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Minkowski decompositions and degenerations of Okounkov bodies

(doctoral thesis)

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Introduction

"It is especially true in algebraic geometry that in this domain the methods employed are at least as important as the results."

Oscar Zariski

Historically Okounkov bodies first appear in papers due to Andrei Okounkov in the middle of 90-ties, see for instance [27]. A few years ago these ideas were expanded independently by Kaveh and Khovanskii [18] and Lazarsfeld and Mustaţă [24]. To give a certain overview now I briefly present the notion of Okounkov bodies. These are convex bodies $\Delta(D)$ in $\mathbb{R}^n$ attached to big divisors D on a smooth projective variety X of dimension n. They depend on the choice of a flag of subvarieties $(Y_n, Y_{n-1}, \ldots, Y_1)$ of codimensions $n, n-1, \ldots, 1$ in X respectively, such that Y_n is a non-singular point of each of the Y_i's – such a flag is called *admissible*. Recently many authors have published several important results, which characterize analytic and geometrical properties of Okounkov bodies, some of the most important results were obtained by D. Anderson, S. Boucksom, A. Küronya, C. Maclean, V. Lozovanu, Shin-Yao Jow, David Witt-Nyström and the advisor of this thesis T. Szemberg, see for instance the following papers [1], [2], [19], [20], [31]. However still we know a little about Okounkov bodies and there are many open problems, which are not solved. These are related to construction problems (as we will see that the construction of Okounkov bodies is not an easy task), analytic properties of functions defined on these bodies, studying invariants of Okounkov bodies, which should measure some asymptotic magnitudes related to linear series etc. My PhD thesis focuses on geometric questions related to Okounkov bodies. I considered the following three tasks. The first task was to present an introduction to the theory of Okounkov bodies, which would also collect many very recent developments in this area of algebraic geometry. The second task was to deliver an expository introduction to study Okounkov bodies for smooth projective surfaces – this introduction contains both folklore and formal facts devoted to Okounkov bodies and very recent results, including my research. The third task was to develop and extend the idea of the Minkowski decomposition in the case of higher-dimensional smooth projective toric varieties. These three scientific tasks also divide this PhD thesis into three main parts. I would like to present the main results of the thesis, which somehow give the insight about my contribution to the study of Okounkov bodies. Some of these results come from joint work with Tomasz Szemberg [29] and other from my joint paper with David Schmitz and Stefano Urbinati [28].

My first result can be viewed as the shape approximation theorem.

Theorem A. Let X be a smooth projective variety of dimension n. Assume that D is a big divisor on X. Then for every $\beta, \beta_1 > 0$ there exists a birational morphism $\eta : \widetilde{X} \to X$, an ample divisor A on $\widetilde{X}$, $\delta > 0$, and any admissible flag $\widetilde{Y}_\bullet$ on $\widetilde{X}$ the Okounkov body $\triangle(\eta^*(D))$ contains $\triangle(A)$ with

$$\mathrm{vol}_{\mathbb{R}^n}(\triangle(\eta^*(D)) \setminus \triangle(A)) < \beta_1$$

and is contained in $\triangle((1+\delta)A)$ with

$$\mathrm{vol}_{\mathbb{R}^n}(\triangle((1+\delta)A)\setminus\triangle(\eta^*(D)))<\beta.$$

Theorem A is proved in Chapter 2 as Theorem 2.2.6.

In the second part I discuss the idea of Minkowski decompositions for smooth projective surfaces with rational polyhedral pseudoeffective cones, which came from [26]. To give an overview in this moment I only mention that the Minkowski decomposition is a construction, which allows us to build Okounkov bodies using smaller indecomposable blocks corresponding to certain nef divisors. The main result tells us how the number of these smaller blocks corresponds to the geometry of the nef cone of smooth projective surfaces which admits the Minkowski decomposition – this number depends on the choice of an admissible flag and the answer gives us the pattern and an idea how to compute this number – Section 4 in Chapter 3 is devoted to this subject and mentioned results appear as Proposition 3.4.4, Theorem 3.4.5, Proposition 3.4.11. The second part can be also viewed as folklore, so I have collected results manifesting the case of projective surfaces. Moreover, I present two results, which are very specific to this case – see Proposition 3.3.4 and Theorem 3.5.3. One of them rephrase the theorem of Jow [31], which shows that Okounkov bodies allow to check the numerical equivalence of big divisors just by comparing the shapes of corresponding Okounkov bodies with respect to all possible admissible flags. In the case of surfaces one can show something more. For a projective surface X denote by $\mathcal{I}(X)$ the set of all irreducible negative curves.

Theorem B. Let X be a smooth complex projective surface. Denote by ρ the Picard number of X. Assume that D_1, D_2 are $\mathbb{R}$-pseudoeffective divisors and let $D_j = P_j + N_j$ be the Zariski decompositions. There exist ample irreducible divisors $A_1, ..., A_\rho$ with general points $x_i \in A_i$, such that D_1, D_2 are numerical equivalent if and only if

- $\triangle_{(x_i,A_i)}(D_1) = \triangle_{(x_i,A_i)}(D_2)$ for every $i \in \{1, ..., \rho\}$,
- $C.N_1 = C.N_2$ for every negative curve $C \in \mathcal{I}(X)$.

This theorem is based on the fact that every real pseudoeffective divisor on a smooth projective surface has the Fujita-Zariski decomposition, see for instance [3] and Definition 3.1.1.

Theorem B is proved in Chapter 3 as Theorem 3.5.3.

The last part of my thesis is devoted to the Minkowski decompositions for toric varieties. The main result tells us that the Minkowski decomposition exists and has a very specific form.

Theorem C. Let X be a smooth projective toric variety of dimension n such that for any (toric) small $\mathbb{Q}$-factorial modification $f : X \dashrightarrow X_i$ the target variety is smooth. Then the set of all T-invariant divisors D such that there exists a small modification $f : X \dashrightarrow X'$ and a divisor D' spanning an extremal ray of $\mathrm{Nef}(X')$ such that $D = f^*(D')$ forms a Minkowski base with respect to T-invariant flags.

It means that Minkowski bases for certain smooth toric varieties consist of vertices of the movable cones. Theorem C is proved in Chapter 4 as Theorem 4.2.4.

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Chapter 1

Preliminaries

We will use the same notation as in Positivity in Algebraic Geometry [23].

Our aim for this chapter is to recall basic definitions and theorems from the theory of linear series and line bundles, which are used in this thesis. Our main source is [23] and we also refer to this book for more details, in particular for proofs of theorems, which will be omitted. In Chapter 4 we give another specific introduction to toric geometry, which is based on two prominent books, [10] and [14].

1.1 Divisors and line bundles

Let X be an irreducible complete variety of dimension n over an arbitrary field $\mathbb{F}$ – at this moment we do not assume anything about characteristic of our field. Denote by $\mathcal{M}_X$ the constant sheaf of rational functions on X. This sheaf contains the structure sheaf $\mathcal{O}_X$and moreover there is an inclusion $\mathcal{O}_X^* \subset \mathcal{M}_X^*$ of sheaves of multiplicative abelian groups. For a sheaf $\mathcal{F}$ we denote by $\Gamma(X, \mathcal{F})$ the set of global sections. For us a divisor D on X always means a Cartier divisor i.e. a divisor represented by data $\{U_i, f_i\}$, where $\{U_\alpha\}$ is an open covering and $f_i \in \Gamma(U_i, \mathcal{M}_X^*)$. The function f_i is called a local equation for D at any point $x \in U_i$. Denote by $Div(X)$ the set of all Cartier divisors. This set $Div(X)$ has the structure of an additive abelian group: if $D, D' \in Div(X)$ and these divisors are represented by data $\{U_i, f_i\}$ and $\{U_i, f_i'\}$ respectively, then $D + D'$ is given by $\{U_i, f_i f_i'\}$. We say that D is effective if $f_i \in \Gamma(U_i, \mathcal{O}_X(U_i))$ is regular on U_i for all i.

A global section $f \in \Gamma(X, \mathcal{M}_X^*)$ determines a divisor

$$D = \mathrm{div}(f) \in Div(X),$$

which is called *principal* and we denote by $\mathrm{Princ}(X)$ the subgroup of all principal divisors on X. We say that two divisors D, D' are *linearly equivalent* ($D \equiv D'$), if $D - D'$ is principal.

We denote by $WDiv(X)$ the set of all Weyl divisors on X, i.e. the set of all locally finite formal sums of codimension one irreducible subvarieties with integer coefficients. We can define the following map

$$(\star) \qquad Div(X) \longrightarrow WDiv(X), \;\; D \mapsto [D] = \sum \mathrm{ord}_V(D)[V],$$

where ord_V means order of vanishing of D along a codimension one subvariety V and this sum is take over all irreducible codimension 1 subvarieties $V \subset X$. This map in general is neither injective nor surjective, but if X is normal, then the map $(\star)$ is one-to-one onto the image, and if X is smooth, then this map is an isomorphism. Consider a divisor $D = \sum a_i V_i$ on a normal projective variety X. Let $f : Y \longrightarrow X$ be a morphism of varieties such that no component of $f(X)$ is contained in any V_i. We define the pull-back $f^*(D)$ by

$$f^*(D) = \sum a_i f^{-1}(V_i).$$

A Cartier divisor $D \in Div(X)$ determines a line bundle $\mathcal{O}_X(D)$ on X: if D is given by data $\{U_i, f_i\}$, then on the overlap of $U_i \cap U_j = U_{ij}$ one can write

$$f_i = g_{ij} f_j \quad \text{for some} \quad g_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X^*),$$

thus the line bundle $\mathcal{O}_X(D)$ can be constructed using the functions g_{ij} as transition functions.

Denote by $Pic(X)$ the Picard group of isomorphism classes of line bundles on X, i.e. $Pic(X) = Div(X)/Princ(X)$.

1.2 Linear series

Let L be a line bundle on X. Denote by $H^0(X, L) = \Gamma(X, L)$ the space of global section of L. Assume that $V \subset H^0(X, L)$ is a finite dimensional subspace. Denote by $\mathbb{P}(V)$ the projective space of one dimensional subspaces of V. If X is a complete variety, then $|V| = \mathbb{P}(V)^*$ is identified with the linear series of divisors of sections from X and in general we refer to $|V|$ as a *linear series*. If X is complete and $V = H^0(X, L)$, then we will say that $|V|$ is a *complete linear series*. Also for a divisor D we write $|D|$ for the complete linear series associated to $\mathcal{O}_X(D)$.

Definition 1.2.1. The *base locus* of linear series $|V|$, denoted by $\mathrm{Bs}(|V|)$, is the (closed) set of points at which all sections in V vanish.

Definition 1.2.2. A linear series $|V|$ is *free* (or basepoint free), if its base locus is empty.

In other words $|V|$ is free if and only if for each point $x \in X$ one can find a section $s = s_x \in V$ such that $s(x) \neq 0$. A divisor D or a line bundle L is *free* if the corresponding complete linear series is free. In the case of line bundles we say synonymously that L is generated by its global sections, or just globally generated.

If X is an irreducible non-singular variety and $|V|$ is a basepoint free linear series, then V determines a morphism

$$\phi : X \xrightarrow{|V|} \mathbb{P}^r$$

setting

$$\phi(x) = [s_0(x) : \ldots : s_r(x)] \in \mathbb{P}^r,$$

where $s_0, \ldots, s_r$ is a basis of V.

Definition 1.2.3. We say that two Cartier divisors $D, D' \in Div(X)$ are *numerically equivalent* $D \equiv_{num} D'$, if for every irreducible curve $C \subset X$ we have $D.C = D'.C$, where . is defined as

$$D.C := \deg_{\tilde{C}} f^* \mathcal{O}_C(D),$$

where $f : \tilde{C} \to C$ is the normalization.

In the same manner we define the numerical equivalence of line bundles. We say that a divisor D (or a line bundle) is numerical trivial, if it is numerical equivalent to zero. Denote by $Num(X) \subset Div(X)$ the subgroup of numerically trivial divisors.

Definition 1.2.4. The Neron-Severi group of X is the group

$$N^1(X) = Div(X)/Num(X),$$

of numerical equivalence classes of divisors on X.

One of the most fundamental theorems about the Neron-Severi group is the following.

Theorem 1.2.5 (Kleiman)**.** *Let X be a projective variety of dimension n over an algebraically closed field $\mathbb{F}$. Then the Neron-Severi group $N^1(X)$ is a finitely generated free abelian group.*

Definition 1.2.6. The rank of the Neron-Severi group $N^1(X)$ is called the Picard number of X, we write $\rho = \rho(X)$.

1.3 Ampleness of line bundles

Definition 1.3.1. Let X be a projective variety and let L be a line bundle on X. We say that L is *very ample*, if there exists a closed embedding $X \subseteq \mathbb{P}^N$ such that

$$f : X \hookrightarrow \mathbb{P}^N \text{ and } L = f^*\mathcal{O}_{\mathbb{P}^N}(1).$$

Definition 1.3.2. In the situation of the previous definition, we say that L is *ample* if mL is very ample for some $m > 0$.

A Cartier divisor D on X is ample (or very ample) if the corresponding line bundle $\mathcal{O}_X(D)$ is so.

Now we present two very fundamental theorems, which allow us to verify ampleness of line bundles numerically.

Theorem 1.3.3 (Nakai - Moishezon - Kleiman Criterion, Theorem 1.2.23, [23], [9])**.** *Let L be a line bundle on a projective variety X of dimension n. Then L is ample if and only if intersection numbers $L^{dimV}.V$ are positive for every positive dimensional irreducible subvariety $V \subset X$.*

In particular for X which is a surface it means that L is ample iff $L^2 > 0$ and for every irreducible curve C one has $L.C > 0$.

Theorem 1.3.4 (Seshadri's Criterion, Theorem 1.4.13, [23])**.** *Let X be a projective variety and let D be a divisor on X. Then D is ample if and only if there exists a positive number ε such that for all points x on X and all (irreducible) curves C passing through x one has*

$$D.C \geq \varepsilon \, \mathrm{mult}_x C.$$

In the above theorem by mult_x we mean the multiplicity of C in the fixed point x, which can be defined in the following way. Let $\mu : \tilde{X} \to X$ by the blow up of X at a point x with the exceptional divisor E. Denote by C' the strict transform of a curve $C \subset X$. Then $\mathrm{mult}_x = C'.E$.

We conclude this section by presenting the following property.

Corollary 1.3.5. *If D_1, D_2 are numerically equivalent Cartier divisors on a projective variety X, then D_1 is ample iff D_2 is ample.*

1.4 Nefness of line bundles

A second important class of divisors is the class of nef divisors. Later on we will also describe nefness of divisors in the context of toric geometry (see Chapter 4).

Definition 1.4.1. Let X be a projective variety and let D be a Cartier divisor. We say that D is *nef* or *numerically effective* if for every (irreducible) curve $C \subseteq X$ we have $D.C \geq 0$.

A line bundle L is nef if the corresponding Cartier divisor D is nef.

Definition 1.4.2. Let X be a projective variety and let D be a Cartier divisor. We say that D is *pseudo-ample* if for every subvariety $Y \subset X$ we have $L^r.Y \geq 0$, where $\dim Y = r$.

Next theorem gives us a certain characterization of nef line bundles.

Theorem 1.4.3 (Kleiman's Theorem, Theorem 1.4.9, [23])**.** *Let X be a projective variety and let L be a line bundle. Then L is nef if and only if L is pseudo-ample.*

1.5 Bigness of line bundles

A third important class of divisors is the class of big divisors. Roughly speaking, big divisors can be characterized in the language of the volume and also in terms of the dimensions of the vector spaces $H^0(X, \mathcal{O}_X(mD))$ for $m >> 0$.

Definition 1.5.1. A line bundle L on an irreducible projective variety X of dimension n is big, if the number of sections $H^0(X, \mathcal{O}_X(mL))$ grows like Cm^n for $m >> 0$ with $C > 0$.

As usually, we will use the notion of bigness in the context of Cartier divisors.

The next result gives us a useful characterization of big divisors.

Theorem 1.5.2 (Corollary 2.2.7, [23])**.** *Let D be a Cartier divisor on an irreducible projective variety X. Then the following conditions are equivalent:*

- *D is big.*
- *For any ample integral divisor A on X, there exists a positive integer $m > 0$ and an effective divisor N such that $mD \equiv_{lin} A + N$.*
- *The same condition as in the above condition holds for a fixed ample divisor A.*
- *There exists an ample divisor A, a positive integer $m > 0$ and an effective divisor N such that $mD \equiv_{num} A + N$.*

Corollary 1.5.3. *The bigness of a divisor D depends only on its numerical equivalence class.*

The following result allows us to check directly whether a nef Cartier divisor D is big.

Theorem 1.5.4. *Let D be a nef divisor on an irreducible projective variety X of dimension n. Then D is big if and only if its top self-intersection is strictly positive, i.e. $(D^n) > 0$.*

At the end of this section let us recall an important result.

Theorem 1.5.5 (Hodge Index Theorem)**.** *Let H be a divisor on a normal projective surface X such that $H^2 > 0$. Then $H.D = 0$ implies that $D^2 \leq 0$. Moreover if $D^2 = 0$, then D is numerical equivalent to the trivial divisor.*

1.6 $\mathbb{Q}$-divisors

Recall that for a projective variety X we denote by $N^1(X)$ the set of numerical equivalence classes of integral divisors. We define the set of $\mathbb{Q}$-numerical equivalence classes as

$$N^1(X)_{\mathbb{Q}} := N^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In the same way we define the set of $\mathbb{R}$-numerical equivalence classes. One can extend the notions of nefness, bigness and ampleness for Cartier $\mathbb{Q}$-divisor (or $\mathbb{R}$-divisor). Also all theorems from the previous sections hold. In this part we would like to recall some facts related to cones in the finite dimensional $\mathbb{R}$-vector space $N^1_{\mathbb{R}}(X)$. By a cone we mean a set $K \subset N^1_{\mathbb{R}}(X)$ stable under multiplication by non-negative scalars from $\mathbb{R}$.

Definition 1.6.1. The ample cone $\mathrm{Amp}(X) \subset N^1_{\mathbb{R}}(X)$ is the convex cone containing all ample $\mathbb{R}$-divisor classes on X.

The nef cone $\mathrm{Nef}(X) \subset N^1_{\mathbb{R}}(X)$ is the convex cone of all nef $\mathbb{R}$-divisor classes on X.

The big cone $\mathrm{Big}(X) \subset N^1_{\mathbb{R}}(X)$ is the convex cone of all big $\mathbb{R}$-divisor classes on X.

The pseudoeffective cone $\overline{\mathrm{Eff}(X)} \subset N^1_{\mathbb{R}}(X)$ is the closure of the convex cone spanned by the classes of all effective $\mathbb{R}$-divisors on X.

If the Picard number $\rho(X) \geq 3$, then the structure of these cones can be quite complicated. In particular these cones may not be polyhedral − the easiest example is the self-product $E \times E$, where E is a (smooth) elliptic curve, see [23, Section 1.5B].

Sometimes it is more convenient to represent a cone by a compact slice − its intersection with an appropriate hyperplane not passing through the origin. We will extensively use this approach in the context of the Bauer - Küronya - Szemberg decomposition in Chapter 2.

The next two results recall important relations between cones mentioned above.

Theorem 1.6.2 (Kleiman, Theorem 1.4.23, [23])**.** *Let X be a irreducible projective variety.*

- *The nef cone is the closure of the ample cone:* $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$.
- *The ample cone is the interior of the nef cone:* $\mathrm{Amp}(X) = \mathrm{int}(\mathrm{Nef}(X))$.

Theorem 1.6.3 (Theorem 2.2.26, [23])**.** *Let X be an irreducible projective variety. The big cone is the interior of the pseudoeffective cone and pseudoeffective cone is the closure of the big cone, i.e.*

$$\mathrm{Big}(X) = \mathrm{int}(\overline{\mathrm{Eff}(X)}), \quad \overline{\mathrm{Eff}(X)} = \overline{\mathrm{Big}(X)}.$$

Now we introduce the notion of the volume for a Cartier divisor D.

Definition 1.6.4. Let X be an irreducible projective variety of dimension n and let L be a line bundle on X. The volume of L is defined as the non-negative real number

$$\mathrm{vol}(D) = \lim_{m \to \infty} \frac{\dim H^0(X, mL)}{m^n/n!}.$$

The volume of a Cartier divisor D is defined similarly by passing to $\mathcal{O}_X(D)$. Note that D is big iff $\mathrm{vol}(D) > 0$. The next simple proposition allows us to compute the volume in the language of the intersection theory.

Proposition 1.6.5. *Let X be a smooth projective variety of dimension n. Assume that D is a nef $\mathbb{Q}$-divisor. Then* $\mathrm{vol}(D) = D^n$ *– the top self-intersection number of D.*

1.7 Stable base locus

Our aim in this section is to recall the Nakamaye theorem on the augmented base locus of a $\mathbb{Q}$-divisor D. We will use this fact in the Chapter 2.

Definition 1.7.1. Let X be a projective variety and let D be a big divisor on V. Then the *stable base locus* $\mathbb{B}(D) \subseteq V$ is defined as

$$\mathbb{B}(D) = \bigcap_{m \geq 1} \mathrm{Bs}(|mD|).$$

The most important inconvenience occurring while working with $\mathbb{B}(D)$ is the fact that it does not depend only on the numerical equivalence class of D. This deficiency disappears if we work with the *augmented base locus.*

Definition 1.7.2. The augmented base locus $\mathbb{B}_+(D) \subseteq X$ of a $\mathbb{Q}$-divisor D is defined as

$$\mathbb{B}_+(D) = \mathbb{B}(D - A),$$

where A is a sufficiently small ample $\mathbb{Q}$-divisor.

One can show that the augmented base locus depends only on the numerical equivalence class of divisors on X. We will use the following important description of $\mathbb{B}_+$ given by Nakamaye. For a big and nef $\mathbb{Q}$-divisor D let us define the null locus $\mathrm{Null}(D) \subset X$ of D by the union of all positive dimensional subvarieties $V \subseteq X$ with

$$D^{\dim V}.V = 0.$$

Theorem 1.7.3 (Theorem 0.3, [22])**.** *Let X be a smooth projective variety of dimension n and let D be a big and nef $\mathbb{Q}$-divisor. Then $\mathbb{B}_+(D) = \mathrm{Null}(D)$.*

For instance by the above theorem we obtain that for an ample $\mathbb{Q}$-divisor A one has $\mathbb{B}_+(A) = \emptyset$.

Finally we recall the definition of the *restricted volume.* Let X be a complex smooth projective variety of dimension n and suppose that V is a d-dimensional subvariety. We set

$$W_m := \mathrm{Im}(H^0(X, \mathcal{O}_X(mD)) \overset{rest.}{\to} H^0(V, \mathcal{O}_V(mD))),$$

to be the space of sections in mD restricted to V.

Definition 1.7.4. The restricted volume of a divisor D to V is defined as

$$\mathrm{vol}_{X|V}(D) = \limsup_{m \to \infty} \frac{\dim_{\mathbb{C}} W_m}{m^d/d!}.$$

Chapter 2

Construction of Okounkov bodies and basic properties

2.1 Definition of Okounkov bodies

Assume that X is an irreducible projective variety of dimension n. In order to define the Okounkov body of a Cartier divisor D we start with the definition of an *admissible flag*. This is a sequence of irreducible subvarieties

$$Y_\bullet \ : \ X = Y_0 \supseteq Y_1 \supseteq ... \supseteq Y_n = \{pt\}, \tag{2.1}$$

such that $\mathrm{codim}_X Y_i = i$ and each Y_i is non-singular at the point $Y_n = \{pt\}$. To a divisor D on a projective variety X we would like to associate a valuation-like function

$$\nu = \nu_{Y_\bullet} = \nu_{Y_\bullet, D} : H^0(X, \mathcal{O}_X(D)) \to \mathbb{Z}^n \cup \{\infty\}$$

$$s \mapsto \nu(s) = (\nu_1(s), ..., \nu_n(s))$$

which will satisfy the following three conditions :

1. $\nu(s) = \infty$ if and only if $s = 0$,

2. ordering $\mathbb{Z}^n$ lexicographically, one has

$$\nu_{Y_\bullet}(s+t) \geq \min\{\nu_{Y_\bullet}(s), \nu_{Y_\bullet}(t)\}$$

   for any non-zero sections $s, t \in H^0(X, \mathcal{O}_X(D))$,

3. given non-zero sections $s \in H^0(X, \mathcal{O}_X(D))$ and $t \in H^0(X, \mathcal{O}_X(E))$,

$$\nu_{Y_\bullet, D+E}(s \otimes t) = \nu_{Y_\bullet, D}(s) + \nu_{Y_\bullet, E}(t).$$

Our aim is to construct each $\nu_i(s)$ inductively by computing the order of vanishing of the section s along each subvariety in the flag. We may assume that each Y_{i+1} is a Cartier divisor on Y_i by appropriately replacing X with an open subset.

