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# Secant varieties, Waring rank and generalizations from algebraic geometry viewpoint

*PhD dissertation*

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Author's declaration:
I hereby declare that this dissertation is my own work.

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# Abstract

This thesis is concerned with secant varieties and their equations. We put everything in the framework of toric varieties, since they can be described by combinatorics. We discuss different choices for the ideal of a subscheme of a toric variety with no torus factors. Three actors appearing throughout the thesis are homogenization, dehomogenization, and derivation. We analyze their interplay.

We describe classical methods of obtaining equations of secant varieties, and explain how they are subject to "barriers", which are connected to irreducibility of Hilbert schemes of points. We elaborate on the applications of Hilbert schemes to the subject of secant varieties.

As an example of how this theory works in practice, we give upper and lower bounds for the rank of monomials on the product of two projective lines, and provide some calculations of the ranks of monomials on other toric surfaces. Some interesting pathologies occur here which do not happen on the projective space.

The final part of this thesis is about breaking the barriers described before in the simplest cases possible. We do this for the fourteenth secant variety of the Segre-Veronese embedding, and for the eight Grassmann secant variety of three-dimensional spaces of the Veronese embedding.

**Keywords:** Waring rank, border rank, cactus rank, Hilbert scheme, secant variety, cactus variety, apolarity, dehomogenization, homogenization, equations of secant varieties.

**AMS MSC 2020 classification:** 14C05, 14M12, 14M25, 14N07.

# Streszczenie

Tematem tej pracy są rozmaitości siecznych i ich równania. Całą teorię rozwijamy dla rozmaitości torycznych, ponieważ mają opis kombinatoryczny. Omawiamy różne alternatywy dla ideału podschematu rozmaitości torycznej bez czynników torusa. Trzy pojęcia, które pojawiają się w całej pracy to ujednorodnienie, odjednorodnienie i różniczkowanie. Analizujemy ich wzajemne relacje.

Opisujemy klasyczne metody uzyskiwania równań rozmaitości siecznych i wyjaśniamy fakt, że te metody są poddane pewnym ograniczeniom, związanym z nieprzywiedlnością schematu Hilberta punktów. Omawiamy zastosowania schematów Hilberta do rozmaitości siecznych.

Jako przykład, do czego można stosować tę teorię, podajemy górne i dolne ograniczenia na rangę jednomianów na iloczynie dwóch prostych rzutowych i przeprowadzamy obliczenia rang jednomianów na innych torycznych powierzchniach. Pojawiają się tu pewne ciekawe patologie, których nie obserwujemy w przypadku przestrzeni rzutowej.

Ostatnia część pracy opisuje, jak przekroczyć ograniczenia opisane wcześniej w najprostszych możliwych przypadkach. Robimy to dla czternastej rozmaitości siecznych do włożenia Segre-Veronese oraz dla ósmej grassmannianowej rozmaitości siecznych przestrzeni trójwymiarowych do włożenia Veronese.

**Słowa kluczowe:** ranga Waringa, ranga brzegowa, ranga kaktusowa, schemat Hilberta, rozmaitość siecznych, rozmaitość kaktusowa, abiegunowość, odjednorodnienie, ujednorodnienie, równania rozmaitości siecznych.

**klasyfikacja AMS MSC 2020:** 14C05, 14M12, 14M25, 14N07.

# Contents

# Chapter 1

# Introduction

It is difficult to trace the origins of secant varieties and ranks in mathematics, but the connected idea of decomposing a complicated object into simple ones has been present therein since ancient times (cf. Pythagorean triples). More recent research includes the observations of Lagrange and Waring. The former considered reducing quadratic forms to a diagonal form, which is the rank decomposition for quadrics. The latter believed that "for all integers $d \geq 2$ there exists a positive integer $g(d)$ such that each $n \in \mathbb{Z}_+$ can be written as $n = a_1^d + \cdots + a_{g(d)}^d$ with $a_i \geq 0$", see [78]. This was proven by Hilbert in 1909.

Moreover, this way of thinking is omnipresent in the natural sciences. Consider for instance the Blind Source Separation problem, which is applied to sonars, radars ([33]), electrocardiography ([40]), speech ([65]).

Let us examine what the situation looks like in the world of multilinear algebra and algebraic geometry. We consider two fundamental problems.

