X_i$ factors through a finite subcoproduct.
Proof. The image of $f$ is compact. Given any subset $K$ of $\coprod X_i$, the family $\{K \cap X_i\}$ forms an open cover of $K$. If $K$ is compact, this open cover must have a finite subcover, and so $K$ lies in a finite subcoproduct. Since any subcoproduct is closed in $\coprod X_i$, $f$ is still continuous when thought of as a map to a finite subcoproduct containing its image. \qed

The same argument will work in $\text{CGTop}$, but we need a slightly subtler argument in the pointed case.

Lemma 7.4.3. Suppose $A$ is a pointed compact topological space, and $X_i$ is a family of pointed cell complexes. Then any map $A \xrightarrow{f} \bigvee X_i$ factors through a finite subwedge.

Proof. Choose a presentation of $* \xrightarrow{r} X_i$ as a relative cell complex, as in Section 2.3.2. Then we get a presentation of $\coprod X_i$ as a relative cell complex, say by choosing a well-ordering on our indexing set. Then the image of $f$ intersects the interiors of only finitely many cells, by Lemma 2.3.2. Each such cell is a cell of some $X_i$, so the image of $f$ lies in a finite subwedge. The inclusion of a finite subwedge is a closed inclusion by Lemma 2.3.1, so $f$ is continuous when thought of as a map into any finite subwedge containing its image. \qed

We then recover the well-known result that finite pointed CW-complexes are small in $\text{HoTop}_\ast$.

To cope with smallness when not every object is fibrant is more subtle, since a map $A \rightarrow \coprod X_i$ in the homotopy category will not necessarily be represented by any map $A \rightarrow \coprod X_i$ in the model category. We therefore make the following definitions.

Definition 7.4.4. (a) Given a partially ordered set $\mathcal{K}$, we say that $\mathcal{K}$ is a countably directed set if both of the following conditions hold:

(i) $\mathcal{K}$ is directed: that is, for any $i, j \in \mathcal{K}$, there is a $k \in \mathcal{K}$ such that $i \leq k$ and $j \leq k$.

(ii) $\mathcal{K}$ admits a linear extension to the nonnegative integers. That is, there is some function $f$ from $\mathcal{K}$ to the nonnegative integers such that if $i < j$ then $f(i) < f(j)$.

(b) Suppose $X$ is an object of a category $\mathcal{C}$ with all small colimits, and $\mathcal{D}$ is a subcategory of $\mathcal{C}$. We say that $X$ is compact relative to $\mathcal{D}$ if $X$ is $\aleph_0$-small relative to $\mathcal{D}$ as in Definition 2.1.3, and if, for every functor $F: \mathcal{K} \rightarrow \mathcal{D}$ from a countably directed set $\mathcal{K}$ to $\mathcal{D}$, the natural map $\text{colim} \mathcal{C}(X, Y_i) \rightarrow \mathcal{C}(X, \text{colim} Y_i)$ is an isomorphism (where colim $Y_i$ is taken in $\mathcal{C}$).

For example, the set $\mathcal{K}$ of all finite subsets of a set $K$ is a countably directed set, and a finite simplicial set $X$ is compact relative to the whole category of simplicial sets. Indeed, given a map $X \rightarrow \text{colim} Y_i$, the image of any nondegenerate simplex must lie in some $Y_i$ and so, as there are only finitely many nondegenerate simplices, the image of $X$ must lie in some $Y_i$. Of course, the map $X \rightarrow Y_i$ may not be a simplicial map, but we only need check finitely many conditions to make it so. Thus we can pass to a $Y_j$ so that the map $X \rightarrow Y_j$ is simplicial. This argument merely requires that $\mathcal{K}$ be directed.

Now, we would like a theorem that says that if $X$ is cofibrant and compact relative to the cofibrations in a model category $\mathcal{C}$, then $X$ is small in $\text{Ho}\mathcal{C}$. In order for this to be true, however, we need to know a great deal about $\mathcal{C}$. One can see this
by considering the model category of chain complexes of abelian groups, with the
usual model structure where cofibrant objects are certain complexes of free abelian
groups (see Section 2.2). Here \( \mathbb{Z} \), thought of as a complex concentrated in degree 0,
is compact relative to the subcategory of cofibrations (even inclusions), and is also
small in the homotopy category. However, if we expand the weak equivalences to
include the mod \( p \) homology isomorphisms, \( \mathbb{Z} \) is no longer small in the homotopy
category (and not every object is fibrant). Yet we have not changed the cofibrations
at all, though we have changed the acyclic cofibrations.