Take a section $0 \neq s \in H^0(X, \mathcal{O}_X(D))$, and define

$$\nu_1 = \nu_1(s) = \mathrm{ord}_{Y_1}(s).$$

After choosing a local equation for Y_1 in X, say t_1, the section s determines a section

$$\widetilde{s_1} = s \otimes t_1^{-\nu_1} \in H^0(X, \mathcal{O}_X(D - \nu_1 Y_1))$$

that does not vanish identically along Y_1, so we get a non-zero section s_1 by restricting $\widetilde{s_1}$ to Y_1, i.e.

$$s_1 = \tilde{s_1}|_{Y_1} \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \nu_1 Y_1)).$$

Then we take $\nu_2 = \nu_2(s) = \mathrm{ord}_{Y_2}(s_1)$.

Generally, fix integers $a_1, ..., a_i \geq 0$ and denote by $\mathcal{O}(D - a_1 Y_1 - ... - a_i Y_i)|_{Y_i}$ the line bundle on Y_i, i.e.

$$\mathcal{O}_X(D)|_{Y_i} \otimes \mathcal{O}_X(-a_1 Y_1)|_{Y_i} \otimes \mathcal{O}_{Y_1}(-a_2 Y_2)|_{Y_i} \otimes ... \otimes \mathcal{O}_{Y_{i-1}}(-a_i Y_i)|_{Y_i}.$$

Suppose inductively that for $i \leq k$ one has constructed non-vanishing sections

$$s_i \in H^0(Y_i, \mathcal{O}(D - \nu_1 Y_1 - ... - \nu_i Y_i)|_{Y_i}),$$

with $\nu_{i+1}(s) = \mathrm{ord}_{Y_{i+1}}(s_i)$, so in particular $\nu_{k+1}(s) = \mathrm{ord}_{Y_{k+1}}(s_k)$. Dividing by the appropriate power of a local equation of Y_{k+1} in Y_k, say t_{k+1}, we obtain a section

$$\widetilde{s_{k+1}} = s_k \otimes t_{k+1}^{-\nu_{k+1}} \in H^0(Y_k, \mathcal{O}(D - \nu_1 Y_1 - ... - \nu_k Y_k)|_{Y_k} \otimes \mathcal{O}_{Y_k}(-\nu_{k+1} Y_{k+1}))$$

not vanishing along Y_{k+1}. Then we take

$$s_{k+1} = \widetilde{s_{k+1}}|_{Y_{k+1}} \in H^0(Y_{k+1}, \mathcal{O}(D - \nu_1 Y_1 - ... - \nu_{k+1} Y_{k+1}))|_{Y_{k+1}})$$

and continue this process up to $n = \dim X$.

Notice, that the sections s_i and $\tilde{s}_i$ depend on the choice of local equations, but the valuation numbers $\nu_i(s) \in \mathbb{Z}_{\geq 0}$ do not! By construction, this valuation-like function satisfies the above three conditions.

The next lemma, whose underlying idea is due to Okounkov, gives us a relation between valuation numbers $\nu_{Y_\bullet}$ and dimensions of certain linear series.

Lemma 2.1.1. *Let $W \subseteq H^0(X, \mathcal{O}_X(D))$ be a subspace. Fix $a = (a_1, ..., a_n) \in \mathbb{Z}^n$ and set*

$$W_{\geq a} = \{s \in W | \nu_{Y_\bullet}(s) \geq a\} \ , \ W_{>a} = \{s \in W | \nu_{Y_\bullet}(s) > a\},$$

where $\mathbb{Z}^n$ is ordered lexicographically. Then

$$\dim(W_{\geq a}/W_{>a}) \leq 1.$$

In particular, if W is finite dimensional, then the number of valuation vectors arising from sections in W is equal to the dimension of W:

$$\#(\mathrm{Im}((W \setminus \{0\}) \xrightarrow{\nu} \mathbb{Z}^n)) = \dim W.$$

The original proof due to Andrei Okounkov can be found in [27].

Since X is a projective variety, the spaces $H^0(X, \mathcal{O}_X(mD))$ are finite dimensional for every $m \in \mathbb{Z}$. In order to define the Okounkov body of D we need to define the graded semigroup of D.

Definition 2.1.2. The graded semigroup of D is the sub-semigroup

$$\Gamma(D) = \Gamma_{Y_\bullet}(D) = \{(\nu_{Y_\bullet}(s), m) \ : \ 0 \neq s \in H^0(X, \mathcal{O}_X(mD)), \, m \in \mathbb{Z}_{\geq 0}\} \subseteq \mathbb{Z}_{\geq 0}^{n+1}.$$

The set $\Gamma(D)$ will be also considered as a subset in $\mathbb{R}^{n+1}$ by the natural inclusion $\mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1}$.

Remark 2.1.3. At this point it is worth emphasizing that the semi-group $\Gamma(D)$ is usually not finitely generated – see [24, Example 1.7].

We abbreviate $\Gamma = \Gamma(D)$, and define

$$\Sigma(\Gamma) \subseteq \mathbb{R}^{n+1}$$

as the closed convex cone generated by Γ – this is the intersection of all closed convex cones containing Γ. The Okounkov body is the slice of the above cone at the level one.

Definition 2.1.4. The Okounkov body of D with respect to an admissible flag $Y_\bullet$ is the compact convex set

$$\triangle_{Y_\bullet}(D) = \triangle(D) = \Sigma(\Gamma) \cap (\mathbb{R}^n \times \{1\}).$$

The convexity of Okounkov bodies is straightforward from the construction, but compactness is not obvious at the first glance, thus we show the following proposition justifying this property.

Proposition 2.1.5. *The Okounkov body of D lies in a bounded subset of the positive orthant of $\mathbb{R}^n$.*

Proof. This proof comes from [24, pages 13 & 14]. That $\triangle(D)$ is a subset of the positive orthant is obvious by the construction, so in order to finish the proof it is enough to show that there exists a sufficiently large number $b(D, Y_\bullet) = b >> 0$ such that $\nu_i(s) \leq mb$ for every $i \in \{1, ..., n\}$, $m > 0$ and every non-zero section $s \in H^0(X, \mathcal{O}_X(mD))$. We pick an ample divisor H on X, and choose a number b_1, large enough, so that

$$(D - b_1 Y_1).H^{n-1} < 0$$

holds. This inequality tells us that $\nu_1(s) \leq mb_1$ holds for any section s chosen as above. Next, we choose b_2 large enough, so that

$$((D - aY_1)|_{Y_1} - b_2 Y_2).H^{n-2} < 0$$

for all $a \in [0, b_1]$. Then $\nu_2(s) \leq mb_2$ for all non-zero section $s \in H^0(X, \mathcal{O}_X(mD))$. Continuing this procedure we construct some $b_i \geq 0$ so that $\nu_i(s) \leq mb_i$ for $i \in \{1, ..., n\}$, and thus it is enough to take $b = \max_{i=1,...,n}\{b_i\}$.

□

Alternatively, we can define the Okounkov body as the closed convex hull of the set of normalized valuation vectors, i.e. with

$$\Gamma(D)_m = \mathrm{Im}((H^0(X, \mathcal{O}_X(mD)) \setminus \{0\}) \stackrel{\nu}{\mapsto} \mathbb{Z}^n),$$

we have

$$\triangle(D) = \text{ closed convex hull } (\bigcup_{m \geq 1} \frac{1}{m} \Gamma(D)_m).$$

At the end of this section we want to emphasize that the construction of Okounkov bodies can be mimicked easily with minor changes for graded linear series - see for instance [24, Section 1.3].

At this moment we omit examples of Okounkov bodies due to the fact that the whole next chapter will be devoted to construction of Okounkov bodies for big divisors on projective surfaces in details and thus we are asking the reader for patience.

2.2 Volume of Okounkov bodies and Fujita-type Approximation

The main goal of this section is to recall the proof the following theorem.

Theorem 2.2.1. *Let D be a big divisor on a projective variety X of dimension n. Then*

$$\mathrm{vol}_{\mathbb{R}^n}(\triangle(D)) = \frac{1}{n!}\mathrm{vol}_X(D),$$

where the Okounkov body $\triangle(D)$ of D is constructed with respect to an admissible flag $Y_\bullet$.

Note that on the left-hand side of the above equality we have the standard volume in $\mathbb{R}^n$, but on the right-hand side we have the volume of a divisor D, which was defined in Section 1.6.

We present the original proof from [24]. Let $\Gamma \subset \mathbb{Z}^{n+1}_{\geq 0}$ be a semigroup and set

$$\Sigma = \Sigma(\Gamma) = \text{ closed convex cone } (\Gamma) \subset \mathbb{R}^{n+1},$$

$$\triangle = \triangle(\Gamma) = \Sigma \cap (\mathbb{R}^n \times \{1\}).$$

Moreover we put for $m \in \mathbb{Z}_{\geq 0}$

$$\Gamma_m = \Gamma \cap (\mathbb{Z}^n_{\geq 0} \times \{m\}).$$

We *do not assume* that Γ is finitely generated, but we will suppose that it satisfies the following three conditions ($\diamond$)

1. $\Gamma_0 = \{0\} \in \mathbb{Z}^n_{\geq o}$,

2. there exist finitely many vectors $(v_i, 1)$ spanning a semi-group $B \subset \mathbb{Z}^{n+1}_{\geq 0}$ such that $\Gamma \subseteq B$,

3. Γ generates $\mathbb{Z}^{n+1}$ as a group.

These conditions imply that the set $\triangle(\Gamma)$ - which we consider in the natural way as a subset of $\mathbb{R}^n$ - is a convex body (the compact convex set). We will need the following two lemmas coming from [24].

Lemma 2.2.2. *Assume that a semi-group Γ satisfies the above three conditions ($\diamond$). Then*

$$\lim_{m\to\infty} \frac{\#\Gamma_m}{m^n} = \mathrm{vol}_{\mathbb{R}^n}(\triangle)$$

.

A proof of the above lemma can be found in [24, page 18].

Lemma 2.2.3. *Let X be a projective variety of dimension n, and let $Y_\bullet$ be any admissible flag of subvarieties of X. If D is a big divisor on X, then the graded semigroup*

$$\Gamma = \Gamma_{Y_\bullet}(D) \subset \mathbb{Z}^{n+1}_{\geq 0}.$$

associated to D satisfies ($\diamond$).

Proof. The first condition $\Gamma_0 = 0$ is clear. In order to prove the second condition we notice that by Proposition 2.1.5 there exists an integer $b >> 0$ with the property that

$$\nu_i(s) \leq mb \quad \text{for every } 1 \leq i \leq n \quad \text{and every } 0 \neq s \in H^0(X, \mathcal{O}_X(mD)).$$

This implies that Γ is contained in the semi-group $B \subseteq \mathbb{Z}^{n+1}_{\geq 0}$ generated by all vectors $(a_1, ..., a_n, 1) \in \mathbb{Z}^{n+1}_{\geq 0}$ with $0 \leq a_i \leq b$. It remains to show that Γ generates $\mathbb{Z}^{n+1}$ as a group.

To this end write $D = A - B$ as the difference of two very ample divisors. By adding a further very ample divisor to both A and B, we can suppose that there exist sections $s_0 \in H^0(X, \mathcal{O}_X(A))$ and $t_i \in H^0(X, \mathcal{O}_X(B))$ for $0 \leq i \leq n$ such that

$$\nu(s_0) = \nu(t_0) = 0,\ \nu_i(t_i) = e_i\ (1 \leq i \leq n),$$

where $e_i \in \mathbb{Z}^n$ is the i-th standard basis vector. In fact, it suffices that t_i is non-zero on Y_{i-1} while the restriction $t_i|_{Y_i}$ vanishes simply along Y_i in a neighborhood of the point Y_n. Next, since D is big, there is an integer $m_0 = m_0(D)$ such that $mD - B$ is linearly equivalent to an effective divisor F_m whenever $m > m_0$. Thus $mD \equiv_{lin} B + F_m$, and if $f_m \in \mathbb{Z}^n$ is the valuation vector of a section defining F_m, then we find that

$$(\star) \quad (f_m, m), (f_m + e_1, m), ..., (f_m + e_n, m) \in \Gamma.$$

On the other hand, $(m+1)D \equiv_{lin} A + F_m$, and so Γ also contains the vector $(f_m, m+1)$. Combined with $(\star)$, this exhibits the standard basis of $\mathbb{Z}^n$ as a subset of the group generated by Γ. ☐

These lead us to the proof of Theorem 2.2.1, which comes from [24].

Proof. Let $\Gamma = \Gamma(D)$ be the graded semigroup of D with respect to $Y_\bullet$. The two above lemmas give us

$$\mathrm{vol}_{\mathbb{R}^n}(\triangle(D)) = \lim_{m\to\infty} \frac{\#\Gamma(D)_m}{m^n}.$$

On the other hand, it follows from Lemma 2.1.1 that we have $\#\Gamma(D)_m = h^0(X, \mathcal{O}_X(mD))$, and then by definition the limit on the right side computes $\frac{1}{n!}\mathrm{vol}_X(D)$. ☐

Since for a big and nef divisor D on a projective variety of dimension n the volume in the sense of Definition 1.6.4 is equal to the self-intersection number D^n, the above theorem gives us an important invariant for Okounkov bodies and allows us to check whether we compute the Okounkov body of D appropriately – we need to compare the naively counted volume of $\triangle(D)$ with the number $\frac{1}{n!}D^n$.

One of the most interesting tools in the theory of linear series is the Fujita approximation, which can be formulated as follows.

Theorem 2.2.4 (Theorem 11.4.4, [23]). *Let ξ be a big class on an irreducible projective variety X of dimension n and fix any positive $\varepsilon > 0$. Then there exists a Fujita approximation on X*

$$\mu : X' \to X \quad , \quad \mu^*(\xi) = a + e$$

having the property that

$$\mathrm{vol}_{X'}(a) > \mathrm{vol}_X(\xi) - \varepsilon.$$

Moreover, if ξ is a rational class, then one can take a and e to be rational as well.

Theorem 2.2.4 can be proved in the language of Okounkov bodies.

Theorem 2.2.5 (Theorem 3.3, [24]). *Let D be a big divisor on an irreducible projective variety X of dimension n and for $p, k > 0$ write*

$$V_{k,p} = \mathrm{Im}\bigg(S^k H^0(X, \mathcal{O}_X(pD)) \to H^0(X, \mathcal{O}_X(pkD)\bigg).$$

Given $\varepsilon > 0$, there exists an integer $p_0 = p_0(\varepsilon)$ having the property that if $p \geq p_0$, then

$$\lim_{k\to\infty} \frac{\dim V_{k,p}}{p^n k^n/n!} \geq \mathrm{vol}_X(D) - \varepsilon.$$

The complete proof can be found in [24], or without using theory of Okounkov bodies, but using multiplier ideals in [23, Theorem 11.4.4 (Part II)].

A natural question arising from the Fujita approximation is whether one can express this theorem in the language of shapes of Okounkov bodies. Using Theorem 2.2.4 we obtain the following variation on the Fujita approximation.

Theorem 2.2.6. *Let X be a smooth projective variety of dimension n. Assume that D is a big divisor on X. Then for every $\beta, \beta_1 > 0$ there exists a birational morphism $\eta : \widetilde{X} \to X$, an ample divisor A on $\widetilde{X}$, $\delta > 0$, and any admissible flag $\widetilde{Y}_\bullet$ on $\widetilde{X}$ the Okounkov body $\triangle(\eta^*(D))$ contains $\triangle(A)$ with*

$$\mathrm{vol}_{\mathbb{R}^n}(\triangle(\eta^*(D)) \setminus \triangle(A)) < \beta_1$$

and is contained in $\triangle((1+\delta)A)$ with

$$\mathrm{vol}_{\mathbb{R}^n}(\triangle((1+\delta)A) \setminus \triangle(\eta^*(D))) < \beta.$$

Proof. By Theorem 2.2.4 we know that for a fixed $\varepsilon > 0$ there exists a birational morphism $\eta : \widetilde{X} \to X$, an ample divisor A and an effective divisor E on $\widetilde{X}$ such that

$$\eta^*(D) = A + E \quad \text{and} \quad \mathrm{vol}(\eta^*(D)) \geq \mathrm{vol}(A) \geq \mathrm{vol}(\eta^*(D)) - \varepsilon.$$

Take any admissible flag $Y_\bullet$ on $\widetilde{X}$ such that Y_1 is not contained in $\mathbb{B}_+(\eta^*(D))$. We obtain $\triangle(\eta^*(D)) = \triangle(A+E) \supseteq \triangle(A)$ since Okounkov bodies are convex and compact and additionally $\mathrm{vol}_{\mathbb{R}^n}(\triangle(\eta^*(D)) \setminus \triangle(A)) < \varepsilon \leq \beta_1$.

To conclude this proof it is enough show that there exists $\delta > 0$ as in the theorem. Notice that if we pass to the limit with $\varepsilon \to 0$, then Theorem 2.2.4 provides that the Okounkov body $\triangle(A)$ approaches to $\triangle(\eta^* D)$ and the difference between volumes of $\triangle(\eta^* D)$ and $\triangle(A)$ tends to 0. Thus for a fixed $\beta > 0$ one can find a sufficiently small $\varepsilon > 0$ and $\delta = \delta(\beta, \varepsilon) > 0$ such that

$$\triangle((1+\delta)A) \supset \triangle(\eta^*(D))$$

and

$$\mathrm{vol}_{\mathbb{R}^n}(\triangle((1+\delta)A) \setminus \triangle(\eta^*(D))) < \beta.$$

This completes the proof. □

2.3 Variation of Okounkov bodies

In this section we are going to deal with Okounkov bodies $\triangle(D)$ viewed as a set-valued function of divisors D. Following [24] we start with showing that the Okounkov body of D depends only on the numerical equivalence class of D.

Theorem 2.3.1 (Proposition 4.1 [24])**.** *Let D be a big divisor on a projective variety X of dimension n. The Okounkov body $\triangle(D)$ depends only on the numerical equivalence class of D.*

For every integer $p > 0$ we have the scaling property

$$\triangle(pD) = p \cdot \triangle(D),$$

where the expression on the right denotes the homothetic image of $\triangle(D)$ under the scaling by the factor p.

Proof. This proof comes from [24]. In order to prove the first property we need to show that $\triangle(D+P) = \triangle(D)$ for any numerically trivial divisor P. This follows from the fact that ampleness is a numerical condition, i.e. $A+C$ is ample for any ample A and $C \equiv_{num} 0$. By the Fujita's vanishing theorem [23, Theorem 1.4.35] there exists a fixed divisor B such that $B+kP$ is ample for every $k \in \mathbb{Z}$. Choose a large integer a such that $aD - B \equiv_{\text{lin}} F$ for some effective divisor (Theorem 1.5.2) and write

$$(m+a)(D+P) \equiv_{lin} mD + (aD - B) + (B + (m+a)P).$$

Based upon representing $B + (m+a)P$ by a divisor not passing through any of the subvarieties Y_i in the flag $Y_\bullet$, we have the following inclusion

$$\Gamma(D)_m + f \subseteq \Gamma(D+P)_{m+a},$$

where f is the valuation vector of the section defining F. Taking the limit $m \to \infty$ it follows that $\triangle(D) \subseteq \triangle(D+P)$. Replacing D by $D+P$ and P by $-P$ we obtain the opposite inclusion.

In order to prove the second property, choose an integer r_0 such that $|rD| \neq \emptyset$ for $r > r_0$ and take q_0 with $q_0 p - (r_0 + p) > r_0$. Then for each $r \in [r_0+1, r_0+p]$ we can fix effective divisors

$$E_r \in |rD|\,,\ F_r \in |(q_0 p - r)D|.$$

For every $r \in [r_0 + a, r_0 + p]$ this gives rise to the inclusion

$$|mpD| + E_r + F_r \subseteq |(mp+r)D| + F_r \subseteq |(m+q_0)D|,$$

and hence also

$$\Gamma(pD)_m + e_r + f_r \subseteq \Gamma(D)_{mp+r} + f_r \subseteq \Gamma(pD)_{m+q_0},$$

where e_r and f_r denote respectively the valuation vectors of E and F. Taking the limit $m \to \infty$ this gives

$$\triangle(pD) \subseteq p\triangle(D) \subseteq \triangle(pD),$$

what ends the proof. □

Now we define explicitly the Okounkov body for a big class $\xi \in \text{Big}(X)_{\mathbb{Q}}$.

Definition 2.3.2. Let $\xi \in \text{Big}(X)_{\mathbb{Q}}$ and choose any effective $\mathbb{Q}$-divisor representing ξ, say D and fix an integer $p >> 0$, which cleans the denominators of D. Then

$$\triangle(\xi) = \frac{1}{p}\triangle(pD).$$

Moreover in the same manner we can compute the volume of $\triangle(\xi)$.

Proposition 2.3.3. *Let $\xi \in \text{Big}(X)_{\mathbb{Q}}$. Then*

$$\text{vol}_{\mathbb{R}^n}(\triangle(\xi)) = \frac{1}{n!}\text{vol}_X(\xi).$$

Note that by Theorem 2.3.1 this is independent of the choice of D and p.

Proof. Choose a $\mathbb{Q}$-divisor D representing ξ and an integer $p >> 0$ clearing the denominators of D. Then the volume of ξ by definition is equal to

$$\text{vol}_X(\xi) = \frac{1}{p^n}\text{vol}_X(pD).$$

In the same manner

$$\text{vol}_{\mathbb{R}^n}(\triangle(\xi)) = \frac{1}{p^n}\text{vol}_{\mathbb{R}^n}(\triangle(pD)).$$

Thus using Theorem 2.2.1 we end the proof. □

We have showed that the definition of Okounkov bodies extends well to big classes $\mathrm{Big}(X)_{\mathbb{Q}}$. The question is whether we can construct the Okounkov body for a big $\mathbb{R}$-divisor D. By the paper [24] of Lazarsfeld and Mustaţă we know that it can be done by the construction of global Okounkov bodies. Now we recall only the main result without proof or details, because global Okounkov bodies do not belong to the main thread of this thesis.

Theorem 2.3.4 (Theorem 4.5, [24]). *Let X be a projective variety of dimension n. There exists a closed convex cone*

$$\triangle(X) \subset \mathbb{R}^n \times N^1(X)_{\mathbb{R}}$$

characterized by the property that in the diagram

$$\begin{array}{ccc} \triangle(X) & \xrightarrow{\quad i \quad} & \mathbb{R}^n \times N^1(X)_{\mathbb{R}} \\ \downarrow & \swarrow_{\mathrm{pr}_2} & \\ N^1(X)_{\mathbb{R}} & & \end{array}$$

the fibre of $\triangle(X)$ over any big class $\xi \in N^1(X)_{\mathbb{R}}$ is the Okounkov body $\triangle(\xi)$, i.e.

$$pr_2^{-1}(\xi) \cap \triangle(X) = \triangle(\xi) \subset \mathbb{R}^n \times \{\xi\} \cong \mathbb{R}^n.$$

Using the above theorem we obtain the following characterization of the volume function.

Theorem 2.3.5 (Corollary 4.12, [24]). *Let X be a projective variety of dimension n and fix an admissible flag $Y_\bullet$. There is a uniquely defined continuous function*

$$\mathrm{vol}_X : \mathrm{Big}(X)_{\mathbb{R}} \to \mathbb{R}$$

that computes the volume of any big real class. This function is homogeneous of degree n and log-concave, which means that for big classes $\xi_1, \xi_2 \in \mathrm{Big}(X)_{\mathbb{R}}$ we have

$$\mathrm{vol}_X(\xi_1 + \xi_2)^{1/n} \geq \mathrm{vol}_X(\xi_1)^{1/n} + \mathrm{vol}_X(\xi_2)^{1/n}.$$

We refer to [24] for the complete proof. It is worth to point out that the log-concavity of the volume function comes from the well-known Brunn-Minkowski inequality, which is a standard tool in convex geometry.

In the last part of this section we would like to look at the so-called slicing theorem. Roughly speaking, this theorem allows us to compute the Okounkov body with respect to a special flag, which contains a divisor $C = Y_1$, by computing the appropriate restricted volume. The idea of this theorem comes from Global Okounkov bodies.