**Problem 1.1.** Let $F$ be a complex homogenous polynomial in $n+1$ variables $x_0, \ldots, x_n$. The Waring rank of $F$ is the smallest number $r$ such that $F$ can be written as a sum of $r$ $d$-th powers of linear forms. We can ask the following questions:

(a) Given $F$, what is the Waring rank of $F$?

(b) What is the Waring rank of a generic polynomial of degree $d$ in $n+1$ variables?

(c) Given a positive integer $r$, what are the equations of the closure of the set of polynomials of rank at most $r$?

**Problem 1.2.** Let $T \in \mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_k+1}$ be a tensor. The tensor rank of $T$ is the smallest number $r$ such that $T$ can be written as a sum of $r$ simple tensors (i.e. tensors of the form $v_1 \otimes \cdots \otimes v_k$ for some $v_1 \in \mathbb{C}^{n_1+1}, \ldots, v_k \in \mathbb{C}^{n_k+1}$). We can ask the following questions:

(a) Given $T$, what is the tensor rank of $T$?

(b) What is the tensor rank of a generic tensor in $\mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_k+1}$?

(c) Given a positive integer $r$, what are the equations of the closure of the set of tensors of rank at most $r$?

*Remark* 1.3. What about fields of characteristic greater than 0? It turns out that the theory can be made to work in such a case. However, in this thesis we always assume that we work over $\mathbb{C}$, since we cite some results from [39], where the authors work over $\mathbb{C}$.

*Remark* 1.4. Problems 1.1(a) and 1.2(a) can be also stated for the real field and this is the case more relevant for the applications. See Remarks 1.8 and 1.10. However, the situation is a little different for Problems 1.1(b,c) and 1.2(b,c). The Euclidean closure of the set of polynomials of rank at most $r$, or tensors of rank at most $r$ is naturally a semialgebraic set, not an algebraic set. Instead of the notion of a general rank, there are (possibly many) "typical ranks", see [11]. When one uses the real field, one needs the notions coming from the world of semialgebraic geometry, which falls outside the scope of the methods used in this thesis.

Problems 1.1 and 1.2 look similarly, and one can wonder whether they can be put into the same setting. This is indeed the case. But before we do this, we have to make a convention about notations.

*Remark* 1.5. By an algebraic set over a field $\Bbbk$ we mean a reduced separated $\Bbbk$-scheme of finite type. By a variety over $\Bbbk$ we mean an irreducible algebraic set over $\Bbbk$. Intuitively, one may think of varieties as something glued from affine varieties, and affine varieties are just irreducible subsets of $\Bbbk^n$ defined by polynomial equations. In order to remain consistent with existing literature, the notions "cactus variety" and "Grassmann cactus variety" are widely accepted exceptions to the above definition of "variety", as typically they are reducible algebraic sets. We introduce cactus varieties in Section 1.3.

Let us consider the following definition.

**Definition 1.6.** Let $W$ be a finite-dimensional vector space over $\mathbb{C}$. Let $X \subseteq \mathbb{P}W$ be a projective variety not contained in a hyperplane. For a non-zero $F \in W$, define the $X$-rank of $F$ as

$$\mathrm{r}_X(F) = \min\{r \in \mathbb{Z}_{\geq 0} \,| \text{there exist } x_1, \ldots, x_r \in X \\ \text{such that } [F] \in \langle x_1, \ldots, x_r \rangle\}.$$

Here $[\cdot]$ denotes the class in the projective space, and $\langle \cdot \rangle$ denotes the projective linear span. When $X$ is fixed and there is no danger of confusion, we write $\mathrm{r}(F)$ instead of $\mathrm{r}_X(F)$ and "rank" instead of "$X$-rank".

Let $V$ be a finite-dimensional vector space over $\mathbb{C}$. If we consider the Veronese embedding $v_d : \mathbb{P}V \to \mathbb{P}\,\mathrm{Sym}^d V$, given by $[l] \mapsto [l^d]$, then the $X$-rank becomes the Waring rank. This is also called the symmetric rank.

Let $V_1, \ldots, V_k$ be finite-dimensional vector spaces over $\mathbb{C}$. If we consider the Segre embedding $\mathrm{Seg} : \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k \to \mathbb{P}(V_1 \otimes \cdots \otimes V_k)$, given by $([v_1], \ldots, [v_k]) \mapsto [v_1 \otimes \cdots \otimes v_k]$, then the $X$-rank becomes the tensor rank.