We make the following definition.

**Definition 7.4.5.** Suppose \( \mathcal{D} \) is a model category. We say that \( \mathcal{D} \) is *compactly
generated* if there are sets \( I \) of cofibrations and \( J \) of acyclic cofibrations satisfying
the following properties:

1. The domains and codomains of \( I \) are compact relative to the subcategory of
   regular \( I \)-cofibrations.
2. The domains of the maps of \( J \) are small relative to the subcategory of
   regular \( J \)-cofibrations and compact relative to the subcategory of regular
   \( I \)-cofibrations.
3. A map is an acyclic fibration (respectively, fibration) if and only if it has
   the right lifting property with respect to \( I \) (respectively, \( J \)).

Note the interaction between the sets \( J \) and \( I \) in part 2 of the above definition.
Obviously every compactly generated model category is cofibrantly generated.
Most standard model categories are compactly generated.

**Lemma 7.4.6.** The model categories \( \mathbf{SSet} \) and \( \mathbf{SSet}^* \) are compactly generated.

**Proof.** We have already seen that finite simplicial sets are compact with re-
spect to the entire category \( \mathbf{SSet} \). The pointed case is similar. \( \square \)

**Lemma 7.4.7.** Suppose \( R \) is a ring. Then the model category \( \mathrm{Ch}(R) \) is com-
pletely generated.

**Proof.** Recall that the generating cofibrations of \( \mathrm{Ch}(R) \) are the maps \( S^{n-1} \rightarrow D^n \)
and the generating acyclic cofibrations are the maps \( 0 \rightarrow D^n \). The initial object
0 is always compact. Recall that a map from \( D^n \) into \( X \) is just an element of \( X_n \).
Using this, it is easy to see that \( D^n \) is compact, just because colimits in \( \mathrm{Ch}(R) \)
are taken dimensionwise. Similarly, a map from \( S^n \) into \( X \) is a cycle in \( X_n \). One can
easily see that taking the cycles in \( X_n \) also commutes with colimits, and so \( S^n \) is
compact. \( \square \)

**Lemma 7.4.8.** Suppose \( B \) is a commutative Hopf algebra over a field. Then
the model category \( \mathrm{Ch}(B) \) is compactly generated.

**Proof.** Use the argument of Lemma 2.4.11. \( \square \)

The model category \( \mathbf{Top} \) does not appear to be compactly generated, though of
course we have already seen how to deal with smallness in this case. Nevertheless,
this seems to be a deficiency in the definition of compactly generated. It would be
much nicer to have one definition that applies to both \( \mathbf{Top} \) and the above examples.
It would also be nice to find an example of an object in some category which is
\( \aleph_0 \)-small yet is not compact.
Localization in the sense of [Hir97] or [Bou79] does not preserve compactness, as one loses control of the acyclic cofibrations. Thus, for example, the model category discussed above of chain complexes of abelian groups and mod $p$ homology isomorphisms is not compactly generated, though it is cofibrantly generated. One could also consider the category of $G$-spaces, where $G$ is a Lie group. There are various notions of weak equivalence on this category. The standard definition of [LMS86] is to let a $G$-map $f: X \to Y$ be a weak equivalence if the induced map $f^H: X^H \to Y^H$ on the $H$-fixed point set is a weak equivalence for all closed subgroups $H$. There is a compactly generated model structure with these as the weak equivalences (though we do not know a published reference for it). However, if one decides to allow all subgroups $H$, rather than just closed ones, there is again a model category structure, but so far as the author knows, there is no compactly generated model structure with these weak equivalences. One would like to take the generating cofibrations to be $G/H \times S^n \to G/H \times D^{n+1}$, but $G/H \times S^n$ need not be compact with respect to any interesting class of maps if $G/H$ is not compact.