Suppose that C is an irreducible (and reduced) Cartier divisor on a smooth projective variety X. We fix an admissible flag

$$Y_\bullet : X \supseteq C \supseteq ... \supseteq Y_{n-1} \supseteq Y_n = \{pt\}.$$

Let $\xi \in N^1(X)_{\mathbb{R}}$ be a big class and consider the Okounkov body

$$\triangle(\xi) \subset \mathbb{R}^n.$$

Denote by $\mathrm{pr}_1 : \triangle(\xi) \to \mathbb{R}$ the projection onto the first coordinate and set

$$\triangle(\xi)_{v_1 = t} = \mathrm{pr}_1^{-1}(t) \subset \{t\} \times \mathbb{R}^{n-1},$$

$$\triangle(\xi)_{v_1 \geq t} = \mathrm{pr}_1^{-1}([t, +\infty)) \subset \mathbb{R}^n.$$

Let $c \in N^1(X)_{\mathbb{R}}$ be the class of C and assume that C is not contained in $\mathbb{B}_+(\xi)$ − this assumption guarantees that $\triangle(\xi)_{v_1=0} \neq \emptyset$. Define the number

$$\mu(\xi, c) = \mu = \sup\{s > 0 : \xi - sc \in \mathrm{Eff}(X)\} = \sup\{s > 0 : \xi - sc \in \mathrm{Big}(X)\}$$

This invariant computes the right-hand side end point of the image $\triangle(\xi)$ under projection pr_1. We are ready to formulate the so-called slicing theorem.

Theorem 2.3.6 (Theorem 4.24, [24])**.** *Under the assumption as above, pick a number $t \in [0, \mu]$. Then*

$$\triangle(\xi)_{v_1 \geq t} = \triangle(\xi - tc) + te_1,$$

where $e_1 = (1, 0, ...0) \in \mathbb{R}^n$. Furthermore

$$\triangle(\xi)_{v_1 = t} = \triangle_{X|C}(\xi - tc),$$

where $\triangle_{X|C}$ means the Okounkov body computed with respect to the flag

$$Y_\bullet|C : C \supseteq Y_2 \supseteq ... \supseteq Y_n$$

on C.

The proof can be found in [24]. The slicing theorem allows to construct Okounkov bodies by calculating restricted volumes. In particular this approach allows us to construct Okounkov bodies on surfaces much simpler (comparing with the procedure given by the construction of Okounkov bodies). We will see how powerful this is in the next chapter.

Chapter 3

Okounkov bodies of big divisors on projective surfaces

3.1 Construction of Okounkov bodies, examples

In this chapter we will consider mainly the case that X is a smooth complex projective surface.

We start with recalling the celebrated Zariski decomposition – see [5] for a modern approach and [3] for the Fujita-Zariski decomposition of $\mathbb{R}$-divisors, which is defined in an analogous way.

Definition 3.1.1 (Zariski decomposition)**.** Let D be a pseudo-effective $\mathbb{Q}$–divisor on a projective surface X. Then there exist $\mathbb{Q}$–divisors P_D and N_D such that

a) $D = P_D + N_D$;

b) P_D is a nef divisor and N_D is either empty or supported on a union of curves $N_1, \ldots, N_r$ with negative definite intersection matrix;

c) $N_i \in \mathrm{Null}(P_D)$ for each $i = 1, \ldots, r$.

One of the most important properties of the Zariski decomposition is that the positive part P_D inherits all sections of D, i.e. there is the isomorphism

$$H^0(X, \mathcal{O}_X(mP_D)) \stackrel{mN_D}{\longrightarrow} H^0(X, \mathcal{O}_X(mD)),$$

for all $m > 0$ sufficiently divisible such that mP_D are Cartier divisors. This isomorphism is given by the multiplication with a section defining mN_D.

In the case of surfaces the so-called Bauer - Küronya - Szemberg decomposition [8] plays an important role. Let us recall the basic properties of this decomposition.

Theorem 3.1.2 (BKS - decomposition)**.** *There exists a locally finite decomposition of the big cone* $\mathrm{Big}(X)$ *into rational locally polyhedral subcones* Σ_i*, such that*

- *in the interior of each subcone* Σ *the support* $\mathrm{Neg}(\Sigma)$ *of the negative part of the Zariski decomposition of the divisors in the subcone is constant,*
- *on each of the subcones the volume function is given by a single polynomial of degree two,*
- *in the interior of each of the subcones the stable base loci are constant.*

In [24] the authors showed how to construct Okounkov bodies using the Zariski decomposition and the BKS - decomposition.

Fix an admissible flag on X

$$X \supset C \ni x,$$

where C is an irreducible curve and $x \in C$ is a smooth point. Recall from Section 2.3 that for a big $\mathbb{Q}$-divisor D we have defined the following number

$$\mu = \sup\{t > 0 : D - tC \in \mathrm{Eff}(X)\} = \sup\{t > 0 : D - tC \in \mathrm{Big}(X)\}.$$

Theorem 3.1.3 (Theorem 6.4, [24])**.** *Let D be a big $\mathbb{Q}$-divisor on X and let (x, C) be an admissible flag as above. Suppose that C is not contained in $\mathbb{B}_+(D)$. Let a be the coefficient of C in the negative part of the Zariski decomposition. For $t \in [a, \mu]$ let us define $D_t = D - tC$, where $0 \leq a \leq \mu$. Consider $D_t = P_t + N_t$ the Zariski decomposition of D_t. Put*

$$\alpha(t) = \mathrm{ord}_x(N_t), \quad \beta(t) = \alpha(t) + \mathrm{vol}_{X|C}(P_t) = \mathrm{ord}_x(N_t) + P_t.C.$$

Then the Okounkov body of D is the region bounded by the graphs of α and β, i.e.

$$\triangle(D) = \{(t, y) \in \mathbb{R}^2 : a \leq t \leq \mu \ \wedge \ \alpha(t) \leq y \leq \beta(t)\}.$$

Moreover, α and β are piecewise linear functions with rational slopes, α is convex and increasing, β is concave.

At the first glance one may think that Okounkov bodies on surfaces can be infinite polygons, for instance for big divisors on the blow up of $\mathbb{P}^2$ in 9 general points. In [20] the authors showed that this may not actually happen.

Theorem 3.1.4 (Theorem B, [20])**.** *Under the assumptions as in the previous theorem, Okounkov bodies with respect to some flag are finite polygons.*

It is worth to point out that Okounkov bodies for big divisors on projective surfaces are almost rational, i.e. described by rational data. The only one number, which may be irrational is μ. Now we show how the number μ is associated with a certain positivity measure, which is called the Seshadri constant.

Definition 3.1.5 (Seshadri constant, [32])**.** Let D be an ample divisor on X and let $\pi : \tilde{X} \to X$ be the blow-up of X at $x \in X$ with the exceptional divisor E. Then the Seshadri constant of D in x is defined as

$$\varepsilon = \varepsilon(D, x) = \sup\{t > 0 : \pi^*(D) - tE \text{ is nef on } \tilde{X}\}.$$

Proposition 3.1.6. *Let D be an ample divisor on X and denote by $\pi : \tilde{X} \to X$ the blow up of X at $x \in X$ with the exceptional divisor E. If $\varepsilon(D, x)$ is irrational, then $\mu(\pi^*(D), E) = \varepsilon(D, x)$ with respect to the flag $x \in E$.*

Proof. The proof partially comes from [20, page 10]. By Theorem 1.3.3 if $\pi^*(D) - tE$ is nef and not ample, then either there is a curve $C \subset \tilde{X}$ such that

$$C.(\pi^*(D) - \varepsilon E) = 0 \ \text{ or } \ (\pi^*(D) - \varepsilon E)^2 = 0.$$

Suppose that $C.(\pi^*(D) - \varepsilon E) = 0$. Since $C.\pi^*(D)$ and $E.C$ are rational numbers it implies that $E.C = 0$ and $\phi^*(D).C = 0$, but by the ampleness of D we get $C = E$, which leads to the contradiction. Thus

$(\pi^*(D) - \varepsilon E)^2 = 0$ and $\pi^*(D) - \varepsilon E$ is not big. This implies $\varepsilon \geq \mu$. In order to show that $\mu \geq \varepsilon$ recall that we have the following well-known inequality $\varepsilon \leq \sqrt{D^2}$, see for instance [32, page 7]. Then for an arbitrary number $\lambda \in [0, \sqrt{D^2})$ divisor $\pi^*(D) - \lambda E$ is big since it is nef with positive self-intersection. Thus $\mu \geq \sqrt{D^2} \geq \lambda$. □

The relationship between ε and μ is quite interesting, but there are no examples of divisors on surfaces with irrational Seshadri constants.

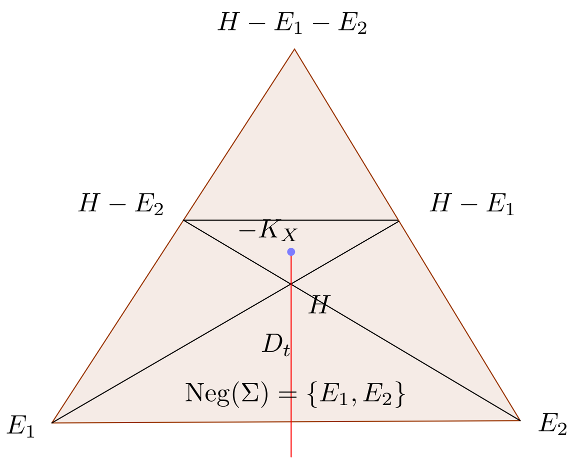
Now we go back to Theorem 3.1.3. Instead of presenting the proof, which is directly based on the slicing theorem and BKS - decomposition, we would like to explain the geometrical meaning of this theorem.

The functions α and β are constructed using the Zariski decomposition of the divisor $D_t = P_t + N_t$ for appropriate $t \in [a, \mu]$, but D_t can be also interpreted as a ray in $\mathrm{Eff}(X)$, which crosses at least one Zariski chamber (the nef chamber). The number μ tells us when the ray D_t leaves $\overline{\mathrm{Eff}}(X)$. Piecewise linearity of functions α and β is the consequence of the fact that on each Zariski chamber the support of the negative part of the Zariski decomposition is constant. It means that P_t and N_t vary linearly on each chamber. Moreover N_t can only increase with t and N_μ exists by the Fujita-Zariski decomposition [3]. Finally, the functions α and β have the so-called breaking points, which correspond to the number of Zariski chambers crossed by the ray D_t. Now we are ready to present how to construct Okounkov bodies using these technics.

Example 3.1.7. Let us consider X_2 the blow up of $\mathbb{P}^2$ in two points with exceptional divisors E_1, E_2. We denote by H the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$. Consider the anticanonical divisor $-K_{X_2} \equiv 3H - E_1 - E_2$. This divisor is ample and effective. We fix the anticanonical flag (x, C), and x is a general point of $C \in |-K_{X_2}|$, which means that x is not an intersection point of C with any negative curve. We want to construct the Okounkov body of $D = 7H - 2E_1 - 2E_2$. We will consider the following ray

$$D_t = D - t(3H - E_1 - E_2) = (7 - 3t)H - (2 - t)E_1 - (2 - t)E_2.$$

The figure below presents the compact slice of $\mathrm{Eff}(X_2)$ by a general hyperplane. By the red line we mark the ray D_t.



It is easy to see that $\mu = 2\frac{1}{3}$, thus $t \in [0, 2\frac{1}{3}]$. Now we want to construct the functions α and β. Since x is a general point, $\alpha(t) = \mathrm{ord}_x(N_t) \equiv 0$ for all $t \in [0, 2\frac{1}{3}]$ and we need to find β. Notice that for all

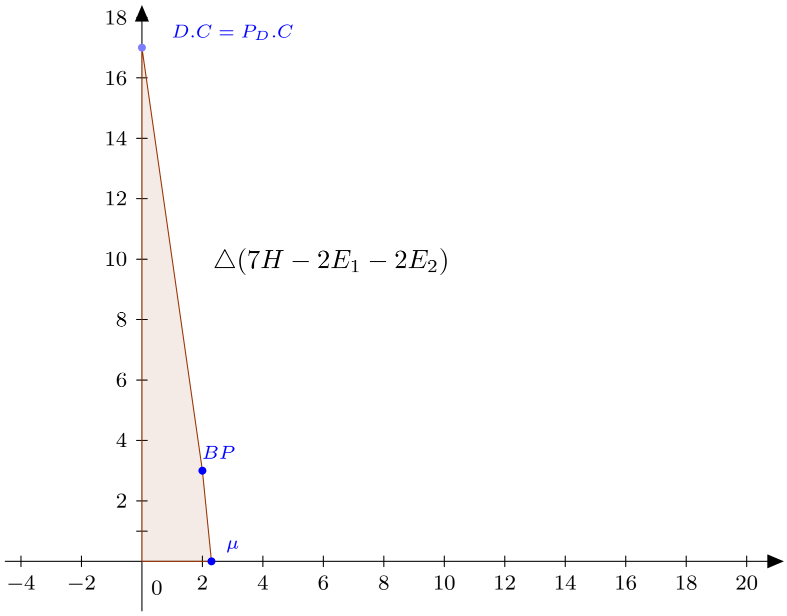
$t \in [0,2]$ our divisor D_t is nef and for $t = 2$ we have $D_2 = H$ – this is a vertex of $\mathrm{Nef}(X_2)$. It means that for $t \in [0,2]$ we have $D_t = P_t$ and we need to find the Zariski decomposition of D_t for $t \in [2, 2\frac{1}{3}]$. However by BKS - decomposition we know that the ray D_t crosses the chamber with the support of the negative part of the Zariski decomposition $\{E_1, E_2\}$. Thus

$$D_t = (7-3t)H + (t-2)E_1 + (t-2)E_2 \quad \text{for} \quad t \in \left[2, 2\frac{1}{3}\right]$$

is the Zariski decomposition with $P_t = (7-3t)H$ and $N_t = (t-2)E_1 + (t-2)E_2$. Summing up, we obtained

$$\beta(t) = \begin{cases} -((7-3t)H - (2-t)E_1 - (2-t)E_2).K_{X_2} = 17 - 7t & \text{for } t \in [0,2] \\ -(7-3t)H.K_{X_2} = 21 - 9t & \text{for } t \in [2, \frac{1}{3}] \end{cases}$$

By Theorem 3.1.3 we know that the Okounkov body $\triangle(D)$ is the region bounded by α and β functions, which is presented below. The breaking point of β is BP $= (2,3)$.



Example 3.1.8. Working under the same assumptions as in the previous example, we will find the Okounkov body of $D = 7H - 2E_1 - 2E_2$ with respect to the flag (x, C), where $x \in C$ is a general point and $C \in |H|$. Consider the ray

$$D_t = (7-t)H - 2E_1 - 2E_2.$$

Since $D_5 = 7H - 2E_1 - 2E_2 - 5H = 2H - 2E_1 - 2E_2$, $\mu = 5$ because $H - E_1 - E_2$ is a vertex of $\mathrm{Eff}(X_2)$. By the same reason as in the previous example $\alpha(t) \equiv 0$ for all $t \in [0,5]$, thus the only thing to find is β. Observe that for $t = 3$ one has

$$D_3 = 4H - 2E_2 - 2E_2 = 2(2H - E_1 - E_2).$$

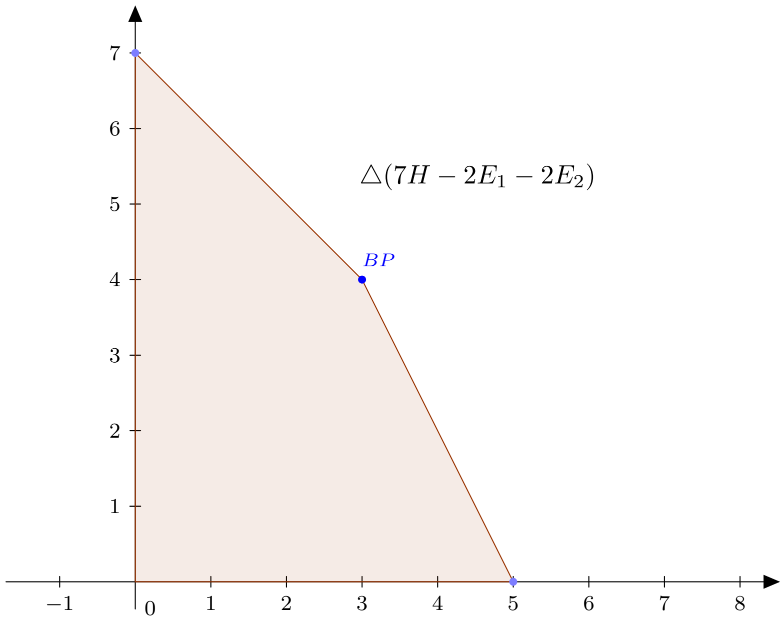
Since $2H - E_1 - E_2$ lies on the boundary of $\mathrm{Nef}(X_2)$ we have $D_t = P_t$ for $t \in [0,3]$. It is enough to compute β for $t \in [3,5]$. By BKS - decomposition we know that for D_t with $t \in [3,5]$ the support of the negative part of the Zariski decomposition is $\{H - E_1 - E_2\}$ – see for example [8]. It is not difficult to see that

$$D_t = (5-t)(2H - E_1 - E_2) + (t-3)(H - E_1 - E_2) \quad \text{for} \quad t \in [3,5]$$

is the Zariski decomposition with $P_t = (5-t)(2H - E_1 - E_2)$ and $N_t = (t-3)(H - E_1 - E_2)$. Thus β has the form

$$\beta(t) = \begin{cases} ((7-t)H - 2E_1 - 2E_2).H = 7-t & \text{for } t \in [0,3] \\ ((5-t)(2H - E_1 - E_2)).H = 10-2t & \text{for } t \in [3,5] \end{cases}$$

The figure below presents the Okounkov body of D, where BP $= (3,4)$ denotes the breaking point of β.



Now we present two examples of Okounkov bodies for divisors on X_2 computed with respect to the flag (x, N), where N is a negative curve and x is a general point.

Example 3.1.9. We find the Okounkov body for $D = 2H - E_1 - E_2$ with respect to the flag (x, E_1), where $x \in E_1$ is a general point. By Theorem 1.7.3 one has

$$\mathbb{B}_+(D) = \text{Null}(D) = H - E_1 - E_2,$$

thus E_1 is not contained in Null(D). Let us consider the ray

$$D_t = 2H - (1+t)E_1 - E_2.$$

Since $D_1 = 1(H - E_1) + 1(H - E_1 - E_2)$ lies on the boundary of $\text{Eff}(X_2)$, thus we get $\mu = 1$. Our task boils down to find the Zariski decomposition of D_t. By an easy inspection we see that the ray D_t crosses the Zariski chamber with the support of the negative part $\{H - E_1 - E_2\}$ and thus

$$D_t = ((2-t)H - E_1 - (1-t)E_2) + t(H - E_1 - E_2) \text{ for } t \in [0,1]$$

is the Zariski decomposition with $P_t = (2-t)H - E_1 - (1-t)E_2$ and $N_t = t(H - E_1 - E_2)$. Combining all data we obtain that $\alpha(t) \equiv 0$ and

$$\beta(t) = ((2-t)H - E_1 - (1-t)E_2).E_1 = 1,$$

hence $\triangle(D) = [0,1]^2 \subset \mathbb{R}^2_{\geq 0}$.

Example 3.1.10. Working under the same assumption we consider $D = 2H - E_2$ on X_2 and the flag (x, C), where x is a general point of $C = H - E_1 - E_2$. By Theorem 1.7.3 one has

$$\mathbb{B}_+(D) = \text{Null}(D) = E_1$$

and $H - E_1 - E_2$ is not contained in $\mathbb{B}_+(D)$. Consider the ray

$$D_t = (2-t)H - (1-t)E_2 + tE_1.$$

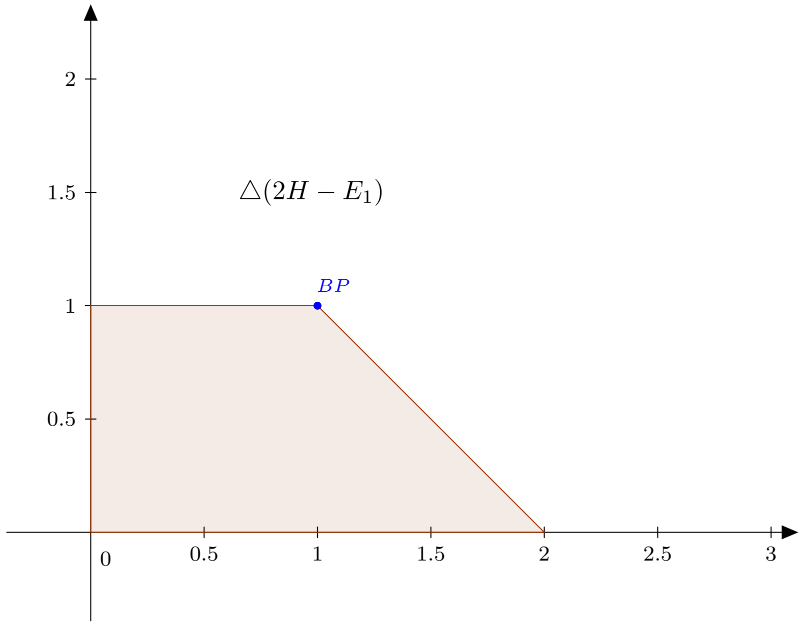
We see that $D_2 = E_1 + 2E_2$ lies on the boundary of $\mathrm{Eff}(X_2)$, thus $\mu = 2$. The ray D_t crosses two Zariski chambers different from $\mathrm{Nef}(X_2)$. Specifically for $t \in [0,1]$ the ray crosses the Zariski chamber Σ_{Q1} with the support of the negative part $\{E_1\}$ and for $t \in [1,2]$ the ray crosses the Zariski chamber Σ_L with the support of the negative part $\{E_1, E_2\}$ – see Example 3.5 from [8]. It is easy to see that the Zariski decomposition has the form

- for $t \in [0,1]$ we have $D_t = ((2-t)H - (1-t)E_2) + tE_1$ with $P_t = ((2-t)H - (1-t)E_2)$ and $N_t = tE_1$,
- for $t \in [1,2]$ we have $D_t = (2-t)H + ((t-1)E_1 + tE_2)$ with $P_t = (2-t)H$ and $N_t = (t-1)E_1 + tE_2$.

By the above considerations one gets $\alpha(t) \equiv 0$ for $t \in [0,2]$ and

$$\beta(t) = \begin{cases} ((2-t)H - (1-t)E_2).(H - E_1 - E_2) = 1 & \text{for } t \in [0,1] \\ (2-t)H.(H - E_1 - E_2) = 2 - t & \text{for } t \in [1,2] \end{cases}$$

The picture below presents the Okounkov body of D, and in that case BP = (1, 1).



As we know it is a tricky task to construct Okounkov bodies in general, but Theorem 3.1.3 in the case of surfaces gives us a systematic method, which allows us to construct Okounkov bodies easier. However it does not mean that this task is easy. The most complicated part relies on determining all negative curves on X, which are crucial in this method.

The natural question arising during the study of the shape of Okounkov bodies is the following.

Question 3.1.11. Let W be a smooth complex projective variety and let $Y_\bullet$ be an admissible flag. Given a big and nef divisor D, is it true that $\triangle(D)$ is always polyhedral?

The answer to the above question is NO, and one of the first examples was established in [20]. What is more surprising, the authors showed that Okounkov bodies for most of ample divisors on certain Fano variety can be non-polyhedral. This leads us to the following question.

Question 3.1.12. Let W be a smooth complex projective variety and let D be a big and nef divisor. Does there exist an admissible flag $Y_\bullet$ such that the Okounkov body $\triangle(D)_{Y_\bullet}$ is polyhedral ?

The answer on the above question is "partially yes". In [2] Anderson, Küronya and Lozovanu showed the following theorems. Recall that for a divisor D on a projective variety W the ring of sections is defined as

$$R(W,D) = \bigoplus_{m\geq 0} H^0(W, mL).$$

Theorem 3.1.13 (Theorem 1, [2]). *Let W be a normal complex projective variety and let L be a big divisor with finitely generated section ring. Then there exists an admissible flag $Y_\bullet$ on W such that the Okounkov body $\triangle(L)_{Y_\bullet}$ is a rational simplex.*

The authors mentioned that according to their knowledge it is an open question whether big and nef (but not semi-ample) divisors can have rational polytopes as Okounkov bodies if the dimension of the underlying variety is at least three. In the case of surfaces the authors give the complete answer.

Theorem 3.1.14 (Proposition 11, [2]). *Let X be a smooth projective surface and L be a big and nef line bundle on X. Then there exists an irreducible curve $Y_1 \subset X$ such that the Okounkov body $\triangle(L)_{Y_\bullet}$ is a rational simplex, where Y_2 is a general point on the curve Y_1.*

For more details we refer to the quoted paper.

Now we would like to focus on properties of Okounkov bodies related to convex geometry. In Sections 3.2, 3.3, 4.2 we will see that the polyhedrality of Okounkov bodies is an important property. Namely under certain mild assumptions we will see that Okounkov bodies for a certain class of surfaces and toric varieties can be decomposed as Minkowski sums of *polyhedral* smaller blocks (simplices, which are indecomposable with respect to the Minkowski sum), and in particular this means that Okounkov bodies are polyhedral.

The first question can be formulated in the following way.

Question 3.1.15. Let W be a smooth complex projective variety such that for a well-chosen admissible flag $Y_\bullet$ all Okounkov bodies are rational simplices. Does there exist a set of smaller blocks (simplices) corresponding to certain divisors such that all Okounkov bodies are Minkowski sums of them?

For two closed and convex sets A, B the Minkowski sum is defined as

$$A + B = \{a + b : a \in A, b \in B\}.$$

We start our investigations in the case of surfaces. We present two examples which show that the problem is not obvious at all.

Example 3.1.16. Let us consider $D = 2H - E_1 - E_2$ on X_2 as in Example 3.1.9. Then we take the "trivial" decomposition

$$2H - E_1 - E_2 = (H - E_1) + (H - E_2).$$

As we can easily see D is big (has positive volume), but $H - E_1$ and $H - E_2$ are nef and not big, thus

$$\triangle(2H - E_1 - E_2) \neq \triangle(H - E_1) + \triangle(H - E_2),$$

because the volume of the Minkowski sum of two segments is zero (both segments are vertical).

Example 3.1.17. Let us come back to Example 3.1.8. The divisor $D = 7H - 2E_1 - 2E_2$ can be written down as

$$D = 3H + 2(2H - E_1 - E_2),$$

and a simple inspection tells us that

$$D = \triangle(3H) + \triangle(2(2H - E_1 - E_2)) = 3\triangle(H) + 2\triangle(2H - E_1 - E_2).$$

In next section we are going to present the idea of the Minkowski decomposition of Okounkov bodies for a certain class of surfaces.

3.2 Minkowski Decompositions

We have previously seen it may happen that for two big and nef divisors D_1, D_2 on a smooth projective surface X we have

$$\triangle(D_1 + D_2) = \triangle(D_1) + \triangle(D_2),$$

where on the right-hand side we have the Minkowski sum.

Using the theory of global Okounkov bodies one may show that for big classes $\xi_1, \xi_2 \in N^1(X)_{\mathbb{R}}$ we have

$$\triangle(\xi_1) + \triangle(\xi_2) \subseteq \triangle(\xi_1 + \xi_2). \tag{3.1}$$

This containment is quite convenient if we want to construct Okounkov bodies by guessing the decomposition of divisors. Under the assumptions as in Example 3.1.17 suppose that for $D = 7H - 2E_1 - 2E_2$ we have the following decomposition

$$D = 3H + 2(2H - E_1 - E_2).$$

By the containment (3.1) of compact and convex sets the only thing to show is that the volumes of the bodies are the same. Since $D = 7H - 2E_1 - 2E_2$ is big and nef, thus its volume is equal to $D^2 = 49 - 8 = 41$ and using Theorem 2.2.1 the volume of the Okounkov body is equal to $\frac{41}{2}$. On the right-hand side we can compute the volume naively, it is equal to $\frac{41}{2}$ and we are done. This approach is naive and the next example shows us that it is quite difficult to obtain such decomposition by hand.

Example 3.2.1. We present Example 3.2 from [25]. Let X_3 be the blow up of $\mathbb{P}^2$ in three general points with exceptional divisors E_1, E_2, E_3. We fix the flag (x, C), where x is a general point on $C \in |H|$, where H is the pull back of $\mathcal{O}_{\mathbb{P}^2}(1)$ Take $D_1 = 3H - 2E_1 - E_2$ and $D_2 = 4H - 2E_1 - 2E_2 - 2E_3$, which are big and nef. It is easy to see that

$$\triangle(D_1 + D_2) \neq \triangle(D_1) + \triangle(D_2).$$

One may ask about conditions for which the following equality holds

$$\triangle(D_1 + D_2) = \triangle(D_1) + \triangle(D_2) \quad D_1, D_2 \in \mathrm{Big}(X).$$

Recently Łuszcz - Świdecka in her doctoral thesis has presented a different approach to the construction of Okounkov bodies in the case of complex smooth projective surfaces X with rational polyhedral pseudoeffective cones $\overline{\mathrm{Eff}}(X)$. Firstly, she has constructed the Minkowski decomposition for the blow up of $\mathbb{P}^2$ in three general points [25], and later together with Schmitz she has expanded her result to the more general setting [26].

Theorem 3.2.2 (Minkowski decomposition)**.** *Let X be a smooth complex projective surface with $\overline{\text{Eff}(X)}$ rational polyhedral. Given a flag (x, C), where x is a general point and C is a big and nef curve on X there exists a finite set of nef divisors $\text{MB}_{(x,C)} = \{P_1, ..., P_r\}$ such that for a big and nef $\mathbb{R}$-divisor D there exist uniquely determined non-negative real numbers $a_i \geq 0$ with*

$$D = \sum_{i=1}^{r} a_i P_i \quad \textit{and} \quad \triangle(D) = \sum_{i=1}^{r} a_i \triangle(P_i),$$

where the first sum indicates the numerical equivalence of divisors and the second sum is the Minkowski sum of convex bodies.

The above theorem justifies the following definition.

Definition 3.2.3 (Minkowski basis)**.** The set $\text{MB}_{(x,C)}$ in Theorem 3.2.2 is called the *Minkowski basis* of X with respect to the flag (x, C).

Remark 3.2.4. Note that in general $\text{MB}_{(x,C)}$ is not a basis of the Neron-Severi space $N^1(X)_{\mathbb{R}}$ (treated as an $\mathbb{R}$–vector space).

Since for surfaces with rational polyhedral pseudoeffective cone $\overline{\text{Eff}(X)}$ the number of negative curves is finite, thus the number of Zariski chambers is finite as well. There is a very nice idea of finding supports of negative parts of the Zariski decomposition for each chamber – see Proposition 1.1 [7]. Roughly speaking this proposition tells us that for surfaces containing finitely many negative curves there is a one-to-one correspondence between supports of the negative part of Zariski decomposition and negative definite principal submatrices of the intersection matrix of these negative curves, so this methods works effectively if we have full information about relations between negative curves, for instance in the case of del Pezzo surfaces.

The basic idea of the Minkowski decomposition is to assign to each Zariski chamber Σ the Minkowski base element M_Σ. Here we present the sketch of this idea.

Let Σ be a Zariski chamber and let $\{N_1, ..., N_k\}$ be a set of negative curves supporting the negative part of the Zariski decomposition on Σ. Fix a flag (x, C), where C is a big and nef curve and x is a general point on C. Now we consider the span of C with negative curves $N_1, ..., N_k$ and $N_1^\perp \cap ... \cap N_k^\perp$, which is the face of $\text{Nef}(X)$. We intersect the span $\langle C, N_1, ..., N_k \rangle$ with $N_1^\perp \cap ... \cap N_k^\perp$ and obtain the rational ray (because $\overline{\text{Eff}(X)}$ is described by the rational data) of the form

$$M_\Sigma = dC + \sum_{i=1}^{k} \alpha_i N_i$$

for certain α_i. Now we show that α_i and d have the same sign and thus M_Σ or $-M_\Sigma$ is nef. To see this it is enough to solve the following system of equations

$$S(\alpha_1, ..., \alpha_k)^t = -d(C.N_1, ..., C.N_k)^t, \tag{3.2}$$

where S is the intersection matrix of $N_1, ..., N_k$, which is negative definite – see Proposition 1.1 [7] or Theorem 3.4.2 in a further part of this thesis. The matrix S^{-1} has only negative entries by Lemma 4.1 in [8], and since C is big and nef we have $C.N_i \geq 0$, thus the above system of equations has the unique solution for fixed d and the sign of d is the same as that of α_i. The only thing which remains to be checked is that M_Σ is a nef divisor, but it follows from the definition of this element.

Now we can say something more about Minkowski basis elements $\text{MB}_{(x,C)}$. For a fixed flag (x, C) the Minkowski basis consists of the curve C, all divisors which are nef and not big (these divisors correspond to Okounkov bodies, which are segments) and elements M_{Σ_i} constructed as above, which correspond to two dimensional simplices, so all elements correspond to indecomposable Okounkov bodies (with respect to the Minkowski sum).

Now we justify only that the Okounkov body of C is indecomposable. We will use Theorem 3.1.3. Using Theorem 1.5.5 and Theorem 1.7.3 the curve C is not contained in $\mathbb{B}_+(C)$. It is easy to see that $\mu = 1$ and for every $t \in [0,1]$ the ray $D_t = C - tC$ is nef. Thus $D_t = P_t$ and

$$\triangle(C) = \{(t,y) \in \mathbb{R}^2 : 0 \le t \le 1 \ \wedge \ 0 \le y \le C^2 - tC^2\},$$

which shows that $\triangle(C)$ is the simplex of height C^2 and length 1.

To conclude we present briefly the algorithm as in [26], which finds a Minkowski decomposition of a fixed big divisor D. Without loss of generality we may consider only the situation with a divisor D, which is at least nef. To justify this claim we recall the following fact.

Proposition 3.2.5 (Corollary 2.2, [25]). *Let X be a smooth complex projective surface and let D be a $\mathbb{Q}$-pseudoeffective divisor with the Zariski decomposition $D = P_D + N_D$. Assume that (x,Y) is an admissible flag, such that Y is not contained in $\mathbb{B}_+(D)$. Then*

$$\triangle(D) = \triangle(P_D) + (0, \mathrm{ord}_x(N_D)).$$

If x is a general point on Y, then of course $\triangle(D) = \triangle(P_D)$.

We have the following possibilities.

1. If D is nef and not big, then $D^2 = 0$ and D is proportional to one of chosen Minkowski basis elements with the same properties (nef and not big), thus it can be expressed as $D = bD'$ and the Okounkov body is a segment
$$\triangle(D) = b\triangle(D')$$
for certain $b > 0$ - see Theorem 2.3.1 and [24, Remark 4.2].

2. Assume that D is a nef and big divisor and let Σ be the corresponding Zariski chamber. Let M_Σ be the Minkowski basis element and set
$$\tau = \sup\{t : D - tM_\Sigma \text{ is nef }\}.$$
Then the $\mathbb{Q}$-divisor $D' = D - \tau M_\Sigma$ lies on the boundary on the face $\mathrm{Nef}(X) \cap \mathrm{Null}(D)$. If $D' = 0$ we are done. Otherwise it can be shown [26] that $\triangle(D) = \tau\triangle(M_\Sigma) + \triangle(D')$. In the next step we repeat this procedure for D'. The algorithm terminates after at most $\rho = \rho(X)$ steps since in every step the dimension of the face of the nef cone in which D lies decreases. Eventually, we end up with either 0 or a divisor spanning an extremal ray of the nef cone and we proceed as in the first case.

Now we present some simple examples, which illustrate the idea of the Minkowski decomposition. In order to compare this approach with Theorem 3.1.3 we will consider the same flags and divisors as in the previous section.

We describe Minkowski bases elements in the case of del Pezzo surfaces X_i as the blow ups of $\mathbb{P}^2$ in i general points with $i \in \{1, ..., 8\}$. Recall that the idea of the construction of Minkowski bases is to assign to each Zariski chamber a Minkowski base element and in order to obtain it we need to solve the system of equations (3.2).

Now we show that actually that $S = -I_k$, where $k = \#\mathrm{Neg}(\Sigma)$. It is enough to consider (2×2) - intersection matrix S_1 for negative curves C_1, C_2 and we need to show that $C_1.C_2 = 0$. Recall also that in the case of del Pezzo surfaces all negative curves are (-1)-curves.

Since S_1 is negative definite thus we have

$$0 > C_1^2 + C_2^2 + 2C_1C_2 = -2 + 2C_1C_2,$$

hence $C_1C_2 < 1$. On the other hand, since C_1, C_2 are effective we get $C_1C_2 \geq 0$ and thus $C_1C_2 = 0$.

Hence we obtain that if $\mathrm{Neg}(\Sigma) = \{N_1, ..., N_k\}$, thus for a fixed big and nef flag (x, C) the Minkowski basis element has the form

$$M_\Sigma = dC + \sum_{i=1}^{k} (N_i.C)N_i. \tag{3.3}$$

Example 3.2.6. We consider del Pezzo surface X_2 with the fixed flag (x, C), where x is a general point on $C \in |H|$. By the construction, the Minkowski basis consists of H, all divisors, which are nef and not big, which is equal to the set $\{H - E_1, H - E_2\}$ and to each Zariski chamber we need to assign the Minkowski basis element. For X_2 there are 5 Zariski chambers (figure on page 22), one of them is $\mathrm{Nef}(X)$ and the remaining four chambers correspond to the following negative supports of the Zariski decomposition:

$$\mathrm{Neg}(\Sigma_1) = \{E_1, E_2\},\ \mathrm{Neg}(\Sigma_2) = \{E_1\},\ \mathrm{Neg}(\Sigma_3) = \{E_2\},\ \mathrm{Neg}(\Sigma_4) = \{H - E_1 - E_2\}.$$

Using (3.3) and fixing $d = 1$ we obtain $M_{\Sigma_1} = M_{\Sigma_2} = M_{\Sigma_3} = H$ and $M_{\Sigma_4} = 2H - E_1 - E_2$. Thus the Minkowski basis is

$$\mathrm{MB}_{(x,H)} = \{H, H - E_1, H - E_2, 2H - E_1 - E_2\}.$$

Example 3.2.7. Under the assumption of the previous example, we find the Minkowski decomposition of $D = 7H - 2E_1 - 2E_2$. Since D is ample we start with subtracting H until we hit the boundary of $\mathrm{Nef}(X)$ and this happens for $t = 3$. Now according to the algorithm we find $\mathrm{Null}(D') = \{H - E_1 - E_2\}$ and the corresponding Minkowski basis element is $2H - E_1 - E_2$. Since $D' - 2(2H - E_1 - E_2) = 0$, thus we obtain the Minkowski decomposition

$$D = 3H + 2(2H - E_1 - E_2) \quad \text{and} \quad \triangle(D) = 3\triangle(H) - 2\triangle(2H - E_1 - E_2).$$

The effectiveness of this approach can be better appreciated for example in the case of the del Pezzo surfaces X_i with $i \in \{6, 7, 8\}$.

3.3 Minkowski chambers and degenerations of Okounkov bodies

In this section we would like to focus on the following problem.

Let X be a smooth complex projective surface and suppose that we chose a "nice" flag (x, C) and an ample divisor A. What can happen with the shape of the Okounkov body of A if we take a small perturbation? More specifically, what are the relations between $\triangle(A)$ and $\triangle(A \pm B)$ for a certain divisor B.

In order to formulate the main proposition we will need some facts from Mori theory. We refer to [13] for a more detailed introduction to the subject.

For a smooth projective surface let us denote by $\overline{\mathrm{NE}(X)}$ the closure of the sef of classes of effective 1-cycles.

Theorem 3.3.1 (Mori's Cone Theorem, Section 5.4, [13])**.** *Let X be a smooth projective surface. There exists at most countable family of irreducible rational curves C_i such that $-3 \leq K_X.C_i < 0$ and*

$$\overline{\mathrm{NE}(X)} = \overline{\mathrm{NE}(X)}_{K_X \geq 0} + \sum_i \mathbb{R}_+[C_i].$$

The rays $\mathbb{R}_+[C_i]$ *are extremal and can be contracted.*

Here by $\overline{\mathrm{NE}(X)}_{K_X \geq 0}$ we mean the set of all effective 1-cycles with non-negative intersection with the canonical divisor K_X. In the case of surfaces, contractions are described by the Castelnuovo theorem.

Theorem 3.3.2 (Castelnuovo)**.** *Let* X *be a smooth projective surface. If* C *is a smooth rational curve with* $C^2 = -1$*, then there exists a smooth projective surface* Y*, a point* $p \in Y$ *and a morphism* $\varepsilon : X \to Y$ *such that* $\varepsilon(C) = p$ *and* ε *is isomorphic to the blow-up of* Y *at* p*.*

We will also need the following description of extremal rays.

Theorem 3.3.3 (Proposition 4.5, [13])**.** *Let* X *be a smooth complex projective surface.*

- *The class of an irreducible curve* $C^2 \leq 0$ *lies on the boundary of* $\overline{\mathrm{NE}(X)}$*.*
- *The class of an irreducible curve* $C^2 < 0$ *spans an extremal ray of* $\overline{\mathrm{NE}(X)}$*.*
- *If* r *spans an extremal ray of* $\overline{\mathrm{NE}(X)}$ *and* $r^2 < 0$*, the extremal ray is spanned by the class of an irreducible curve.*

Now we want to show how Okounkov bodies of certain divisors are related to types of extremal rays of the pseudoeffective cone.

Suppose that C is a big and nef curve and $x \in C$ is a general point. Let us take an ample divisor A. We will consider rays of the form

$$D_t = A - tC = P_t + N_t,$$

with $t \in [0, \mu]$ and μ as in Theorem 3.1.3. There are two main cases.

Suppose that there is $t_0 > 0$ such that the ray D_{t_0} hits the boundary of $\mathrm{Nef}(X)$ and $D_{t_0}^2 = 0$. By Theorem 3.3.3 it means that D_{t_0} lies on the boundary of $\overline{\mathrm{Eff}}(X)$ and $t_0 = \mu$. By Theorem 3.1.3 Okounkov bodies are described by functions α and β, i.e.

$$\triangle(A) = \{(t, y) \in \mathbb{R}^2 : 0 \leq t \leq \mu \ \wedge \ \alpha(t) \leq y \leq \beta(t)\}.$$

Since x is a general point, thus $\alpha(t) \equiv 0$ for all $t \in [0, \mu]$. Now we want to see how the function $\beta(t) = P_t.C$ varies. Suppose that $\beta(\mu) = 0$. By Theorem 1.5.5 it means that $(A - \mu C)^2 \leq 0$. There are two subcases.

1) $(A - \mu C)^2 = 0$, thus $A \equiv_{num} \mu C$ and $\triangle(A) =$ convex hull $\{(0, 0), (0, A^2), (1, 0)\}$.

2) $(A - \mu C)^2 < 0$, but this is absurd since $(A - \mu C)$ is nef.

The second subcase implies that if A is not numerical equivalent to μC, then $\beta(\mu) > 0$ and the Okounkov body has the form

$$\triangle(A) = \mu \triangle(C) + \gamma \triangle(A - \mu C)$$

for a certain $\gamma > 0$. Notice that in this particular case the assumption that x is a general point does not matter since for every $t \in [0, \mu]$ we have $N_t \equiv 0$.

Assume now that there is a number $t_1 > 0$ such that $D_{t_1}^2$ is an extremal ray with negative self-intersection. Thus $\mu = t_1$ and by the above proposition D_{t_1} is spanned by the irreducible curve. Now we want to show that $\beta(\mu) = 0$ and $\triangle(A)$ is a Minkowski sum of regular two dimensional simplices. In order to finish our consideration we need to see that $D_{t_1} = N_{t_1}$, which means that the positive part P_{t_1} is trivial, but this is pretty obvious since this extremal ray is an irreducible negative curve. Thus $\beta(\mu) = 0$ and in particular this implies that a segment line is not a Minkowski summand of $\triangle(D)$. All these considerations can be summed up by the following proposition.

Proposition 3.3.4. *Let X be a smooth complex projective surface. Fix a flag (x, C), where $x \in C$ is a general point and C is big and nef. Let A be an ample divisor and denote by $D_t = A - tC$ the ray in $\overline{\mathrm{Eff}}(X)$.*

1. *If there exists a number $t_0 > 0$ and D_{t_0} hits the boundary of $\mathrm{Nef}(X)$ with $D_{t_0}^2 = 0$, then $\mu = t_0$ and either*
$$\triangle(D) = \text{ convex hull } \{(0,0), (0, A^2), (1,0)\}$$
or
$$\triangle(A) = \mu\triangle(C) + \gamma\triangle(A - \mu C).$$
2. *If there exists a number $t_1 > 0$ such that $D_{t_1}^2$ is an extremal ray with negative self-intersection, then $\mu = t_1$ and the Okounkov body of D is the finite Minkowski sum of regular two dimensional simplices.*

Let us present a certain example, which shows that the assumption in (2) that D_μ is an extremal ray is essential.

Example 3.3.5. Let us consider X_2 and fix a flag $x \in C \in |H|$, where x is a general point. Consider the divisor $D = 4H - 2E_1 - E_2$ and as usually let

$$D_t = D - tC = (4-t)H - 2E_1 - E_2.$$

Notice that $D_2 = 2H - 2E_2 - E_2$ lies on the boundary of $\overline{\mathrm{Eff}}(X)$ and $D_2^2 < 0$. Moreover $\mu = 2$. A simple computation shows that $D_2 = (H - E_1) + (H - E_1 - E_2)$ is the Zariski decomposition and

$$\beta(2) = P_2.C = (H - E_1).H = 1.$$

Using the theory of Minkowski decompositions it is easy to see that

$$\triangle(D) = \triangle(H) + \triangle(2H - E_1 - E_2) + \triangle(H - E_1).$$

Finally we would like to see a relation between Okounkov bodies and types of contractions in the following example.

Example 3.3.6. Let us consider X_2. In this case we have 5 extremal rays, $E_1, E_2, H - E_1 - E_2$ which span $\overline{\mathrm{Eff}}(X)$ and $H - E_1$, $H - E_2$, which are vertices of $\mathrm{Nef}(X)$. Rays $H - E_1$ and $H - E_2$ can be contracted to $\mathbb{P}^1$. Contractions of E_1, E_2 lead us to X_1 (the blow up of $\mathbb{P}^2$ at single point) and the last divisor $H - E_1 - E_2$ gives us the contraction to $\mathbb{P}^1 \times \mathbb{P}^1$.

The above considerations deliver an interesting picture, which tells us that certain Okounkov bodies encode some information about the birational geometry of surfaces.

Problem 3.3.7. Is it possible to generalize the above consideration for higher dimensional smooth projective varieties ?

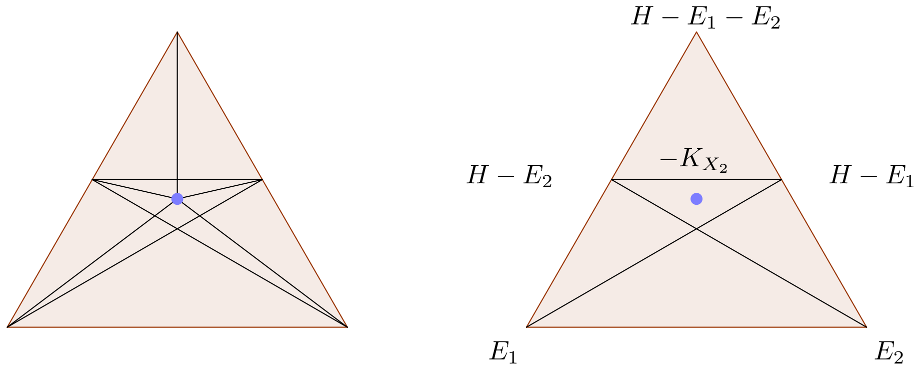
Proposition 3.3.4 is related to the Minkowski chamber decomposition [30], which we would like to introduce now.

Let X be a smooth projective surface with the rational polyhedral pseudoeffective cone. Then the cone $\overline{\mathrm{Eff}}(X)$ can be decomposed into a finite number of Minkowski chambers in the following way. Each chamber, which is a simplicial cone, is spanned by exactly ρ of the Minkowski basis elements $(D_{i_1}, ..., D_{i_\rho})$, which are vertices. If there is a Minkowski basis element M not contained in one of the rays spanning

$\overline{\text{Eff}}(X)$ then decompose $\overline{\text{Eff}}(X)$ into subcones spanned by the sides of $\overline{\text{Eff}}(X)$ and the ray spanned by M. Repeat the process for each subcone until no Minkowski base elements apart from spanning ones lie in each cone. Now, we can pass to a triangulation of each subcone into simplicial cones without having to add any new rays. We call the resulting subcones *Minkowski chambers* of $\overline{\text{Eff}}(X)$. Note that the chambers are rational cones (generated by the corresponding Minkowski base elements) and that the coefficients of the Minkowski decomposition are unique and vary linearly on the closure of each chamber, provided we allow only decompositions with respect to the base elements spanning the chamber. The chamber decomposition needs not to be unique, because during this process we choose generators of each cone.

For details see [30] page 3.

Example 3.3.8. Consider X_2 with (x, C), where $C \in |-K_{X_2}|$. Then one can divide the $\overline{\text{Eff}}(X)$ into the following subcones.



According to the above theorem there are five special rays, i.e. those joining $-K_{X_2}$ with $H - E_1$, $H - E_2$, $H - E_1 - E_2$, E_1 and E_2. Of course the Minkowski chamber decomposition depends on the choice of an admissible flag and generators of each subcones, but still it is a nice tool, which gives information about possible shapes of Okounkov bodies.