It is useful to know that we can always choose the generators $J$ in a compactly generated model category to be regular $I$-cofibrations.

**Lemma 7.4.9.** Suppose $\mathcal{C}$ is a compactly generated model category with generating cofibrations $I$ and generating acyclic cofibrations $J$. Then there is a set $J'$ of acyclic cofibrations satisfying the following properties.

1. Every map of $J'$ is a regular $I$-cofibration.
2. The domains of the maps of $J'$ are compact relative to the subcategory of regular $I$-cofibrations (and in particular small relative to the subcategory of regular $J'$-cofibrations).
3. A map is a fibration if and only if it has the right lifting property with respect to $J'$.

**Proof.** For each map $f: A \to B$ in $J$, use the small object argument to factor $f$ as a regular $I$-cofibration $f': A \to B'$ followed by an acyclic fibration. Let $J'$ be the set of such maps $f'$. Since $f$ is a weak equivalence, so is $f'$, and so $J'$ consists of acyclic cofibrations. Since the domain of $f'$ is the same as the domain of $f$, certainly the domains of the maps of $J'$ are compact relative to the subcategory of regular $I$-cofibrations. Thus it suffices to show that any map which has the right lifting property with respect to $J'$ is a fibration. But $f$ is a retract of $f'$, as one can see by lifting in the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f'} & B' \\
\downarrow f & & \downarrow \\
B & \xrightarrow{f} & B
\end{array}
$$

Hence any map which has the right lifting property with respect to $J'$ also has the right lifting property with respect to $J$, so is a fibration.

Our goal is then the following theorem.

**Theorem 7.4.10.** Suppose $\mathcal{C}$ is a compactly generated model category with generating cofibrations $I$ and generating acyclic cofibrations $J$. Suppose $A$ is cofibrant and compact with respect to regular $I$-cofibrations. Then $A$ is small in $\text{Ho} \mathcal{C}$.

This theorem has the following corollary.
Corollary 7.4.11. Suppose $\mathcal{C}$ is a pointed compactly generated model category. Let $\mathcal{G}$ be the set of cofibers of the generating cofibrations $I$. Then $\mathcal{G}$ is a set of small weak generators for the pre-triangulated category $\text{Ho}\mathcal{C}$.

Proof. We have already seen in Theorem 7.3.1 that $\mathcal{G}$ is a set of weak generators for $\text{Ho}\mathcal{C}$. Using Theorem 7.4.10, we need to check that the cofibers of the maps of $I$, which are obviously cofibrant, are also compact relative to regular $I$-cofibrations. This follows by commuting colimits, using the fact that the domains and codomains of the maps of $I$ are compact relative to regular $I$-cofibrations.

Before we can prove Theorem 7.4.10, we need to study diagrams in a model category that are based on countably directed sets, such as the directed set of all finite subsets of a given set. Note that a countably directed set $K$ is an example of a direct category (Definition 5.1.1), so we have a model category $\mathcal{C}K$ of functors from $K$ to a model category $\mathcal{C}$ as in Theorem 5.1.3. The weak equivalences and fibrations are the objectwise ones, and the cofibrations are detected by the latching spaces (see Definition 5.1.2).

We need to know that the model structure on $\mathcal{C}K$ is cofibrantly generated when $\mathcal{C}$ is. Note that if $B$ is any small category and $i$ is an object of $B$, the evaluation functor $\text{Ev}_i: \mathcal{C}B \to \mathcal{C}$ defined by $\text{Ev}_iX = X_i$ has a left adjoint $F_i: \mathcal{C} \to \mathcal{C}B$. In case $B$ is a direct category, this left adjoint is particularly simple to describe: $(F_iX)_j$ is either $X$ if there is a map from $i$ to $j$, or else is the initial object. The maps of $F_iX$ are all either identities or maps from the initial object.

Proposition 7.4.12. Suppose $\mathcal{C}$ is a cofibrantly generated model category, with generating cofibrations $I$ and generating acyclic cofibrations $J$. Let $B$ be a direct category. Then the model structure on $\mathcal{C}B$ of Theorem 5.1.3 is cofibrantly generated. Indeed, we can let the set of generating cofibrations $I'$ be the collection $F_i f$, where $i$ is in $B$ and $f \in I$. Similarly, we can let the set of generating acyclic cofibrations $J'$ be the set of $F_i g$, where $g \in J$.