3.4 On the cardinality of Minkowski bases

In this section we would like to consider the question related to the number of a Minkowski basis elements with respect to a fixed flag (x, C) with C big and nef. Actually the answer to this problem is a nice combination of a purely geometrical picture with relations between Zariski chambers. Our aim is to show that on a smooth projective surface with $\overline{\text{Eff}}(X)$ rational polyhedral if (x, A) is taken to be a flag with A ample the number of the Minkowski basis elements is the largest possible.

To begin with, we define two numbers

$$\text{NnB}(X) = \#\{D \in N^1(X) : D \text{ is nef and not big }\},$$

$$\text{Zar}(X) = \text{ number of Zariski chambers }.$$

In our convention the number $\text{Zar}(X)$ *does not count the nef cone.* Also it is more convenient for us to consider the compact slice of $\text{Nef}(X)$, thus for a fixed ample divisor H we define

$$\text{Nef}_H = \{D \in \text{Nef}(X) : D.H = 1\}.$$

It is natural to ask how many elements are there in the Minkowski basis. We will show here that the answer depends on the choice of the flag and that the number

$$1 + \text{NnB}(X) + \text{Zar}(X) \tag{3.4}$$

is a sharp upper bound for the number of elements in the Minkowski basis. The number of negative curves on surfaces with $\overline{\mathrm{Eff}(X)}$ rational polyhedral is finite, hence the number of Zariski chambers on such surfaces is finite as well. This number can be large. For example for del Pezzo surfaces X_i (blow ups of $\mathbb{P}^2$ at i general points) we have

i	1	2	3	4	5	6	7	8
$\mathrm{Zr}(X_i)$	2	5	18	76	393	2 764	33 645	1 501 681

, (3.5)

where $\mathrm{Zr}(X_i) = \mathrm{Zar}(X_i) + 1$ – see [7] for details.

In this section we will use the following results due to Bauer, Funke and Neumann [7, Proposition 1.1].

Theorem 3.4.1. *Let X be a smooth projective surface with finitely many negative curves. Suppose that S is the intersection matrix of these negative curves. Then the number of Zariski chambers supported on these negative curves is equal the number of negative definite principal submatrices of the matrix S.*

Theorem 3.4.2. *Under the above assumption, the set of Zariski chambers of X that are different from* $\mathrm{Nef}(X)$ *is in bijective correspondence with the sets of reduced divisors on X whose intersection matrices are negative definite.*

Now, we explain that the second summand in (3.4) is also finite.

Lemma 3.4.3 (Nef, non-big divisors)**.** *Let X be a smooth projective surface with* $\overline{\mathrm{Eff}(X)}$ *rational polyhedral. Then there is only a finite number of nef and non-big divisors in* $\mathrm{Nef}_H(X)$.

Proof. Assume to the contrary that there are two divisors N_1, N_2, which are nef and not big, such that for all $t \in [0, 1]$ the divisors $tN_1 + (1 - t)N_2$ lie on the common face (here the rational polyhedrality assumption comes into the play). Thus $(tN_1 + (1 - t)N_2)^2 = 0$ for every $t \in [0, 1]$, what implies that $N_1.N_2 = 0$. It means that the intersection matrix of N_1, N_2 is the zero matrix of size 2×2, which contradicts the Hodge Index Theorem. □

Now we relate the number in (3.4) to the geometry of the cone $\mathrm{Nef}_H(X)$. We denote by f_i the number of i–dimensional faces of Nef_H for $i = 0, \ldots, \rho(X) - 1$. Moreover we write $f_0 = (f_0)_b + (f_0)_{nb}$, where $(f_0)_b$ is the number of big vertices in Nef_H and $(f_0)_{nb}$ is the number of non-big vertices.

Proposition 3.4.4. *Let X be a smooth complex projective surface with* $\overline{\mathrm{Eff}(X)}$ *rational polyhedral. Then*

$$\sum_{i=0}^{\rho-1} f_i = 1 + \mathrm{NnB}(X) + \mathrm{Zar}(X). \tag{3.6}$$

Proof. Let G be a face of $\mathrm{Nef}_H(X)$. If $G = \mathrm{Nef}_H(X)$ then this corresponds to $f_{\rho-1} = 1$ and is accounted for by 1 on the right hand side in the formula (3.6). Otherwise we distinguish two cases: either G is a vertex of $\mathrm{Nef}_H(X)$ which is not big, hence $G^2 = 0$ or G is a big vertex or a face of dimension ≥ 1.

The first case occurs $(f_0)_{nb}$ times and is accounted for by the second summand on the right in (3.6).

The second case corresponds to the third summand in (3.6). Indeed, given a nef and big divisor D there exists a Zariski chamber Σ_D with $\mathrm{Neg}(\Sigma_D) = D^{\perp}$. This follows from Nakamaye's result [22, Theorem 1.1]. Thus the inequality $\leq$ in (3.6) is established.

For the reverse inequality it suffices to show that distinct Zariski chambers determine distinct faces of $\mathrm{Nef}_H(X)$. To this end let Σ be a Zariski chamber. By [8, page 6] there is a face of $\mathrm{Nef}_H(X)$ orthogonal to the support of $\mathrm{Neg}(\Sigma)$. The injectivity of this assigment $\Sigma \to \mathrm{Neg}(\Sigma)^{\perp}$ follows again from the aforementioned result of Nakamaye. □

Now we are in a position to prove our main result.

Theorem 3.4.5. *Let X be a smooth complex projective surface with $\overline{\mathrm{Eff}(X)}$ rational polyhedral. Given a flag (x, A), where A is an ample curve and x is a smooth point on A, there is*

$$\#\mathrm{MB}_{(x,A)} = 1 + \mathrm{NnB}(X) + \mathrm{Zar}(X).$$

Proof. Given Zariski chambers Σ_1, Σ_2 with $\mathrm{Neg}(\Sigma_1) = \{N_1, \ldots, N_n\}$ and $\mathrm{Neg}(\Sigma_2) = \{N_{n+1}, \ldots, N_m\}$ – these chambers can have common elements. One associates to them the Minkowski basis elements

$$M_{\Sigma_1} = b_1 A + \sum_{j=1}^{n} a_j N_j \quad \text{and} \quad M_{\Sigma_2} = b_2 A + \sum_{j=n+1}^{m} a_j N_j.$$

Suppose that $\mathrm{Neg}(\Sigma_1) \neq \mathrm{Neg}(\Sigma_2)$ and assume to the contrary that $M_{\Sigma_1} = M_{\Sigma_2}$. Furthermore we may assume after reordering that the symmetric difference between these negative supports is $\{N_k, ..., N_m\}$. By the construction of Minkowski basis elements [26, pages 6 & 7] we know that

$$M_{\Sigma_1} = M_{\Sigma_2} \in N_i^{\perp} \quad \text{for all} \quad i \in \{1, ..., m\}.$$

This implies that for every N_i we have that $N_i.M_{\Sigma_1} = N_i.M_{\Sigma_2} = 0$. Let us take one of the elements from $N_r \in \{N_k, ..., N_m\}$. The condition

$$N_r.M_{\Sigma_1} = N_r.M_{\Sigma_2} = 0$$

implies in particular that $N_r.A = 0$, a contradiction. Proceeding in the same spirit one shows that the symmetric difference is empty and $\mathrm{Neg}(\Sigma_1) = \mathrm{Neg}(\Sigma_2)$, which ends the proof. □

Example 3.4.6 (Del Pezzo surfaces)**.** Using the above theorem we can compute the cardinality of Minkowski basis for del Pezzo surfaces X_i with respect to a fixed ample flag (x, A). To this end we need to compute the number of nef non-big curves on X_i. Let $C = aH - \sum b_j E_j$ be such a curve. First we observe that C is a rational curve. This follows from the adjunction since

$$2(p_a(C) - 1) = K_{X_i} \cdot C + C^2 = K_{X_i} \cdot C < 0$$

implies $p_a(C) = 0$. Hence

$$2 = -K_{X_i} \cdot C = 3a - \sum b_j. \tag{3.7}$$

On the other hand

$$0 = C^2 = a^2 - \sum b_j^2. \tag{3.8}$$

It is elementary to check that (3.7) and (3.8) have only finitely many integral solutions, listed (up to permutation of b_i's) in the following table

	a	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
$C(1)$	1	1	0	0	0	0	0	0	0
$C(2)$	2	1	1	1	1	0	0	0	0
$C(3)$	3	2	1	1	1	1	1	0	0
$C(4)$	4	2	2	2	1	1	1	1	0
$C(5)$	5	3	2	2	2	1	1	1	1
$C(6)$	6	3	3	2	2	2	2	1	1
$C(7)$	7	3	3	3	3	2	2	2	1
$C(8_a)$	8	4	3	3	3	3	2	2	2
$C(8_b)$	8	3	3	3	3	3	3	3	1
$C(9)$	9	4	4	3	3	3	3	3	2
$C(10)$	10	4	4	4	4	3	3	3	3
$C(11)$	11	4	4	4	4	4	4	4	3

.

Note that all solutions can be obtained from $C(1)$ applying standard Cremona transformations. This verifies again that an irreducible nef non-big curve on a del Pezzo surface is rational.

Counting all curves $C(j)$ on the appropriate surface X_i and taking (3.5) into account we have

i	1	2	3	4	5	6	7	8
$\#\mathrm{MB}_{(x,A)}$	3	7	21	81	403	2 797	33 764	1 503 721

Remark 3.4.7. During the proof we showed that nef divisors with the intersection $D^2 = 0$ have the property that $D.K_{X_i} = -2$. Such curves are called "conics", which is justified by Theorem 3.4.8.

Theorem 3.4.8. *Let Q be a divisor with $Q^2 = 0$ and $Q.K_{X_i} = -2$. Then Q is nef and the linear system $|Q|$ has no base points and determines a morphism $X_i \to \mathbb{P}^1$, which is a conic bundle.*

The proof is a standard consequence of the Riemann-Roch theorem and the Kawamata-Viehweg vanishing theorem. It turns out that conic divisors play an important role in theory of Cox rings.

Remark 3.4.9. Let X_i be del Pezzo surface and let C be a curve in the anti-canonical system $-K_{X_i}$. There is a Weyl group action on $\overline{\mathrm{Eff}(X_i)}$, which fixes the anti-canonical class, see [21, page 139]. In this situation there is a Weyl group invariant Minkowski basis $\mathrm{MB}_{(x,C)}$. Indeed, it can be constructed taking for each $j = 0, \ldots, \rho(X_i) - 1$ an element M_j corresponding to a j–dimensional face of $\mathrm{Nef}_H(X_i)$ and then all of its images under the action of the Weyl group, see also [8, Section 3.1].

Now we show that for a special choice of a flag (x, C), it might happen that the number of divisors in the Minkowski basis is strictly smaller than the number in (3.4). In fact we get Minkowski bases with any number of elements between 3 and 7 on the del Pezzo surface X_2.

Example 3.4.10. For del Pezzo surface X_2 we have the following possibilities:

- Fix a toric flag for X_2, i.e. (x, L_1) with $L_1 \in |H - E_1|$ and $x = L_1 \cap L_2$ for a fixed line $L_2 \in |H - E_2|$. Then by [28]
$$\mathrm{MB}_{(x,L_1)} = \{H, H - E_1, H - E_2\}.$$
We will consider the toric case in details in Chapter 4.

From now on x denotes a general point on the flag curve C.

- For the flag (x, C), where $C \in |H|$, we have
$$\mathrm{MB}_{(x,C)} = \{H, H - E_1, H - E_2, 2H - E_1 - E_2\}.$$
- For a curve $C \in |2H - E_1|$, we get
$$\mathrm{MB}_{(x,C)} = \{2H - E_1, H - E_1, H - E_2, H, 3H - 2E_1 - E_2\}.$$
- For a curve $C \in |2H - E_1 - E_2|$ we have
$$\mathrm{MB}_{(x,C)} = \{2H - E_1 - E_2, H - E_1, H - E_2, 2H - E_1, 2H - E_2, H\}.$$
- For the anticanonical flag (x, C) with a curve $C \in | - K_{X_2}|$ we have
$$\mathrm{MB}_{(x,C)} = \{-K_{X_2}, H, H - E_1, H - E_2, 2H - E_1 - E_2, 3H - E_1, 3H - E_2\}.$$

The next question arising immediately is about the precise formulae, which allows us to compute the cardinality of Minkowski bases with respect to any big and nef flag. It turns out that the answer to this question is deeply encoded in relations between Zariski chambers, more precisely in the relations between the stable base loci of Zariski chambers.

For a big divisor D let us define the following number

$$\mathrm{NZ}(D) = \#\{\Sigma : \mathrm{Neg}(\Sigma) \cap \mathrm{Null}(D) \neq \emptyset\}.$$

Recall that by Theorem 1.5.5 for a big divisor D we have that $\mathrm{Null}(D)$ is a subset of the set of all negative curves on a smooth projective surface X containing D, thus the above number is well-defined. In the case of surfaces containing only (-1)-curves we can prove the following.

Proposition 3.4.11. *Let X be a smooth projective surface which contains only finitely many (-1)-curves and let (x, C) be an admissible flag, where C is big and nef. Then*

$$\#\mathrm{MB}_{(x,C)} = 1 + \mathrm{NnB}(X) + \mathrm{Zar}(X) - \mathrm{NZ}(C).$$

Proof. Since the number $1 + \mathrm{NnB}(X)$ does not depend on the choose of an admissible flag, thus the only one thing is to compute the number $\mathrm{Zar}(X) - \mathrm{NZ}(C)$. Suppose that $\mathrm{Neg}(\Sigma) = \{N_1, ..., N_k\}$ and let $M_\Sigma = dC + \sum_{j=1}^{k} a_j N_j$ be a Minkowski basis element with fixed $d \neq 0$. Since there are only (-1)-curves, thus we have $a_j = N_j.C$. Assume that $N_{s+1}, .., N_k \in \mathrm{Null}(C)$ and $N_1, ..., N_s \notin \mathrm{Null}(C)$. By the construction of Minkowski basis elements we have that $M_\Sigma \in \mathrm{Neg}(\Sigma)^\perp$. This implies that for every $N_i \in \mathrm{Null}(C)$ one has

$$0 = M_\Sigma.N_i = \sum_j a_j N_j.N_i = -a_i.$$

Thus we obtain

$$M_\Sigma = dC + \sum_{j=1}^{s} a_j N_j$$

with $a_j > 0$. Since for all such surfaces the intersection matrix of curves in the negative part of the Zariski decomposition is $-I_r$ with $r = \#\mathrm{Neg}(\Sigma')$, thus the corresponding intersection matrix is $-I_s$, and on the other hand this matrix corresponds to the other Zariski chamber due to Theorem 3.4.2. This completes the proof. □

In the case of del Pezzo surfaces it is easy to find the cardinality of a Minkowski basis with respect to a fixed admissible flag (x, C), where C is big and nef curve. By the previous consideration the main task is to compute $\mathrm{Zar}(X) - \mathrm{NZ}(C)$.

The algorithm works as follows:

Input: the flag curve C and the intersection matrix S of all negative curves on X_i.

Now we construct the matrix S' – we add to matrix S one column and one row $S'[0, j] = S'[j, 0]$ with the entries $s'_{00} = C^2$ and $s'_{0j} = C.N_j$. In next step we construct the matrix $\tilde{S}$ by removing all columns and rows such that $C.N_j = 0$ plus first row and first column from S'. Using Theorem 3.4.2 our task boils down to compute the number of negative definite principal submatrices of $\tilde{S}$. The number of such matrices is equal to $\mathrm{Zar}(X) - \mathrm{NZ}(C)$.

Output: The number $\mathrm{Zar}(X) - \mathrm{NZ}(C)$ and Minkowski basis elements corresponding to the computed principal submatrices $\tilde{S}$.

Example 3.4.12. Let us consider del Pezzo surface X_3. There are 6 negative curves, i.e.

$$E_1, E_2, E_3, H_{12} := H - E_1 - E_2, H_{23} := H - E_2 - E_3, H_{13} := H - E_1 - E_3.$$

Let us choose the flag (x, C), where $x \in C \in |H|$ is a general point. The matrix S' has the following form

	C	E_1	E_2	E_3	H_{12}	H_{13}	H_{23}
C	1	0	0	0	1	1	1
E_1	0	−1	0	0	1	1	0
E_2	0	0	−1	0	1	0	1
E_3	0	0	0	−1	0	1	1
H_{12}	1	1	1	0	−1	0	0
H_{13}	1	1	0	1	0	−1	0
H_{23}	1	0	1	1	0	0	−1

.

According to our algorithm one has to remove the first 4 rows and the first 4 columns corresponding to C, E_1, E_2, E_3 to obtain $\tilde{S}$. The matrix $\tilde{S}$ has 7 negative definite principal submatrices, which correspond to 7 Zariski chambers, and these chambers give us 7 distinct Minkowski basis elements.

Thus the cardinality of $\mathrm{MB}_{(x,C)}$ equals to 11.

This argumentation does not work for surfaces containing negative curves other than (-1)-curves. However one can obviously give the following naive upper-bound.

Let C be a big and nef curve. Using the curves in $\mathrm{Null}(C)$ one can find $\mathrm{NZ}(X) \geq 1$ subsets such that the corresponding intersection matrices of these curves are negative definite – it means that these sets are supports of the negative parts of Zariski chambers. Then for every projective surface with $\overline{\mathrm{Eff}}(X)$ rational polyhedral and a flag (x, C) one has

$$\#\mathrm{MB}_{(x,C)} \leq 1 + \mathrm{NnB}(X) + \mathrm{Zar}(X) - \mathrm{NZ}(X). \tag{3.9}$$

Now we present a non-del Pezzo example for which the above inequality is sharp. This example is based on a certain $K3$ surface, thus we would like to recall the definition of such surfaces.

Definition 3.4.13. A complex $K3$ surface is a complete non-singular variety X of dimension 2 such that $K_X \equiv_{lin} \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

We refer to [17] for a very detailed introduction to the subject. It is worth pointing out that for $K3$ surfaces the self-intersection of negative curves is at least -2 – this can be showed using the adjunction formula.

Example 3.4.14. This construction comes from [6]. Let X be a smooth quartic surface in $\mathbb{P}^3$, which contains a hyperplane section that decomposes into two lines L_1, L_2 and an irreducible conic C. The existance of such surfaces was proved for instance in [6, Proposition 3.3]. There exists such a surface with the Picard number 3 and the pseudo-effective cone is generated by L_1, L_2 and C. Curves L_1, L_2, C have the following intersection matrix

$$\begin{pmatrix} -2 & 1 & 2 \\ 1 & -2 & 2 \\ 2 & 2 & -2 \end{pmatrix}.$$

The BKS decomposition consists of five chambers, namely the nef chamber, which is spanned by $\{L_1 + C, L_2 + C, C + 2L_1 + 2L_2\}$, one chamber corresponding to each (-2)-curve and one chamber with support $\{L_1, L_2\}$. Fix the flag (x, D), where $D \in |C + 2L_1 + 2L_2|$ and $x \in D$ is a general point. Of course D is big. Simple computations shows that

$$(C + 2L_1 + 2L_2).L_1 = CL_1 + 2L_1L_1 + 2L_1L_2 = 2 - 4 + 2 = 0,$$

$$(C + 2L_1 + 2L_2).L_2 = 0,$$

thus

$$\text{Null}(C + 2L_1 + 2L_2) = \{L_1, L_2\}.$$

Zariski chambers corresponding to $\{L_1\}$, $\{L_2\}$, $\{L_1, L_2\}$ have the same Minkowski basis element D. Since $(C + 2L_1 + 2L_2).C = -2 + 4 + 4 = 6$, thus by the construction of Minkowski basis elements one has

$$M = C + 2L_1 + 2L_2 + 3C = 4C + 2L_1 + 2L_2.$$

Summarizing up all computations, the Minkowski basis with respect to the flag (x, D) is

$$\text{MB}_{(x,D)} = \{C, L_1 + C, L_2 + C, 4C + 2L_1 + 2L_2\},$$

and the number of elements is equal to

$$1 + \text{NnB}(X) + \text{Zar}(X) - \text{NZ}(X) = 4.$$

It is quite easy to see that in general the inequality (3.9) gives us a coarse estimate, thus we can formulate the following problem.

Problem 3.4.15. Find other geometric conditions, which allow to compute the cardinality of Minkowski bases.

Our first attempt in this direction shows that without a special input (or another geometric information) in order to compute the cardinality of a Minkowski basis we need to compute all Minkowski basis elements, which is quite disappointing.

In the last part of this section we would like to focus on the question about Minkowski bases with the minimal number of elements. Previously we have showed that if we want to construct a Minkowski basis with the maximal number of elements it is enough to choose an ample flag curve. The opposite question seems to be a little bit more complicated.

The first impression suggests that in order to construct a Minkowski basis with the minimal number of elements it is enough to choose a big and nef flag curve (divisor), for which $\mathbb{B}_+$ contains the largest number of irreducible negative curves. However if we look at the case of toric surfaces, then the interesting phenomenon happens. In next chapter we prove in particular that for a smooth projective toric surface with a fixed toric flag (a complete intersection of T-invariant divisors) there is the unique Minkowski basis consisting of only the vertices of the nef cone. In order to emphasize discrepancies between cardinalities let us consider X_3 the blow up of $\mathbb{P}^2$ in three non-collinear points. Then for an arbitrary toric flag the number of Minkowski basis elements is equal to 5, but on the other hand if we choose a flag $x \in C \in |H|$ one has $\#\text{MB}_{(x,C)} = 11$. However if we consider only big and nef flag curves, then the argumentation from the beginning of this paragraph should be valid.

3.5 Numerical equivalence of divisors on projective surfaces

In Section 2.3 we saw that Okounkov bodies are numerical in nature, what means that if big divisors D_1, D_2 are numerically equivalent, then $\triangle_{Y_\bullet}(D_1) = \triangle_{Y_\bullet}(D_2)$ for an admissible flag $Y_\bullet$. It is not clear whether one can read off all numerical invariants of a given big divisor from its Okounkov bodies with respect to any flag. In [31] the author proves the following very interesting theorem.

Theorem 3.5.1. *Let X be a normal complex projective variety of dimension n. If D_1, D_2 are two big divisors on X such that*

$$\triangle_{Y_\bullet}(D_1) = \triangle_{Y_\bullet}(D_2)$$

for every admissible flag $Y_\bullet$ on X, then D_1 and D_2 are numerically equivalent.

The proof is based on sophisticated methods related to restricted complete linear series and restricted volumes. However in the case of complex projective surfaces one can prove that it is enough to consider strictly smaller number of admissible flags in order to obtain the same result as [31].

Proposition 3.5.2. *Let X be a smooth complex projective surface. Denote by ρ the Picard number of X. Then there exists a set of irreducible ample divisors $\{A_1, ..., A_\rho\}$ and a set of general points $\{x_1, ..., x_\rho\}$ with $x_i \in A_i$, such that for two big $\mathbb{R}$-divisors D_1, D_2 if*

$$\triangle_{(x_i, A_i)}(D_1) = \triangle_{(x_i, A_i)}(D_2)$$

for every $i \in \{1, ..., \rho\}$, then the positive parts of the Zariski decompositions P_1, P_2 of D_1, D_2 are numerical equivalent.

Proof. Let us choose an ample base $\mathcal{B} = \{A_1, ..., A_\rho\}$ of the $\mathbb{R}$-vector space of all 1-cycles and pick general points $x_i \in A_i$.

Fix a flag (x_i, A_i). By Theorem 3.1.3 we know that for a big divisor D

$$\triangle(D) = \{(t, y) \in \mathbb{R}^2 : 0 \leq t \leq \mu \ \& \ \alpha(t) \leq y \leq \beta(t)\}.$$

Since x_i is a general point, thus $\alpha(t) \equiv 0$ and $\beta(t) = P_t.A_i$, where P_t is the positive part of the Zariski decomposition of $D_t = D - tA_i$. Combining this with the condition $\triangle_{(x_i, A_i)}(D_1) = \triangle_{(x_i, A_i)}(D_2)$ one obtains that P_1 and P_2 are numerical equivalent, which ends the proof. □

In order to finish our idea it is enough to construct a test configuration for negative parts N_1 and N_2. We have thus to use all irreducible negative curves $\mathcal{I}(X) = \{C_j\}_{j \in I}$ on X. Then the condition $C_j.N_1 = C_j.N_2$ for every $j \in I$ implies that N_1 and N_2 are numerical equivalent.

Theorem 3.5.3. *Let X be a smooth complex projective surface. Denote by ρ the Picard number of X. Assume that D_1, D_2 are $\mathbb{R}$-pseudoeffective divisors and let $D_j = P_j + N_j$ be the Zariski decompositions. There exist irreducible ample divisors $A_1, ..., A_\rho$ with general points $x_i \in A_i$, such that D_1, D_2 are numerical equivalent if and only if*

- $\triangle_{(x_i, A_i)}(D_1) = \triangle_{(x_i, A_i)}(D_2)$ *for every $i \in \{1, ..., \rho\}$,*
- $C.N_1 = C.N_2$ *for every negative curve $C \in \mathcal{I}(X)$.*

Of course our approach is based on the nature of projective surfaces, i.e. the existence of the Fujita-Zariski decomposition and Theorem 3.1.3, which do not exist for higher dimensional projective varieties. However we can ask a similar question.