Proof. Note that a map $X \to Y$ in $\mathcal{C}B$ has the right lifting property with respect to $F_i f$ if and only if $X_i \to Y_i$ has the right lifting property with respect to $f$. It follows that a map has the right lifting property with respect to $I'$ ($J'$) if and only if it is an acyclic fibration (a fibration).

To complete the proof, we must verify the required smallness conditions. The evaluation functor $\text{Ev}_j$ commutes with colimits. If $f \in I$, then $\text{Ev}_j F_i f$ is either $f$ itself or the identity on the initial object. It follows that, if $h$ is a regular $I'$-cofibration, then $\text{Ev}_j h$ is a regular $I$-cofibration. Hence if $A$ is small relative to regular $I$-cofibrations, then $F_i A$ is small relative to regular $I'$-cofibrations. Hence the domains and codomains of the maps of $I'$ are small relative to regular $I'$-cofibrations. Similarly, the domains of $J'$ are small relative to regular $J'$-cofibrations.

It is proved in [DHK] that if $\mathcal{C}$ is a cofibrantly generated model category, and $B$ is any small category, then there is a cofibrantly generated model structure on $\mathcal{C}B$ where the weak equivalences and fibrations are the obvious ones. We do not need this greater generality.

Now we need to understand a little more about the cofibrations in $\mathcal{C}^X$.

Proposition 7.4.13. Suppose $\mathcal{C}$ is a cofibrantly generated model category with generating cofibrations $I$, and $X$ is a countably directed set. Suppose $f: A \to B$
is a regular $I'$-cofibration in the model structure on $\mathcal{C}^\mathcal{K}$ discussed above. Then, for $i \leq j \in \mathcal{K}$, the map $B_i \amalg A_j \rightarrow B_j$ is a regular $I'$-cofibration in $\mathcal{C}$.

This proposition has an obvious corollary.

**Corollary 7.4.14.** Suppose $\mathcal{C}$ is a cofibrantly generated model category with generating cofibrations $I$, and $\mathcal{K}$ is a countably directed set. Suppose $A$ is an $I'$-object in $\mathcal{C}^\mathcal{K}$, so that the map $0 \rightarrow A$ is a regular $I'$-cofibration. Then for $i \leq j \in \mathcal{K}$, the map $A_i \rightarrow A_j$ is a regular $I$-cofibration in $\mathcal{C}$.

**Proof of Proposition 7.4.13.** Suppose $A_f \rightarrow B$ is a regular $I'$-cofibration in $\mathcal{C}^\mathcal{K}$, and $i < j \in \mathcal{K}$. We must show that the map $h: C = Ev_i B \amalg Ev_i A \rightarrow Ev_j B$ is a regular $I$-cofibration.

We can write $f$, up to isomorphism, as the transfinite composition of a $\lambda$-sequence $X: \lambda \rightarrow \mathcal{C}^\mathcal{K}$ for some ordinal $\lambda$, such that each map $X_\beta \rightarrow X_\beta+1$ for $\beta+1 < \lambda$ is a pushout of a map of $I'$. More specifically, we have a pushout square

$$
\begin{array}{ccc}
F_{k_\beta} W_\beta & \rightarrow & X_\beta \\
\downarrow & & \downarrow \\
F_{k_\beta} Z_\beta & \rightarrow & X_{\beta+1}
\end{array}
$$

for some $g_\beta$ in $I$. Commuting colimits and using the fact that $Ev_i$ commutes with colimits, we find that the map $h: C \rightarrow Ev_j B$ is isomorphic to the colimit of the maps

$$
h_\beta: C_\beta = Ev_i X_\beta \amalg Ev_i A \rightarrow Ev_j X_\beta.
$$

Here both $C_\beta$ and $Ev_j X_\beta$ are themselves $\lambda$-sequences.