Problem 3.5.4. Are there exist testing configurations for normal complex projective varieties of dimension $n \geq 3$?

Chapter 4

Minkowski Decompositions for toric varieties

This chapter is devoted to the construction of Okounkov bodies for toric varieties. Our main aim is to present a theorem coming from [28], which states that for every toric variety Minkowski decompositions of T-invariant divisors with respect to T-invariant flags exists, and Minkowski bases elements for such varieties are vertices of the movable cone. In next sections we introduce remaining facts and theorems related to toric varieties and higher dimensional birational geometry and after that we present the main results in details. The last section is devoted to several interesting examples of such decompositions.

4.1 Preliminaries on toric and birational geometry

We start with definition of toric varieties. This introduction is based on "Introduction to Toric Varieties" due to W. Fulton [14] and "Toric varieties" due to D. Cox, J. Little and H. Schenck [10].

Let N be a lattice, which is isomorphic with $\mathbb{Z}^n$ for a certain $n > 0$. Then $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ denotes the real vector space. Let $M = \mathrm{Hom}(N, \mathbb{Z})$ denote the dual lattice with the dual paring $\langle\, , \rangle$. In the same spirit we define $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ the dual vector space. Let us recall the following definition (just to complete the toric setting).

Definition 4.1.1. A convex polyhedral cone is a set

$$\sigma = \{r_1 v_1 + ... + r_s v_s \in N_{\mathbb{R}} : r_i \geq 0\}$$

generated by any finite set of vectors $v_1, ..., v_s \in N_{\mathbb{R}}$. Elements $v_1, ..., v_s$ are called generators for the cone σ. We say that a cone σ is strongly convex if $\sigma \cap -\sigma = \{0\}$.

The dimension of σ is the dimension of the linear space $\mathbb{R} \cdot \sigma$ spanned by σ. The dual cone $\sigma^{\vee}$ of σ is the set of equations of supporting hyperplanes, i.e.

$$\sigma^{\vee} = \{u \in M_{\mathbb{R}} : \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\}.$$

It can be shown that dual of a convex polyhedral cone is a convex polyhedral cone.

A face τ of σ is the intersection of σ with any supporting hyperplane

$$\tau = \sigma \cap u^{\perp} = \{v \in \sigma : \langle u, v\rangle = 0\}$$

for some $u \in \sigma^{\vee}$.

Proposition 4.1.2 (Section 1.2, Proposition 2, [14]). *Let σ be a rational convex polyhedral cone and let u be an element in $S_\sigma = \sigma^\vee \cap M$. Then $\tau = \sigma \cap u^\perp$ is a rational convex polyhedral cone. All faces of σ have this form and $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u)$.*

Definition 4.1.3. A fan Σ in N is a nonempty set of rational strongly convex polyhedral cones σ in $N_\mathbb{R}$ such that

- each face of a cone in Σ is also a cone in Σ,
- the intersection of two cones in Σ is a face of each.

We will assume that all fans Σ are finite, i.e. the number of cones is finite.

Now we want to construct the toric variety $X(\Sigma)$ for a fixed fan Σ. At the beginning we construct affine toric varieties in the following way.

One of the crucial facts related to convex polyhedral cones is the following Gordon lemma.

Lemma 4.1.4 (Gordon, Section 1.2, Proposition 1, [14]). *If σ is a rational convex polyhedral cone, then $S_\sigma = \sigma^\vee \cap M$ is a finitely generated semigroup.*

It is well-known that for a semigroup S one can construct a group ring $\mathbb{C}[S]$, which is commutative $\mathbb{C}$-algebra. As a complex vector space it has a basis χ^u with $u \in S$ and the multiplication is determined by the addition in S, i.e.

$$\chi^u \cdot \chi^v = \chi^{u+v}.$$

It is also well-known that a finitely generated commutative $\mathbb{C}$-algebra $\mathcal{A}$ determines a complex affine scheme, which is usually denoted by $\mathrm{Spec}(A)$.

Coming back to the toric situation, since σ is a convex polyhedral cone, thus by Gordon lemma $S_\sigma = \sigma^\vee \cap M$ is finitely generated semigroup. Now we can consider $\mathcal{A}_\sigma = \mathbb{C}[S_\sigma]$, which is finitely generated and commutative and thus

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$$

corresponds to an affine variety, which is called the affine toric variety indicated by σ.

All of semigroups will be a subsemigroups of $M = S_{\{0\}}$. If $e_1, ..., e_n$ are generators of N, then $e_1^*, ..., e_n^*$ generate M and $\mathbb{C}[M] = \mathbb{C}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}] = \mathbb{C}[x_1, ..., x_n]_{x_1 \cdot ... \cdot x_n}$, which is the ring of Laurent polynomials in n variables. Thus we obtain

$$U_{\{0\}} = (\mathbb{C}^*)^n,$$

which is the affine algebraic torus. The torus can be also a fan described as $T_N = N \otimes_\mathbb{Z} \mathbb{C}^*$.

To obtain the toric variety $X(\Sigma)$ corresponding to Σ we need to glue a disjoint union of affine toric varieties as follows. Let σ and ν be two cones intersecting along a common face $\tau = \sigma \cap \nu$. Then U_τ is canonically identified as a principal open subvariety of both U_σ and U_ν. Glue U_σ and U_ν by this identification on these open subvarieties.

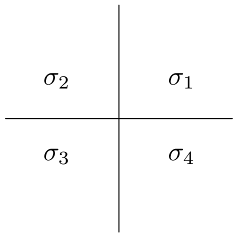
In particular one can show that for two cones σ and ν of a fan Σ we have the identity $U_\sigma \cap U_\nu = U_{\sigma\cap\nu}$.

Now we are in a position to formulate the following theorem, which is also a definition of toric varieties.

Theorem 4.1.5 (Definition of toric varieties, Section 1.4, [14]). *Let Σ be a fan in $N_\mathbb{R}$. Consider the disjoint union $\bigcup_{\sigma\in\Sigma} U_\sigma$, where two points $x \in U_\sigma$ and $x' \in U_{\sigma'}$ are identified if they agree on the gluing of U_σ and $U_{\sigma'}$. The resulting space $X(\Sigma)$ is called a toric variety. It is a topological spaces endowed with an open covering by the affine toric varieties $X_\sigma = U_\sigma$ for each $\sigma \in \Sigma$.*

To illustrate all of the above notions we present now an easy example.

Example 4.1.6. Let us consider the following fan $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ in $N_{\mathbb{R}}$.



Denote by $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Then $\sigma_1 = \text{span}\{e_1, e_2\}$, $\sigma_2 = \text{span}\{-e_1, e_2\}$, $\sigma_3 = \text{span}\{-e_1, -e_2\}$ and $\sigma_4 = \text{span}\{-e_2, e_1\}$. Now we want to construct semigroup rings S_{σ_i}. Denote by e_1^*, e_2^* the dual basis elements, i.e. $e_i^*(e_j) = \delta_{ij}$. It is easy to see that dual cones are the following: $\sigma_1^\vee = \text{span}\{e_1^*, e_2^*\}$, $\sigma_2^\vee = \text{span}\{-e_1^*, e_2^*\}$, $\sigma_3^\vee = \{-e_1^*, -e_2^*\}$, $\sigma_4^\vee = \{-e_2^*, e_1^*\}$.

The associated semigroup $\mathbb{C}$-algebras and affine toric varieties are the following:

- $S_{\sigma_1} = \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}] = \mathbb{C}[x_1, x_2], \quad U_{\sigma_1} = \text{Spec}(\mathbb{C}[x_1, x_2]) = \mathbb{C}^2,$
- $S_{\sigma_2} = \mathbb{C}[\chi^{-e_1^*}, \chi^{e_2^*}] = \mathbb{C}[x_1^{-1}, x_2], \quad U_{\sigma_2} = \text{Spec}(\mathbb{C}[x_1^{-1}, x_2]) = \mathbb{C}^* \times \mathbb{C},$
- $S_{\sigma_3} = \mathbb{C}[\chi^{-e_1^*}, \chi^{-e_2^*}] = \mathbb{C}[x_1^{-1}, x_2^{-1}], \quad U_{\sigma_3} = \text{Spec}(\mathbb{C}[x_1^{-1}, x_2^{-1}]) = (\mathbb{C}^*)^2,$
- $S_{\sigma_4} = \mathbb{C}[\chi^{e_1^*}, \chi^{-e_2^*}] = \mathbb{C}[x_1, x_2^{-1}], \quad U_{\sigma_4} = \text{Spec}(\mathbb{C}[x_1, x_2^{-1}]) = \mathbb{C} \times \mathbb{C}^*.$

After the gluing process (use Proposition 4.1.2) we obtain $\mathbb{P}^1 \times \mathbb{P}^1$.

Now let us recall briefly geometrical properties of toric varieties. We need the following definitions.

Definition 4.1.7. A cone σ defined by vectors $\{v_1, ..., v_s\}$ is a simplex iff the vectors v_i are linearly independent. A fan Σ is simplicial if all cones of Σ are simplices.

Definition 4.1.8. A vector $v \in \mathbb{Z}^n$ is primitive if its coordinates are coprime. A cone is regular if $\{v_1, ..., v_s\}$ spanning the cone are primitive and there exists primitive vectors $\{v_{s+1}, ..., v_n\}$ such that $\{v_1, ..., v_n\}$ is a basis of lattice N. A fan Σ is regular if all its cones are regular.

Definition 4.1.9. A fan Σ is complete if its cones cover $\mathbb{R}^n$.

A fan Σ is polytopal if there exists a polytope P such that $0 \in P$ and Σ is spanned by the faces of P (by polytope P we mean the convex hull of a finite number of points).

Theorem 4.1.10 (Geometric characterization)**.** *Let Σ be a fan. Then*

- *Σ is complete if and only if $X(\Sigma)$ is compact,*
- *Σ is regular if and only if $X(\Sigma)$ is smooth,*
- *Σ is polytopal if and only if $X(\Sigma)$ is projective.*

A complete proof of this theorem can be found in [14].

In next step we would like to define the set of T-invariant divisors, i.e. divisors invariant under the action of the torus T. Consider a fan Σ and toric variety $X(\Sigma)$. Then the primitive T-invariant divisors correspond to rays of the fan Σ – to see this it is enough to verify that each ray corresponds to $(n-1)$-dimensional subvariety [14]. Let us order the edges $\tau_1, ..., \tau_d$ and let v_i be the first lattice point meeting along the ray τ_i. These divisors correspond to the closure of orbits $D_i = V(\tau_i)$. Thus the T-invariant Weil divisors are formal combinations $\sum_i a_i D_i$.

To characterize T-invariant Cartier divisors consider $X = U_\sigma$ with $\dim(\sigma) = n$. Let D be a divisor that is preserved by T, corresponding to the fractional ideal $I = H^0(X, \mathcal{O}(-D))$. It can be shown that I is generated by χ^u for a unique element $u \in \sigma^\vee \cap M$.

Proposition 4.1.11 (Section 3.3, [14])**.** *Assume that X is a normal toric variety. Let $u \in M$ and v be the first lattice point along the edge τ. Then*

$$[\mathrm{div}(\chi^u)] = \sum_i \langle v_i, u\rangle D_i.$$

The last part of the introduction is a characterization of line bundles and their spaces of sections in the language of toric polyhedrons.

For a toric variety $X(\Sigma)$ denote by $\mathrm{Pic}(X(\Sigma))$ the group of all line bundles up to isomorphism. It is quite interesting that in such case $\mathrm{Pic}(X(\Sigma))$ can be computed using only the set of T-invariant Cartier divisors $\mathrm{Div}_T(X(\Sigma))$. Denote by $\mathrm{Weil}_T(X(\Sigma))$ the set of all T-invariant Weil divisors.

Proposition 4.1.12 (Section 3.4, [14])**.** *Let $X(\Sigma)$ be a toric variety with Σ not contained in any proper subspace of $N_\mathbb{R}$. Then we have the following exact sequences*

$$0 \to M \to \mathrm{Div}_T(X(\Sigma)) \to \mathrm{Pic}(X(\Sigma)) \to 0,$$

$$0 \to M \to \bigoplus_{i=1}^{d} \mathbb{Z}[D_i] \to \mathrm{Weil}_T(X(\Sigma)) \to 0.$$

In particular $\mathrm{rk}(\mathrm{Pic}(X(\Sigma))) \leq \mathrm{rk}(\mathrm{Weil}_T(X(\Sigma))) = d - n$ *where d is the number of edges in the fan. In addition* $\mathrm{Pic}(X(\Sigma))$ *is free abelian.*

In order to define *support functions* we will need the following description of T-invariant Cartier divisors.

Theorem 4.1.13 (Theorem 4.2.8, [10])**.** *Let $X(\Sigma)$ be a toric variety with the fan Σ and let $D = \sum_p a_p D_p$. Then the following conditions are equivalent:*

- *D is Cartier,*
- *D is principal on the affine open subset U_σ for all $\sigma \in \Sigma$,*
- *for each $\sigma \in \Sigma$ there is $m_\sigma \in M$ with $\langle m_\sigma, u_p\rangle \geq a_p$ for all $p \in \{1, ..., d\}$, where u_p is a primitive generator and d is the number of rays of Σ.*

Let us denote by $|\Sigma|$ the support of Σ.

Definition 4.1.14. Let Σ be a fan in $N_\mathbb{R}$. A *support function* is a function $\phi : |\Sigma| \to \mathbb{R}$ that is linear on each cone of Σ. The set of all support function is denoted by $\mathrm{SF}(\Sigma)$.

A support function ϕ is integral with respect to the lattice N if

$$\phi(|\Sigma| \cap N) \subseteq \mathbb{Z}.$$

Let $D = \sum_p a_p D_p$ be a Cartier and let $\{m_\sigma\} \in_{\sigma\in\Sigma}$ be the data describing Cartier divisors of D. Using the above characterization we have that $\langle m_\sigma, i_\sigma \rangle = -a_p$ for all $p \in \{1, ..., d\}$. Now we can describe Cariter divisors in the language of support functions.

Theorem 4.1.15 (Theorem 4.2.12, [10])**.** *Let Σ be a fan in $N_\mathbb{R}$. Then*

- *For a given divisor $D = \sum_p a_p D_p$ with Cartier data $\{m_\sigma\}_{\sigma\in\Sigma}$, the function*

$$\phi_D : |\Sigma| \ni u \mapsto \langle m_\sigma, u \rangle \in \mathbb{R} \text{ when } u \in \sigma$$

  is well-defined support function that is integral with respect to N

- *$\phi_D(u_p) = -a_p$ for all $p \in \{1, ..., d\}$ so that*

$$D = -\sum_p \Phi_D((u_p))D_p.$$

We present two approaches to describe the the global sections of a sheaf $\mathcal{O}_{X(\Sigma)}(D)$, where D is T-invariant.

Theorem 4.1.16 (Proposition 4.3.2, [10])**.** *If D is T-invariant Weil divisor on $X(\Sigma)$, then*

$$H^0(X(\Sigma), \mathcal{O}_{X(\Sigma)}(D)) = \bigoplus_{\operatorname{div}(\chi^m)+D\geq 0} \mathbb{C}\chi^m.$$

Other approach is based on the construction of the toric polyhedron. Let $D = \sum_p a_p D_p$ be a T-invariant Cartier divisors and $m \in M$. Condition $\operatorname{div}(\chi^m) + D \geq 0$ is equivalent to

$$\langle m, u_p \rangle + a_p \geq 0 \text{ for all } p \in \{1, ..., d\}.$$

Now we can define the *toric polyhedron* of D:

$$P_D = \{m \in M_\mathbb{R} : \langle m, u_p \rangle \geq -a_p \text{ for } p \in \{1, ..., d\}\}.$$

The set P_D is a polyhedron since it is an intersection of finitely many closed half spaces. The following proposition decodes information about the global sections in the language of toric polyhedrons.

Theorem 4.1.17 (Proposition 4.3.3, [10])**.** *If D is a T-invariant Cartier divisor on $X(\Sigma)$, then*

$$H^0(X(\Sigma), \mathcal{O}_{X(\Sigma)}) = \bigoplus_{m\in P_D\cap M} \mathbb{C}\chi^m,$$

where $P_D \subset M_\mathbb{R}$ is the toric polyhedron defined above.

Toric polyhedrons have nice linear properties, i.e. if we replace T-invariant divisor D by a linearly equivalent divisor, then

$$P_{D+\operatorname{div}(\chi^m)} = P_D - m \text{ for } m \in M,$$

and moreover for $p \in \mathbb{Z}_{\geq 0}$ we have

$$P_{pD} = pP_D.$$

These properties allow us to define toric polyhedrons for T-invariant $\mathbb{Q}$-divisors. The key result in the context of Minkowski decompositions tells us that toric polyhedrons for globally generated T-invariant divisors are additive.

Proposition 4.1.18 ([14]). *Let $X(\Sigma)$ be a projective toric variety and let D_1, D_2 be a T-invariant divisors such that $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$ are globally generated. Then*

$$P_{D_1+D_2} = P_{D_1} + P_{D_2},$$

where on the right-hand side we have the Minkowski sum.

We will use also normal fans and their properties. Denote by $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

Definition 4.1.19. Let $P \subset (\mathbb{R}^*)^n$ be a convex polytope. Then the polar polytope P° in $\mathbb{R}^n$ to P is defined as

$$P^\circ = \{v \in \mathbb{R}^n : \langle u, v\rangle \geq -1, \text{ for all } u \in P\}.$$

One can show that if P° is a convex polytope and if P is rational, then P° is a lattice polytope.

Definition 4.1.20. A face F of P is defined as

$$F = \{u \in P : \langle u, v\rangle = r \text{ where } v \in \mathbb{R}^N \text{ is such that } \langle u, v\rangle \geq r \text{ for all } u \in P\}.$$

Suppose that $\{0\} \in \text{int}(P)$. Then for every face F of P we can define a polar face of P°

$$F^* = \{v \in P^\circ : \langle u, v\rangle = -1 \text{ for all } u \in F\}.$$

Now we construct the fan Σ_P associated to a polytope P. Let F be a face of P. Then one can define

$$\sigma_F = \{v \in N_{\mathbb{R}} : \langle u, v\rangle \leq \langle u', v\rangle \text{ for all } u \in F \text{ and } u' \in P\}.$$

The cone $\sigma_F \subset \mathbb{R}^n$ is generated by F^*.

Proposition 4.1.21 ([14]). *Let P be a convex polytope. Then the cones σ_F form a fan Σ_P and if $\{0\} \in \text{int}(P)$, then Σ_P is made of the cones based on the faces of the polar polytope P°.*

As we can see one can construct toric varieties associated to polytopes – we construct a fan Σ_P corresponding to the polytope P. It turns out that normal fans have also nice additive properties.

Proposition 4.1.22 (Proposition 6.2.13, [10]). *Let P and Q be lattice polytopes in $M_{\mathbb{R}}$ with $0 \in \text{int}P, \text{int}Q$. Then:*

1. *Q is an $\mathbb{N}$-Minkowski summand of P if and only if Σ_P refines Σ_Q.*

2. *Σ_{P+Q} is the coarsest common refinement of Σ_P and Σ_Q.*

Corollary 4.1.23 (Proposition 6.2.15, [10]). *Let P be a full dimensional lattice polytope in $M_{\mathbb{R}}$. Then a polytope $Q \subseteq M_R$ is an $\mathbb{N}$-Minkowski summand of P if and only if there is a torus invariant basepoint free Cartier divisor A on X_P such that $Q = P_A$.*

Another important application of the above notions is the following.

Theorem 4.1.24 (Theorem 6.2.8, [10]). *Let D be a basepoint free Cartier divisor on a complete toric variety $X(\Sigma)$, and let X_{P_D} be the toric variety of the polytope $P_D \subseteq M_{\mathbb{R}}$ with $0 \in \text{int}P_D$. Then the refinement Σ of Σ_{P_D} induces a proper toric morphism*

$$\varphi : X(\Sigma) \to X_{P_D}.$$

Furthermore, D is linearly equivalent to the pullback via φ of the ample divisor on X_{P_D} coming from P_D.

In particular we can deduce the following.

Remark 4.1.25. The polytope of the pullback of a T-invariant divisor by a toric morphism is isomorphic to the original polytope.

Now we are going to recall basic facts of higher dimensional birational geometry. Assume for a moment that X is a projective normal variety.

Definition 4.1.26. A divisor D on X is *movable* if its stable base locus $\mathbb{B}(D)$ has codimension ≥ 2.

The movable cone $\mathrm{Mov}(X)$ is the convex cone in $N^1(X)$ generated by classes of movable divisors – this cone is not closed in general.

We have the following inclusion

$$\overline{\mathrm{Mov}(X)} \subseteq \mathrm{Nef}(X) \subset \overline{\mathrm{Eff}(X)}.$$

Now assume in addition that X is also $\mathbb{Q}$-factorial, which means that the canonical divisor K_X is $\mathbb{Q}$-Cartier.

Definition 4.1.27. A contraction on X is a surjective morphism with connected fibres $f : X \to Y$, where Y is normal and projective. Moreover f is said to be of fiber-type if dim $X >$ dim Y.

Definition 4.1.28. A small $\mathbb{Q}$-factorial modification (abb. *SQM*) of X is a birational map $g : X \dashrightarrow \widetilde{X}$, where $\widetilde{X}$ is normal, projective and $\mathbb{Q}$-factorial and additionally g is an isomorphism in codimension 1.

One can show that if g is *SQM*, then it induces an isomorphism

$$g^* : N^1(\widetilde{X}) \to N^1(X)$$

and g^* preserves the effective and movable cones. Now we are in a position to introduce the notion of Mori Dream Spaces.

Definition 4.1.29 (Definition 1.8, [16])**.** Let X be a normal and $\mathbb{Q}$-factorial projective variety. We say X is a Mori Dream Spaces (abb. *MDS*) if the following properties hold:

1. $\mathrm{Pic}(X)$ is finitely generated,

2. $\mathrm{Nef}(X)$ is generated by the classes of finitely many semiample divisors,

3. there is a finite collection of *SQM*s $g_i : X \dashrightarrow X_i$ for $i = 1, ..., r$, such that every X_i satisfies previous conditions and
$$\mathrm{Mov}(X) = \bigcup_{i=1}^{r} g_i^*(\mathrm{Nef}(X_i)).$$

Definition 4.1.30. Varieties X_i appearing in Definition 4.1.29 will be called *modifications* of X.

Now we formulate some remarks about *MDS*.

- If X is *MDS*, then $\mathrm{Nef}(X)$ is a rational polyhedral cone and every nef divisors in X is semiample.

- If X is *MDS*, then X_i is a *MDS* as well for every $i = 1, ..., r$.

- One can show [33] that every smooth projective rational surface with $-K_X$ big is a *MDS*.

- If X is normal and $\mathbb{Q}$-factorial projective variety with the Picard number $\rho = 1$, then X is *MDS* if $\mathrm{Pic}(X)$ is finitely generated. Hence for instance $\mathbb{P}^1$ is *MDS*, but any elliptic curve is not *MDS*.

Theorem 4.1.31 (Corollary 2.4, [16])**.** *Let X be a $\mathbb{Q}$-factorial projective toric variety. Then X is* MDS.

In particular when X is toric, then SQM's are described in a very simple and combinatorial way – for details we refer to the nice lecture notes [34].

The last ingredient we need is a generalization of the Zariski decomposition for higher dimensional varieties.

Definition 4.1.32 (Cutkosky - Kawamata - Moriwaki Zariski decomposition)**.** Let X be a smooth projective variety with $\dim X \geq 3$. Then D has rational Zariski decomposition in the sense of Cutkosky - Kawamata - Moriwaki if there exists a smooth birational modification $\mu : X' \to X$ together with an effective divisor N' on X', such that

- $P' = \mu^* D - N'$ is nef $\mathbb{Q}$-divisor on X', and
- the natural maps
$$H^0(X', \mathcal{O}_{X'}([mP'])) \to H^0(X', \mathcal{O}_{X'}(\mu^*(mD)))$$
are bijective for every $m \geq 1$.

Cutkosky showed that such decompositions cannot exist in general. In fact, if D is a big divisor which admits a rational CKM decomposition, then the volume $\mathrm{vol}_X(D)$ is rational. Therefore a big divisor D with irrational volume does not admit a rational CKM decomposition. For details we refer to [11].

4.2 Minkowski decompositions for T-invariant divisors

4.2.1 Okounkov bodies for toric varieties

We start with recalling some facts and tools related to the study of Okounkov bodies for toric varieties. In first step we would like to generalize the notion of Minkowski basis in the general setting (not only for toric varieties).

Definition 4.2.1. Let X be a smooth projective variety of dimension n and $Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{pt\}$ an admissible flag on X. A collection $\{D_1, \ldots, D_r\}$ of pseudo-effective divisors on X is called a *Minkowski base* of X with respect to $Y_\bullet$ if

- for any pseudo-effective divisor D on X there exist non-negative numbers $\{a_1, \ldots, a_r\}$ and a translation $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that
$$\phi(\Delta_{Y_\bullet}(D)) = \sum a_i \Delta_{Y_\bullet}(D_i)$$
- the Okounkov bodies $\Delta_{Y_\bullet}(D_i)$ are indecomposable in the sense of Minkowski decomposition.

The difference is significant since we allow to use not only nef divisors. In fact we will see that in the toric case Minkowski bases elements are movable divisors. We will use the following theorem due to Lazarsfeld and Mustaţă.

Proposition 4.2.2 (Proposition 6.1, [24])**.** *Let X be a smooth projective toric variety, and let $Y_\bullet$ be a flag given as a complete intersection of a set of T-invariant divisors generating a maximal cone σ. Given any big line bundle $\mathcal{O}_X(D)$ on X, such that $D|_{\mathcal{U}_\sigma} = 0$, then*

$$\Delta(\mathcal{O}_X(D)) = \varphi(P_D),$$

where φ is an $\mathbb{R}$-linear map.

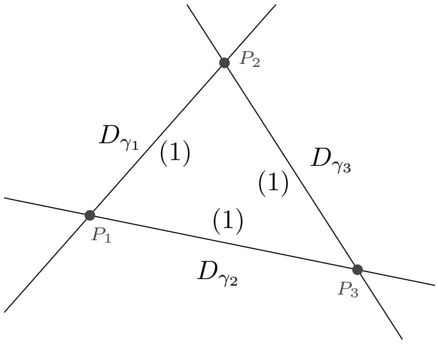
Example 4.2.3. Now we present a complete example, which shows us how to construct the blow-up of $\mathbb{P}^2$ at two T-invariant points, a toric polyhedron and how to use Proposition 4.2.2.

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ be the standard basis for the lattice $\mathbb{Z}^2$. Now we will treat $\mathbb{P}^2$ as a toric surface. The fan Σ corresponding to $\mathbb{P}^2$ is generated by three cones,

$$\sigma_1 = \text{span}\{e_1, e_2\}, \sigma_2 = \text{span}\{e_2, -e_1 - e_2\}, \sigma_3 = \{e_1, -e_1 - e_2\}.$$

The fan Σ has three rays, i.e. $\gamma_1 = \mathbb{R}_+ e_1$, $\gamma_2 = \mathbb{R}_+ e_2$, $\gamma_3 = \mathbb{R}_+(-e_1 - e_2)$, and these rays correspond to three T-invariant divisors.

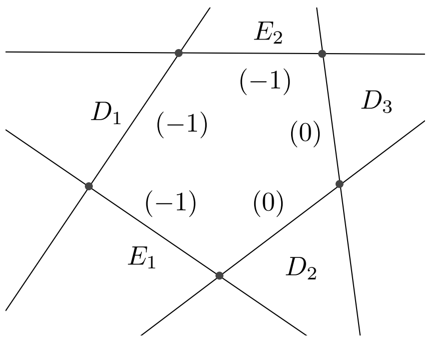
Now we show how to construct the toric blow up of $\mathbb{P}^2$ at two points. Let us choose two T-invariant points – in this case we have actually three T-invariant points, which corresponds to the intersection of T-invariant divisors, i.e. $P_1 = D_{\gamma_1} \cap D_{\gamma_2}$, $P_2 = D_{\gamma_1} \cap D_{\gamma_3}$, $P_3 = D_{\gamma_2} \cap D_{\gamma_3}$.



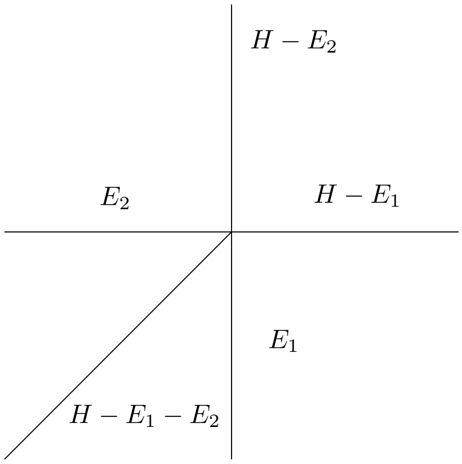
In parentheses we denote the self-intersection of the T-invariant divisors.

The blow-up of $\mathbb{P}^2$ at P_1 is obtained by adding the exceptional curve E_1 with the self-intersection -1, of course the self-intersections of $D_{\gamma_1}, D_{\gamma_2}$ drops by 1.

Proceed in the same manner one constructs the blow-up of $\mathbb{P}^2$ at P_1, P_2, and resulting blow up is still a toric variety. Thus the blow up of $\mathbb{P}^2$ at four points is not a toric surface since there are only three T-invariant points to blow-up.



In the language of fans, blowing up means adding certain rays to a fan. In our case we add two rays, $\mathbb{R}_+(-e_1)$ corresponding to E_2 and $\mathbb{R}_+(-e_2)$ corresponding to E_1. The resulting fan Σ has the following form.



Denote for brevity $\rho_1 = H-E_1, \rho_2 = H-E_2, \rho_3 = E_2, \rho_4 = H-E_1-E_2$ and $\rho_5 = E_1$. Now we would like to construct the Okounkov body for $D = 2H - E_1 - E_2$ with respect to the flag $\{E_2, E_2 \cap C\}$ where $C \in |H - E_2|$. According to Proposition 4.2.2 we need to present D as a linear combination of ρ_1, ρ_4 and ρ_5, thus

$$D = \sum_{i=1}^{5} a_i \rho_i = 1 \cdot \rho_1 + 0 \cdot \rho_2 + 0 \cdot \rho_3 + 1 \cdot \rho_4 + 1 \cdot \rho_5.$$

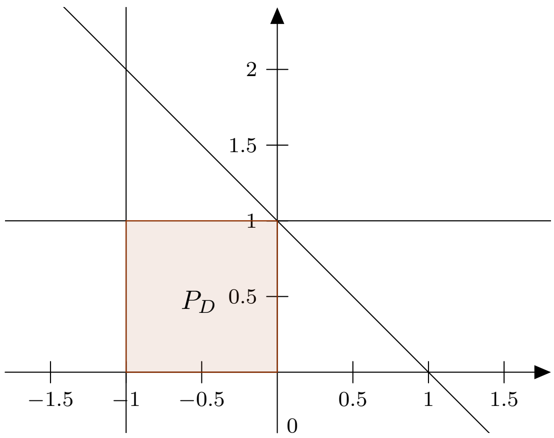
In order to find the toric polyhedron P_D we need to solve the following system of inequalities

$$(x, y) \circ f_i \geq -a_i \text{ for } i \in \{1, ..., 5\}.$$

Thus we get

$$\begin{cases} x \geq -1 \\ y \geq 0 \\ x \leq 0 \\ y \leq 1 - x \\ y \leq 1 \end{cases},$$

where f_i are generators of rays. The figure below presents P_D.



Of course P_D is not the Okounkov body for D, but due to Proposition 4.2.2 there exists a linear transform such that $\phi(P_D) = \triangle_{(E_2, C\cap E_2)}(D)$. It is easy to see that ϕ is given by the matrix

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and $\phi(P_D) = [0,1]^2$.

4.2.2 Characterization of Minkowski base elements

Let X be a smooth projective toric variety of dimension n admitting only smooth modifications as in Definition 4.1.30. First note that a Minkowski base for any given T-invariant flag will also be a Minkowski base for any other T-invariant flag. This is due to Proposition 4.2.2 which says that the Okounkov body of a big divisor D on X with respect to a flag $Y_\bullet$ is a linear transform of the polytope P_D. Minkowski summation respects linear transformations, so both properties of a Minkowski base can be checked immediately on the level of T-invariant polytopes, which is what we will always do in the rest of this chapter. Our aim is to prove the following.

Theorem 4.2.4. *The set of all T-invariant divisors D on a toric variety X as above such that there exists a small modification $f : X \dashrightarrow X'$ and a divisor D' spanning an extremal ray of $\mathrm{Nef}(X')$ such that $D = f^*(D')$ forms a Minkowski base with respect to T-invariant flags.*

Proof. Let D be a T-invariant pseudo-effective divisor. There exists a small modification $f : X \dashrightarrow X'$ such that D admits a CKM-Zariski decomposition considered as a divisor on X'. We can thus assume that D admits a CKM-Zariski decomposition $D = P + N$ on X (since a small modification does not change the polytope). Now, all sections in $H^0(X, \mathcal{O}_X(D))$ come from $H^0(X, \mathcal{O}_X(P))$ and so the Okounkov body of D is merely a translate of the body of P, which is a nef divisor. We conclude the proof of the first part of the theorem by proving that the polytope corresponding to P can be decomposed as a Minkowski sum of divisors spanning extremal rays of $\mathrm{Nef}(X)$. But this immediately follows from the fact that we can write P as a non-negative $\mathbb{Q}$-linear combination of these divisors together with Proposition 4.1.18, stating the additivity of polytopes for nef divisors.

The theorem then follows from Proposition 4.2.5. □

Let us prove that the set of divisors spanning extremal rays of the nef cone on some small modification also satisfies the second condition for a Minkowski base.

Proposition 4.2.5. *For a T-invariant movable Cartier divisor D the polytope P_D is indecomposable if and only if there exists a small modification $f : X \dashrightarrow X'$ and divisor D' spanning an extremal ray of $\mathrm{Nef}(X')$ with $D = f^*(D')$.*

Remark 4.2.6. This result is slightly stronger than what we need in order to prove the theorem. In fact, it shows that if we require elements of a Minkowski base to be movable, then the Minkowski base is unique. Therefore, finding all indecomposable polytopes coming from movable divisors recovers this Minkowski base. This is exactly what the algorithm given in the next section does.

Proof. First note that we can assume D to be nef on X, since being movable, it is the pullback of a nef divisor under a small modification $f : X \dashrightarrow X'$ and the polytopes agree, since f does not alter the rays in the fan defining X, but only changes higher dimensional cones.

Let us consider X_{P_D}, the variety given by the normal fan of P_D and consider a proper toric morphism $\varphi : X \to X_{P_D}$ given by a certain multiple of D.

Let us now suppose that D spans an extremal ray of $\mathrm{Nef}(X)$ and that $P = Q + N$ is a Minkowski decomposition. Since P_D is a full dimensional polytope with respect to X_{P_D}, this means that there exist nontrivial basepoint free T-invariant, hence nef, Cartier divisors A and B on X_{P_D} such that $Q = P_A$ and $N = P_B$.

But now we have that $\varphi^*(A)$, $\varphi^*(B)$ are nef divisors on X and since toric polytopes remain invariant via pullback, we have

$$P_D = P_{\varphi^*(A)} + P_{\varphi^*(B)} \quad \text{and} \quad D = \varphi^*(A) + \varphi^*(B)$$

hence we get a contradiction.

For the opposite implication, assume D is nef and it does not lie in an extremal ray of $\mathrm{Nef}(X)$. We can then find non-trivial nef divisors A and B such that $D = A + B$. Now for globally generated divisors we know that the polytope of a sum coincides with the Minkowski sum of the polytopes, i.e.,

$$P_D = P_A + P_B,$$

showing that P_D is decomposable. □

4.2.3 How to find a Minkowski basis

Let us consider a smooth toric projective variety $X = X(\Sigma)$, $\dim(X) = n$ and let $d = \#(\Sigma(1)) - n$, where $\Sigma(1)$ denotes the set of rays in Σ. We will first informally describe the idea of the algorithm determining the Minkowski base of $X(\Sigma)$ consisting of movable divisors. A formal description of the algorithm follows afterwards. In the following, R_m will always be a set of rays of the fan Σ and R_m^* denotes the set of half-spaces dual to given rays in R_m. Furthermore, P_m will be the set of points corresponding to the vertices of R_m^*. Δ_i = conv hull(P_m) will denote the convex hull generated by the points in P_m.

Let $H_t(\rho_i) = \{x \in N_{\mathbb{R}} | \langle x, \rho_i\rangle \geq -t\}$ and $\bar{H}_t(\rho_i) = \{x \in N_{\mathbb{R}} | \langle x, \rho_i\rangle = -t\}$.

- Fix a cone $\sigma \in \Sigma(n)$ and consider its dual cone $\sigma^\vee$.

  Let $R_1 = \{\rho \in \Sigma(1) \backslash \sigma(1)\}$ $\qquad R_1^* = \{\rho^\vee | \rho \in \Sigma(1)\}$.
  Set $d = \#(R_1)$, $R_1 = \{\rho_i\}_i$.

- $(\star)$ For every $\rho_i \in R_1$, we consider the half-spaces $H_1(\rho_i)$.

  Define : $P_2 = \bar{H}_0(\rho_i) \cap \{\rho | \rho \in \Sigma(1)\}$; $\quad R_2 = R_1 \backslash \{\rho_i\}$; $\quad m = 2$.

- $(\star\star)$ For every $\rho_i \in R_m$ do either of the following:

  – Let $t = \max(0, \min\{t' \in \mathbb{R} | H_{t'}(\rho_i) \supseteq \Delta_{m-1}\})$;
    set $R_{m+1} = R_m \backslash \{\rho_i\}$ $\quad R_m^* = R_{m-1}^* \cap H_t(\rho_i)$;
    If $m = d$, check if Δ_m corresponds to a divisor. If it does, return the divisor (this is a Minkowski base element).
    If $m < d$, set $m = m + 1$ and go to $(\star\star)$.

  – Set $R_{m+1} = R_m \backslash \{\rho_i\}$ $\quad R_m^* = R_{m-1}^* \cap H_0(\rho_i)$;
    If $m = d$, check if Δ_m corresponds to a divisor. If it does, return the divisor (this is a Minkowski base element)
    If $m < d$, set $m = m + 1$ and go to $(\star\star)$.

- Iterate until all possible combinations in $(\star)$ and $(\star\star)$ are exhausted.

We now give a formal description of the above sketch which we in fact implemented for actual computations.

Algorithm 4.2.7. *Algorithm* TMB

Input: fan Σ with $n+d$ rays

Output: Minkowski base for $X(\Sigma)$

Variables:

R, an array of length d; each entry is a set of rays.

M, an array of length d; each entry is a set of rays.

P, an array of length d; each entry is a set of points.

Δ, an array of length d; each entry is a polyhedron.

Uses:

CorrespondsToDivisor checks if the given polytope arise as a polytope of a T-invariant divisor

for k from 1 to d do
    $R_k \leftarrow \Sigma(1)$;
    $M_k, \Delta_k \leftarrow \emptyset$;
    $P_k \leftarrow \emptyset$;
end do;
for l from 1 to d do
    $R_1 \leftarrow \Sigma(1) \backslash \rho_l$;
    $\Delta_1 \leftarrow H_1(\rho_l) \cap \sigma^\vee$;
    if $(\mathit{CorrespondsToDivisor}(\Delta_1))$ then
        $TMB \leftarrow TMB \cup \{\Delta_1\}$;
    end if;
    $P_2 \leftarrow \mathit{Vertices}(\Delta_1)$;
    while $(R_2 \neq \emptyset)$ do
        $m \leftarrow \max\{n | P_n \neq \emptyset\}$;
        if $(M_m = \emptyset)$ then
            Pick $\rho \in R_m$;
            $R_m \leftarrow R_m \backslash \rho$;
            for k from $m+1$ to d do
                $R_k \leftarrow R_{m-1} \backslash \rho$;
                $M_k \leftarrow \emptyset$;
            end do;
            $P_m \leftarrow \mathit{Vertices}(\Delta_{m-1})$;
            $M_m \leftarrow \rho$;
        end if;
        $t \leftarrow \min_{p \in P_m} \left(\mathit{Solves}_s(p \in \overline{H}_s(\rho))\right)$;
        if $(t > 0)$ then
            $\Delta_m \leftarrow \Delta_{m-1} \cap H_t(\rho)$;
            if $(\mathit{CorrespondsToDivisor}(\Delta_m))$ then
                $TMB \leftarrow TMB \cup \{\Delta_m\}$;
            end if;
            $P_m \leftarrow \{0\}$;
        else
            $\Delta_m \leftarrow \Delta_{m-1} \cap H_0(\rho)$;

if ($CorrespondsToDivisor(\Delta_m)$) then
    $TMB \leftarrow TMB \cup \{\Delta_m\}$;
end if;
$s \leftarrow \max\{n \leq m-1 | R_n \neq \emptyset\}$;
if ($m < s$) then
    $P_m \leftarrow \emptyset$;
    $P_{m+1} \leftarrow Vertices(\Delta_m)$;
    $M_m \leftarrow \emptyset$;
else
    $M_m \leftarrow \emptyset$;
    $P_m \leftarrow \emptyset$;
    if ($P_s = \emptyset$) then
        $P_s \leftarrow Vertices(\Delta_{s-1})$;
    end if;
end if;
end if;
end do;
end do;
return TMB;

4.2.4 How the algorithm works

In this subsection we will prove correctness of the algorithm. Note that it clearly terminates.

Proposition 4.2.8. *Every polytope Δ in the output of the algorithm is indecomposable (as a sum of polytopes having the origin as a vertex).*

Proof. The idea behind the algorithm is quite straightforward: every *slope* appearing in some toric polyhedron has to occur in the polytope of at least one of the Minkowski base elements. Let us recall the steps of the algorithm:

- We fix a T-invariant flag, i.e., we choose a set of $d = \dim(X)$ rays that generate one of the cones $\sigma \in \Sigma(d)$. This cone will correspond to a cone $\sigma^\vee$ in the dual space $M_\mathbb{R}$. Let us denote by Σ_0 the fan (not complete) generated by the rays of σ.

- In the first step we fix a slope, i.e., a ray $\rho_1 \in \Sigma(1) \backslash \sigma(1)$. To this ray corresponds an hyperplane $H_1(\rho_1)$.

- $H_1(\rho_1)$ intersects an edge of $\sigma^\vee$ if and only if the corresponding hyperplanes generate a convex cone in $N_\mathbb{R}$. If it intersects all the rays, then we found a simplex, hence a Minkowski base element. Either way, we call the resulting polyhedron Δ_1 and the corresponding fan Σ_1.

- If Δ_1 is not a (bounded) polytope we need to add an additional ray, say ρ_2, into Σ_1, i.e., intersect our polyhedron with an additional half-space. There are two different ways in which we intersect in the algorithm: either take Δ_2 to be the intersection of Δ_1 with the half-space $H_t(\rho_2)$, where t is the minimal positive number such that all vertices of Δ_1 are contained in $H_t(\rho_2)$ or with the half-space $H_0(\rho_2)$.

- We keep intersecting until all $\rho \in \Sigma_1 \backslash \sigma(1)$ are exhausted. Let us denote by Δ_k the region bounded by the already existing hyperplanes after k steps.

We first prove inductively that in each step of the algorithm, the fan Σ_k remains minimal, in the sense that it is not the refinement of any other fan with the same convex span. In particular this means that as soon Δ_k is bounded, then it is indecomposable by Proposition 4.1.22.

$\triangle_0$ is clearly minimal. Let us suppose that Σ_k for $k \geq 0$ is a minimal fan. When adding a face to the dual polyhedron Δ_k that is tangent to a vertex of Δ_k, this is equivalent to:

- cancel in the fan all the cones corresponding to faces completely contained in the complement of the half-space, so that the resulting fan is minimal again,
- cancel in the fan the cone corresponding to the vertex (and all of its faces but the rays),
- add a ray that is not contained in the convex span of the rays we are left with,
- construct all the possible cones with the new ray and the rays of the last cone we canceled,
- if there is a containment of cones, then the ray of the inner one that is properly contained gets canceled from the new fan (we do not admit star subdivisions).

Is the new fan minimal? Yes, in fact:

- by construction we cannot cancel any ray for the new cones we have, since we already canceled all the star subdivisions,
- none of the other rays can get canceled by minimality of the fan at the previous stage.

□

Remark 4.2.9. Of course, not every indecomposable T-invariant polytope is found by the algorithm, but only the convex ones with a vertex on the origin. In case the variety admits a flip then it is possible that not all the indecomposable polytopes corresponding to extremal rays of the movable cone will have the origin as a vertex for a fixed flag. To overcome this issue, it will be enough to go through all the possible flags, given by a different T-invariant maximal cone, run again the algorithm and compare the possibly extra elements with the base found in the previous steps (Example 4.3.2).

Proposition 4.2.10. *Let X be a toric variety and let $Y_\bullet$ be a flag given by the complete intersection of the generators of a maximal subcone. Let D be a T-invariant divisor such that no element of the chosen flag is contained in its base locus. Then if P_D is indecomposable then D is in the output of the algorithm.*

Proof. This is given by the construction of the polytopes in the algorithm.

Denote by σ the cone in $M_{\mathbb{R}}$ corresponding to the flag $Y_\bullet$. Without loss of generality, we can assume D to be a member of its linear series with $D_{|\sigma} = 0$. This guarantees P_D to be contained in the dual cone $\sigma^\vee$. Write $D = \sum a_i D_i$ where the D_i's are the prime T-invariant divisors corresponding to the rays $\rho_1, \ldots, \rho_r$ of Σ_X. We can assume by reordering that the first s of the D_i's are those which correspond to facets of P_D. We describe how the algorithm finds P_D.

Let us define Δ_1 to be $\sigma^\vee$. As the first step we intersect Δ_1 with all the half-spaces $H_0(\rho_i)$ where P_D is contained in $\overline{H}_0(\rho_i)$, say for all $s+1 \leq i \leq s'$ to obtain Δ_2. Note that P_D is an indecomposable full dimensional lattice polytope in the intersection of $N_{\mathbb{R}}$ with these half-spaces, thus in the dual lattice M' of this intersection its normal fan is not the refinement of any complete fan by Proposition 4.1.22.

Let now Δ_3 be the intersection of Δ_2 with $H_{a_j}(\rho_j)$ for which $a_j = 0$ and $j \leq s$. This will give the smallest cone containing P_D.

Since the fan is convex and complete, we can choose a ray ρ_1 that is contained in a cone adjacent to $\Delta_3^\vee$ and such that the induced completion in the span of the fan is contained in Σ_{P_D}. Up to rescaling we can assume that $a_1 = 1$. This will be the starting point for the iterative part of the algorithm. Let us define $P_1 = \{\text{vertices of } \bar{H}_1(D_1) \cap \Delta_3\}$. Then, choosing the ray ρ_2 with the same criterion, if $t = \inf\{s | H_s(\rho_2) \supseteq P_1\}$, then by construction $t = a_2$. Iterating this step with increasing each time the number of intersection points the polytope will be fully reconstructed. □

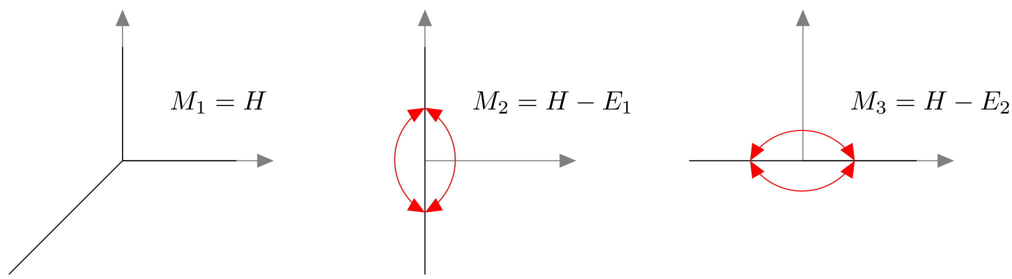
4.2.5 How to find Minkowski decompositions

Let us now turn to the problem of finding the Minkowski decomposition of a given T-invariant big divisor D once the Minkowski base for X is determined by the algorithm.

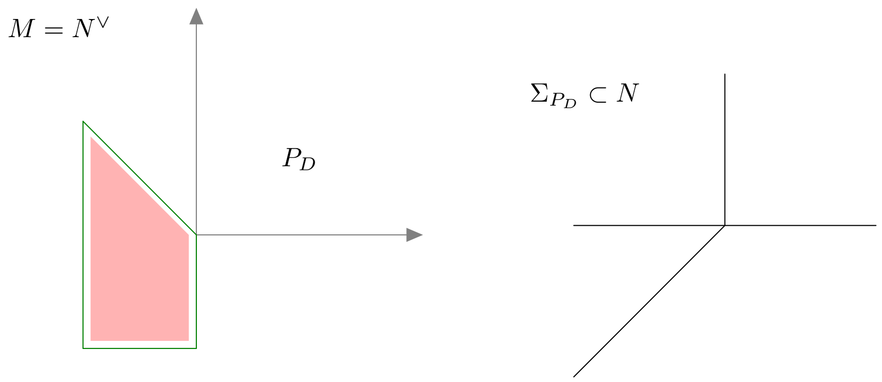
Our aim is to use the knowledge of the Okounkov body $\Delta_{Y_\bullet}(D)$ with respect to some T-invariant flag and go backwards by finding Minkowski summands of this polytope. In order to do it we will use Proposition 4.1.22 as follows.