We must construct a factorization of $h$ (up to isomorphism) into the transfinite composition of a $\lambda$-sequence $D_\beta$ of regular $I$-cofibrations (in fact, each map $D_\beta \rightarrow D_{\beta+1}$ will either be an isomorphism or a pushout of a map of $I$). To do so, we let $D_\beta$ be the pushout in the diagram

$$
\begin{array}{ccc}
C_\beta & \rightarrow & C \\
\downarrow h_\beta & & \downarrow \\
Ev_j X_\beta & \rightarrow & D_\beta
\end{array}
$$

Then $D_0 = C$ and by commuting colimits we see that the $D_\beta$ form a $\lambda$-sequence whose transfinite composition is $h$. Thus it suffices to show that each map $D_\beta \rightarrow D_{\beta+1}$ is a regular $I$-cofibration.

To see this, we consider three cases. In case $k_\beta \leq j$, we have $C_{\beta+1} = C_\beta$ and $Ev_j X_{\beta+1} = Ev_j X_\beta$. Hence the map $D_\beta \rightarrow D_{\beta+1}$ is an isomorphism, so in particular is a regular $I$-cofibration.

In case $k_\beta \leq i$, then both $Ev_i X_{\beta+1}$ and $Ev_j X_{\beta+1}$ are obtained by pushing out $W_\beta \rightarrow Z_\beta$. It follows that $C_{\beta+1}$ is obtained from $C_\beta$ by pushing out the same map $W_\beta \rightarrow Z_\beta$, so we have a commutative diagram

$$
\begin{array}{ccc}
W_\beta & \rightarrow & C_\beta & \rightarrow & Ev_j X_\beta \\
g_\beta & & \downarrow h_\beta & & \downarrow \\
Z_\beta & \rightarrow & C_{\beta+1} & \rightarrow & Ev_j X_{\beta+1}
\end{array}
$$
where the left and outer squares are both pushout squares. It follows that the right square is also a pushout square. Thus we have a commutative diagram

\[
\begin{array}{ccc}
C_\beta & \longrightarrow & C_{\beta+1} \\
\downarrow h_\beta & & \downarrow h_{\beta+1} \\
Ev_jX_\beta & \longrightarrow & Ev_jX_{\beta+1}
\end{array}
\]

where the left and outer squares are pushout squares. It follows that there is a map

\[Ev_jX_{\beta+1} \rightarrow D_\beta\]

making the diagram commute, and hence making the right square into a pushout square. Since the pushout of the right square is by definition \(D_{\beta+1}\), the map \(D_\beta \rightarrow D_{\beta+1}\) is again an isomorphism, and thus a regular \(I\)-cofibration.

In case \(k_\beta \leq j\) but \(k_\beta \not\leq i\), we have \(C_{\beta+1} = C_\beta\), but \(Ev_jX_{\beta+1} = Ev_jX_\beta \amalg W_\beta Z_\beta\). It follows that

\[D_{\beta+1} = C \amalg C_{\beta+1} \amalg (Ev_jX_\beta \amalg W_\beta Z_\beta) = D_\beta \amalg W_\beta Z_\beta\]

Hence \(D_\beta \rightarrow D_{\beta+1}\) is a pushout of \(W_\beta \rightarrow Z_\beta\), and so is a regular \(I\)-cofibration.

Thus the map \(C \rightarrow Ev_jB\) is a transfinite composition of regular \(I\)-cofibrations, and so is a regular \(I\)-cofibration. \(\square\)

We now show that, in a compactly generated model category, filtered colimits of fibrant objects are fibrant.

**Lemma 7.4.15.** Suppose \(\mathcal{C}\) is a cofibrantly generated model category with generating cofibrations \(I\) and generating acyclic cofibrations \(J\). Let \(\mathcal{K}\) be a countably directed set. Suppose \(X \in \mathcal{C}^\mathcal{K}\) is fibrant and that each map \(X_i \rightarrow X_j\) for \(i \leq j \in \mathcal{K}\) is a regular \(I\)-cofibration. If the domains of the maps of \(J\) are compact relative to regular \(I\)-cofibrations, then \(\text{colim} \, X\) is fibrant in \(\mathcal{C}\).