Given the divisor D, we consider its polytope P_D and the corresponding dual fan Σ_D. Now, a Minkowski base element B is a summand in the Minkowski decomposition of D if and only if its fan Σ_B is refined by Σ_D. This is a straightforward check. Note also that the fans Σ_D and Σ_B only dependent on the linear equivalence classes. We thus obtain all classes of Minkowski base elements with representative B_i having positive coefficient a_i in the Minkowski decomposition $M_D = \sum a_j B_j$.

Example 4.2.11. Let us consider X the blow-up of $\mathbb{P}^2$ in two T-points. We will use the same notation as in Example 4.2.3. It is easy to see that the nef cone $\text{Nef}(X)$ is spanned by the classes $M_1 := H$, $M_2 := H - E_1$ and $M_3 := H - E_2$. By Theorem 4.2.4 these classes form a Minkowski base with respect to torus-invariant flags. The normal fans of their polytopes are depicted below.



Let us now decompose the T-invariant divisor $D := \rho_1 + \rho_2 + \rho_5$. It has the following polytope P_D and corresponding normal fan $\triangle_{P_D}$.

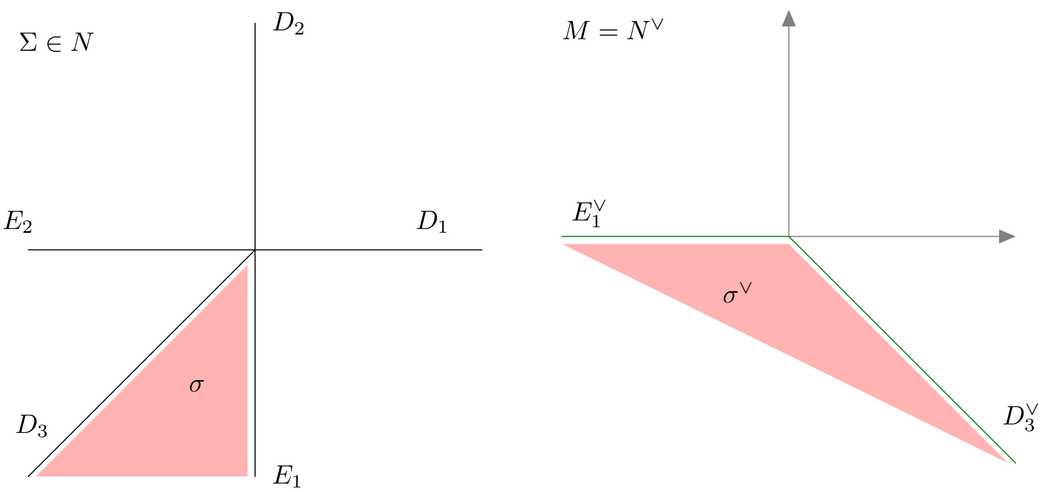


Now, Σ_{P_D} is the refinement of the fans Σ_{M_1} and Σ_{M_3} but not of Σ_{M_2}. Thus D decomposes as a positive linear combination of M_1 and M_3. We easily obtain the decomposition $D \sim M_1 + M_3 = H + (H - E_2)$.

4.3 Examples

4.3.1 The blow-up of $\mathbb{P}^2$ in two points

Let us consider the blow-up op $\mathbb{P}^2$ in two points. The fan Σ is given by 5 rays, corresponding to divisor classes $D_1 = H - E_1$, $D_2 = H - E_2$, $D_3 = H - E_1 - E_2$, E_1 and E_2. Let us fix the flag given by $Y_\bullet = \{D_3, D_3 \cap E_1\}$.

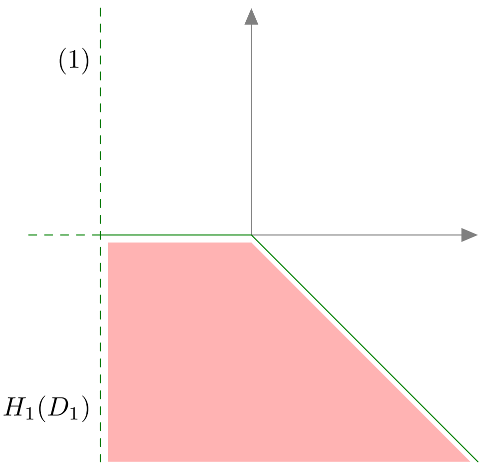


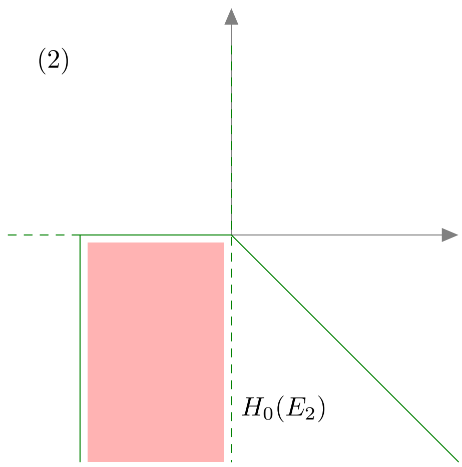
We can now run the algorithm. In the following table it is recorded what is saved in each step in R, M, P and D.

When there is no output for D (in the table: —), it means that there is no divisor corresponding to the particular configuration of generating hyperplanes.

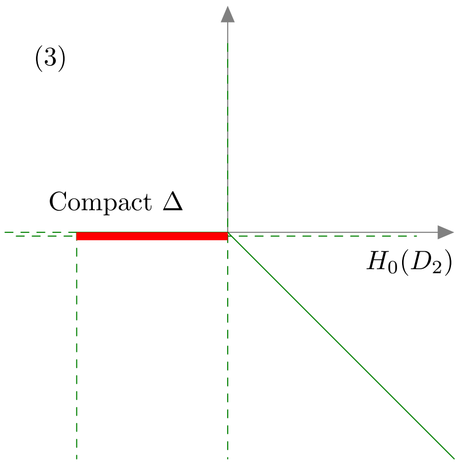
To make the table more readable, we made pictures for the steps marked with a number.

R	M	P	D
$[[D_1, D_2, E_2], [D_1, D_2, E_2], [D_1, D_2, E_2]]$	$[[\emptyset], [\emptyset], [\emptyset]]$	$[\emptyset]$	
$[[D_2, E_2], [D_2, E_2], [D_2, E_2]]$	$[[D_1], [\emptyset], [\emptyset]]$	$[[\emptyset], [(0,0), (-1,0)], [\emptyset]]$	$-$
$[[D_2, E_2], [E_2], [E_2]]$	$[[D_1], [D_2], [\emptyset]]$	$[[\emptyset], [(0,0)], [(0,0), (-1,0)]]$	$-$ (1)
$[[D_2, E_2], [E_2], [\emptyset]]$	$[[D_1], [D_2], [E_2]]$	$[[\emptyset], [(0,0)], [(0,0)]]$	D_1
$[[D_2, E_2], [E_2], [\emptyset]]$	$[[D_1], [D_2], [E_2]]$	$[[\emptyset], [(0,0)], [\emptyset]]$	D_1
$[[D_2, E_2], [E_2], [E_2]]$	$[[D_1], [D_2], [\emptyset]]$	$[[\emptyset], [\emptyset], [(0,0), (-1,0)]]$	$-$
$[[D_2, E_2], [E_2], [\emptyset]]$	$[[D_1], [D_2], [E_2]]$	$[[\emptyset], [\emptyset], [(0,0)]]$	D_1
$[[D_2, E_2], [E_2], [\emptyset]]$	$[[D_1], [D_2], [E_2]]$	$[[\emptyset], [\emptyset], [\emptyset]]$	D_1
$[[D_2, E_2], [D_2], [D_2]]$	$[[D_1], [E_2], [\emptyset]]$	$[[\emptyset], [(0,0)], [(0,0)]]$	$-$ (2)
$[[D_2, E_2], [D_2], [\emptyset]]$	$[[D_1], [E_2], [D_2]]$	$[[\emptyset], [(0,0)], [(0,0)]]$	D_1 (3)
$[[D_2, E_2], [D_2], [\emptyset]]$	$[[D_1], [E_2], [D_2]]$	$[[\emptyset], [(0,0)], [\emptyset]]$	D_1
$[[D_2, E_2], [D_2], [D_2]]$	$[[D_1], [E_2], [\emptyset]]$	$[[\emptyset], [\emptyset], [(0,0)]]$	$-$
$[[D_2, E_2], [D_2], [\emptyset]]$	$[[D_1], [E_2], [D_2]]$	$[[\emptyset], [\emptyset], [(0,0)]]$	D_1
$[[D_2, E_2], [D_2], [\emptyset]]$	$[[D_1], [E_2], [D_2]]$	$[[\emptyset], [\emptyset], [\emptyset]]$	D_1
$[[D_1, E_2], [D_1, E_2], [D_1, E_2]]$	$[[D_2], [\emptyset], [\emptyset]]$	$[[\emptyset], [(0,0), (1,-1)], [\emptyset]]$	$-$ (4)
$[[D_1, E_2], [E_2], [E_2]]$	$[[D_2], [D_1], [\emptyset]]$	$[[\emptyset], [(0,0)], [(0,0), (1,-1)]]$	$-$ (5)
$[[D_1, E_2], [E_2], [\emptyset]]$	$[[D_2], [D_1], [E_2]]$	$[[\emptyset], [(0,0)], [(0,0)]]$	$D_2 + E_2$
$[[D_1, E_2], [E_2], [\emptyset]]$	$[[D_2], [D_1], [E_2]]$	$[[\emptyset], [(0,0)], [\emptyset]]$	D_2
...	...	...	...
$[[D_1, D_2], [D_2], [\emptyset]]$	$[[E_2], [D_2], [\emptyset]]$	$[[\emptyset], [\emptyset], [(0,0)]]$	—
...	...	...	...



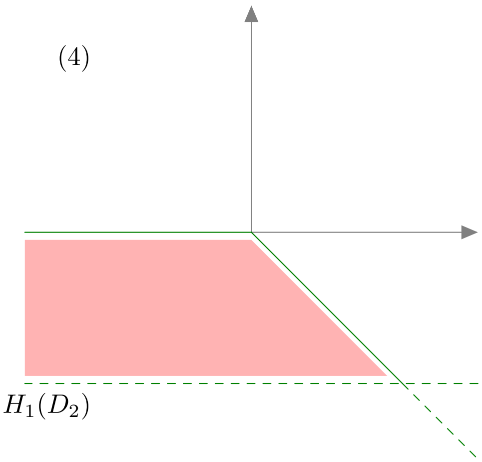


(3)

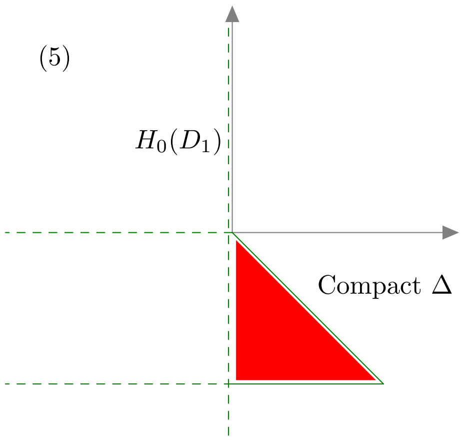


With $\Delta = P_D$, $D = D_1 \in$ MBE.

(4)



(5)

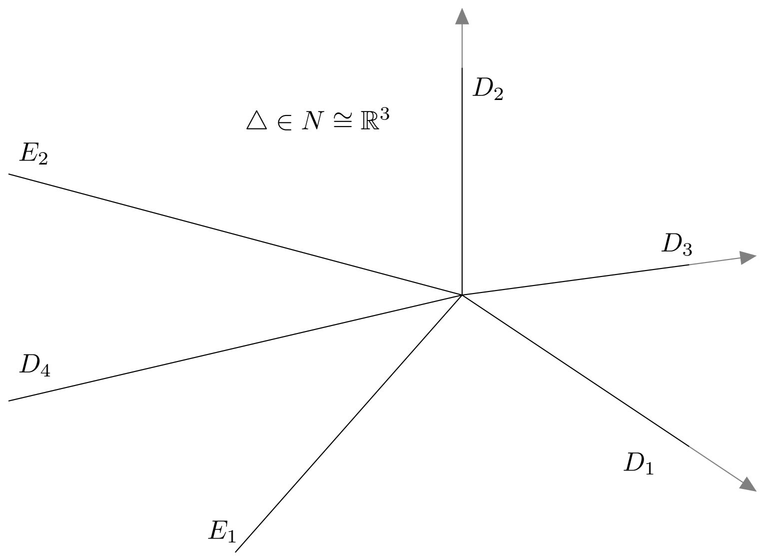


In this case $\Delta = P_D$ with $D = D_2 + E_2 \in$ MBE.

In the 2-dimensional case there is no need of repeating the algorithm since the nef and the movable cone coincide.

4.3.2 The blow-up of $\mathbb{P}^3$ in two intersecting lines

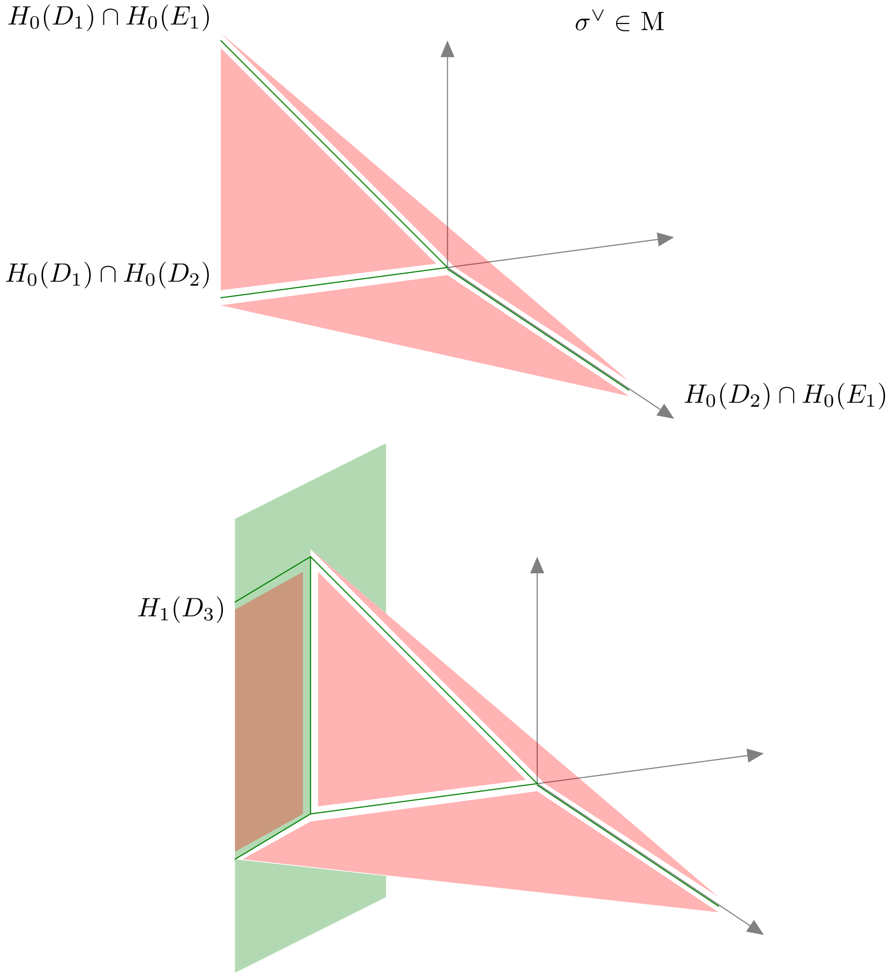
Let us consider the blow-up of $\mathbb{P}^3$ in two intersecting lines. The fan Σ contains 6 rays which are spanned by the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(-1, -1, -1)$, $(0, -1, -1)$, $(-1, 0, -1)$, and correspond to divisor classes $D_1 = H - E_1$, $D_2 = H - E_2$, $D_3 = H$, $D_4 = H - E_1 - E_2$, E_1 and E_2 respectively. Let us fix the flag given by $Y_\bullet = \{D_1, D_1 \cap D_2, D_1 \cap D_2 \cap E_1\}$.



Since this variety admits SQMs (there are two SQMs corresponding to the order of blowing up of lines) and we have chosen a cone that can be flipped, we expect not to find all the generators of the moving cone in the first stage. In order to find other the remaining generators we have to take a blow up in the revers order and proceed in the same way.

R	M	P	D
$[[D_3, D_4, E_2], [D_3, D_4, E_2], [D_3, D_4, E_2]]$	$[[\emptyset], [\emptyset], [\emptyset]]$	$[\emptyset]$	
$[[D_4, E_2], [D_4, E_2], [D_4, E_2]]$	$[[D_3], [\emptyset], [\emptyset]]$	$[[\emptyset], [\vec{0}, (0,0,-1), (0,1,-1)], [\emptyset]]$	$-$ (6)
$\dots$	$\dots$	$\dots$	$\dots$
$[[D_4, E_2], [E_2], [\emptyset]]$	$[[D_3], [D_4], [E_2]]$	$[[\emptyset], [\vec{0}], [\vec{0}]]$	D_3
$[[D_4, E_2], [E_2], [\emptyset]]$	$[[D_3], [D_4], [E_2]]$	$[[\emptyset], [\vec{0}], [\emptyset]]$	D_3
$\dots$	$\dots$	$\dots$	$\dots$
$[[D_4, E_2], [E_2], [\emptyset]]$	$[[D_3], [D_4], [E_2]]$	$[[\emptyset], [\emptyset], [\vec{0}]]$	D_3
$[[D_4, E_2], [E_2], [\emptyset]]$	$[[D_3], [D_4], [E_2]]$	$[[\emptyset], [\emptyset], [\emptyset]]$	D_3
$\dots$	$\dots$	$\dots$	$\dots$
$[[D_4, E_2], [D_4], [\emptyset]]$	$[[D_3], [E_2], [D_4]]$	$[[\emptyset], [\vec{0}], [\vec{0}]]$	$D_4 + D_3$
$[[D_4, E_2], [D_4], [\emptyset]]$	$[[D_3], [E_2], [D_4]]$	$[[\emptyset], [\vec{0}], [\emptyset]]$	D_3
$\dots$	$\dots$	$\dots$	$\dots$
$[[D_4, E_2], [D_4], [\emptyset]]$	$[[D_3], [E_2], [D_4]]$	$[[\emptyset], [\emptyset], [\vec{0}]]$	$D_4 + D_3$
$[[D_4, E_2], [D_4], [\emptyset]]$	$[[D_3], [E_2], [D_4]]$	$[[\emptyset], [\emptyset], [\emptyset]]$	D_3
$\dots$	$\dots$	$\dots$	$\dots$
$[[D_3, E_2], [E_2], [\emptyset]]$	$[[D_4], [D_3], [E_2]]$	$[[\emptyset], [\vec{0}], [\vec{0}]]$	$D_4 + E_2$
$[[D_3, E_2], [E_2], [\emptyset]]$	$[[D_4], [D_3], [E_2]]$	$[[\emptyset], [\vec{0}], [\emptyset]]$	—
$\dots$	$\dots$	$\dots$	$\dots$

Table 4.1: Minkowski basis elements



The Minkowski basis element (6)

References

[1] D. Anderson, Okounkov bodies and toric degenerations, *Math. Ann.* **356**: 1183 – 1202 (2013).

[2] D. Anderson & A. Küronya & V. Lozovanu, Okounkov bodies of finitely generated divisors, *Internat. Math. Res. Notices* **vol. 2014 (9)**: 2343 – 2355 (2014).

[3] L. Bădescu, *Algebraic surfaces*, Universitext, Springer-Verlag, New York, 2001.

[4] Th. Bauer, Seshadri constants of quartic surfaces, *Math. Ann.* **309**: 475 – 481 (1997).

[5] Th. Bauer, A simple proof for the existence of Zariski decompositions on surfaces, *J. Alg. Geom.* **18**: 789 – 793 (2009).

[6] Th. Bauer & M. Funke, Weyl and Zariski chambers on K3 surfaces, *Forum Mathematicum* vol. **24(3)**: 609 – 625 (2012).

[7] Th. Bauer & M. Funke & S. Neumann, Counting Zariski chambers on Del Pezzo surfaces, *Journal of Algebra* **324**: 92–101 (2010).

[8] Th. Bauer & A. Küronya & T. Szemberg, Zariski chambers, volumes, and stable base loci, *J. Reine Angew. Math.* **576**: 209 – 233 (2004).

[9] F. Campana & T. Peternell, Algebraicity of the ample cone of projective varieties, *J. Reine Angew. Math.* **407**: 160 – 166 (1990).

[10] D. Cox & J. Little & H. Schenck, *Toric varieties*, Springer-Verlag, New York, 2011.

[11] S. D. Cutkosky, Zariski decomposition of divisors on algebraic varieties, *Duke Math. J.* **53**: 149 – 156 (1986).

[12] O. Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, 2001.

[13] O. Debarre, *Introduction to Mori Theory*, lecute notes, (on-line avaliabe: `http://www.math.ens.fr/~debarre/M2.pdf`).

[14] W. Fulton, *Introduction to Toric Varieties, Annals of Mathematics Studies Number **131***, Princeton University Press, Princeton, 1993.

[15] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.

[16] Y. Hu & S. Keel, Mori dream spaces and GIT, *Michigan Mathematical Journal* **48**: 331 – 348 (2000).

[17] D. Huybrechts, *Lectures on K3 surfaces*, lecture notes, (on-line avaliabe: `http://www.math.uni-bonn.de/people/huybrech/K3Global.pdf`).

[18] K. Kaveh & A. Khovanskii, Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory, *Ann. of Math.* **176**: 1 - 54 (2012).

[19] A. Küronya, *Partial positivity concepts in projective geometry*, Habilitationsschrift, Albert-Ludwigs-Universität Freiburg, 2011 (on-line avaliabe: `www.math.bme.hu/~kalex/Papers/hab.pdf`).

[20] A. Küronya & V. Lozovanu & C. Maclean, Convex bodies appearing as Okounkov bodies of divisors, *Adv. Math.* **229**: 2622 – 2639 (2012).

[21] Y. Manin, *Cubic Forms. Algebra, Geometry, Arithmetic*, North-Holland Mathematical Library. Vol. 4. North-Holland, 1974.

[22] M. Nakamaye, Stable base loci of linear series, *Math. Ann.* **318**: 837–847 (2000).

[23] R. Lazarsfeld, *Positivity in Algebraic Geometry, I & II Ergebnisse der Mathematik und ihrer Grenzgebiete*, Vols. 48 & 49, Springer Verlag, Berlin, 2004.

[24] R. Lazarsfeld & M. Mustaţă, Convex bodies associated to linear series, *Ann. Sci. École Norm. Sup.(4)* **42**: 783 – 835 (2009).

[25] P. Łuszcz - Świdecka, On Minkowski decomposition of Okounkov bodies on a Del Pezzo surface, *Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica* **10**: 105 – 115 (2011).

[26] P. Łuszcz - Świdecka & D. Schmitz, Minkowski decomposition of Okounkov bodies on surfaces, *Journal of Algebra* **414**: 159 – 174 (2014).

[27] A. Okounkov, Brunn - Minkowski inequality for multiplicities, *Invent. Math.* **125**: 405 – 411 (1996).

[28] P. Pokora & D. Schmitz & S. Urbinati, Minkowski decomposition and generators of the moving cone for toric varieties, **arXiv:1310.8505**.

[29] P. Pokora & T. Szemberg, Minkowski bases on algebraic surfaces with rational polyhedral pseudo-effective cone, *Electron. Res. Announc. Math. Sci.* **21**: 126 - 131 (2014).

[30] D. Schmitz & H. Seppänen, On the polyhedrality of global Okounkov bodies, **arXiv:1403.4517**, to appear in *Advances in Geometry.*

[31] Shin-Yao Jow, Okounkov bodies and restricted volumes along very general curves, *Adv. Math.* **223**: 1356 – 1371 (2010).

[32] B. Strycharz-Szemberg & T. Szemberg, Remarks on the Nagata Conjecture, *Serdica Math. J.* **30**: 405 – 430 (2004).

[33] D. Testa & A. Varilly-Alvarado & M. Velasco, Big rational surfaces, *Mathematische Annalen* **351**: 95 – 107 (2011).

[34] J. A. Wiśniewski, Toric Mori Theory and Fano Manifolds, *Seminaires & Congres* **6**: 249 – 272 (2002).