**Proof.** We will show that the map \(\text{colim} \, X \rightarrow 0\) has the right lifting property with respect to the maps of \(J\). Suppose \(f: A \rightarrow B\) is a map of \(J\), and \(g: A \rightarrow \text{colim} \, X\) is a map. We must find an extension \(h: B \rightarrow \text{colim} \, X\) such that \(hf = g\).

Since \(A\) is compact relative to regular \(I\)-cofibrations, there is an \(i \in \mathcal{K}\) and a factorization \(A \xrightarrow{g_i} X_i \xrightarrow{m_i} \text{colim} \, X\) of \(g\). Since \(X_i\) is fibrant, there is an extension \(h_i: B \rightarrow X_i\) such that \(h_if = g_i\). It follows that \(h = m_ih_i\) is the required extension of \(g\). \(\square\)

We can now finally prove Theorem 7.4.10.

**Proof of Theorem 7.4.10.** Suppose \(A\) is cofibrant and compact relative to regular \(I\)-cofibrations in the compactly generated model category \(\mathcal{C}\). Suppose as well that \(\{X_\alpha \mid \alpha \in K\}\) is a set of objects of \(\text{Ho} \, \mathcal{C}\). We must show that the natural map \(\text{colim}_{\alpha \in K, S \text{ finite}} [A, \coprod_{\alpha \in S} X_\alpha] \xrightarrow{\delta} [A, \coprod_{\alpha \in K} X_\alpha]\) is a bijection, where of course the coproduct is taken in \(\text{Ho} \, \mathcal{C}\). Note first that \(\delta\) is clearly injective. Hence it suffices to show that every map \(A \rightarrow \coprod_{\alpha \in K} X_\alpha\) factors through a finite coproduct. We can also assume that each \(X_\alpha\) is cofibrant.

Let \(\mathcal{K}\) denote the directed set of finite subsets of \(K\). We then have a diagram \(Z: \mathcal{K} \rightarrow \mathcal{C}\), where \(Z_S = \coprod_{\alpha \in S} X_\alpha\). The diagram \(Z\) has a number of good properties. The colimit of \(Z\) is \(\coprod_{\alpha \in K} X_\alpha\). The diagram \(Z\) is cofibrant in the model category structure on \(\mathcal{C}^\mathcal{K}\), since \(LZ_S = Z_S\) unless \(S\) has only one element \(\alpha\), in which case the map \(LZ_S \rightarrow Z_S\) is the map \(0 \rightarrow X_\alpha\).
Now, use the small object argument in $\mathcal{C}^X$ to factor $0 \to Z$ into a regular $I'$-cofibration $0 \to QZ$ followed by an acyclic fibration $QZ \to Z$. Choose generating acyclic cofibrations $J'$ for $\mathcal{C}^X$ as in Lemma 7.4.9. Then use the small object argument to factor $QZ \to 0$ into a regular $J'$-cofibration $QZ \rightarrowtail RQZ$ followed by a fibration $RQZ \twoheadrightarrow 0$. Since every map in $J'$ is a regular $I'$-cofibration, every regular $J'$-cofibration is a regular $I'$-cofibration, and hence the map $0 \to RQZ$ is a regular $I'$-cofibration.

Since the colimit functor preserves weak equivalences between cofibrant objects by Corollary 5.1.7, the map $\text{colim } QZ \to \text{colim } Z = \coprod_{\alpha \in K} X_\alpha$ is a weak equivalence. Similarly the map $\text{colim } QZ \to \text{colim } RQZ$ is a weak equivalence. Furthermore, since $0 \to RQZ$ is a regular $I'$-cofibration, each map in the diagram $RQZ$ is a regular $I$-cofibration by Corollary 7.4.14. Hence by Lemma 7.4.15, colim $RQZ$ is fibrant. Hence any map $A \to \coprod_{\alpha \in K} X_\alpha$ in Ho $\mathcal{C}$ is represented by a map $A \to \text{colim } RQZ$ in $\mathcal{C}$. Since $A$ is compact relative to regular $I$-cofibrations, this map factors through $\text{Ev}_S RQZ$ for some finite subset $S$ of $K$. Since $\text{Ev}_S RQZ$ is weakly equivalent to $\coprod_{\alpha \in S} X_\alpha$, the theorem follows. \qed