In the remaining part of this section, we give an overview of the partial answers to the Problems 1.1 and 1.2.

**Problem 1.1(a)** This is a very hard problem, but at least the Waring rank is know for monomials [26], forms of degree 2 (this is equivalent to diagonalization of a quadratic form), and forms in 2 variables (Sylvester's theorem [73], which is described using modern notation in [57, Theorem 1.44]).

**Problem 1.1(b)** This is solved by the Alexander-Hirschowitz Theorem [3], see another proof in [14].

**Problems 1.1(c) and 1.2(c)** This is the topic of Sections 3.4–3.5 and Chapter 6. We review it in Section 1.2 in more detail.

**Problem 1.2(a)** This also is a very hard problem. For tensors of order 2, the rank of a tensor is just the rank of a matrix, which can be easily computed using Gaussian elimination or SVD. In [51], [58],[74], the authors compute the ranks of tensors in $\mathbb{C}^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^l$, using the Kronecker normal form.

**Problem 1.2(b)** The problem has not been solved completely yet. There are some partial results, like [32], where the authors consider the case $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$, or [2], where the authors use induction and describe some cases. In the latter paper, Abo, Ottaviani, and Peterson conjecture that a general tensor in $\mathbb{C}^{n_1+1} \otimes \mathbb{C}^{n_2+1} \otimes \cdots \otimes \mathbb{C}^{n_k+1}$, with $n_1 \leq n_2 \leq \ldots \leq n_k$, has rank equal to

$$\left\lceil \frac{(n_1+1)\cdots(n_k+1)}{n_1+\ldots+n_k+1} \right\rceil,$$

except possibly for the following cases:

- $n_k - 1 \geq \prod_{i=1}^{k-1}(n_i+1) - \sum_{i=1}^{k-1} n_i$,
- $k=3$, and $(n_1,n_2,n_3) = (2,n,n)$ for some even $n$,
- $k=3$, and $(n_1,n_2,n_3) = (2,3,3)$,
- $k=4$, and $(n_1,n_2,n_3,n_4) = (1,1,1,1)$.

This conjecture is still open.

## 1.1 Calculation of rank for other varieties

It is an important problem to calculate the rank of specific tensors or polynomials. A crucial result in this area is the Carlini, Catalisano, Geramita Theorem on the rank of monomials. Let $V$ be an $(n+1)$-dimensional vector space with basis $x_0,\ldots,x_n$. Let $a_0 \leq \cdots \leq a_n$ be positive integers with sum $d$. We calculate the rank with respect to the Veronese embedding $v_d : \mathbb{P}V \to \mathbb{P}\operatorname{Sym}^d V$ (as in Problem 1.1).

**Theorem 1.7** ([26]). *The following equality holds*

$$\mathrm{r}_{v_d(\mathbb{P}V)}(x_0^{a_0}x_1^{a_1}\cdots x_n^{a_n}) = (a_1+1)(a_2+1)\cdots(a_n+1).$$

*Remark* 1.8. It should be noted that the rank of monomials over the reals is usually different than the rank over the complex numbers, see [27].

However, the problem of calculating the rank of monomials for the Segre-Veronese embedding remains open (the rank in this case is called the partially symmetric rank). If $V_1,\ldots,V_k$ are finite-dimensional complex vector spaces, and $d_1,\ldots,d_k$ are positive integers, the Segre-Veronese embedding is defined by

$$v_{d_1,\ldots,d_k} : \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k \to \mathbb{P}(\operatorname{Sym}^{d_1} V_1 \otimes \cdots \otimes \operatorname{Sym}^{d_k} V_k),$$
$$([v_1],\ldots,[v_k]) \mapsto [v_1^{d_1} \otimes \cdots \otimes v_k^{d_k}].$$

The simplest example is $\mathbb{P}^1 \times \mathbb{P}^1$. For positive integers $d, e$, we obtain the map

$$\begin{aligned} v_{d,e} : \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}(\operatorname{Sym}^d \mathbb{C}^2 \otimes \operatorname{Sym}^e \mathbb{C}^2), \\ ([v_1], [v_2]) &\mapsto [v_1^d \otimes v_2^e]. \end{aligned}$$

Elements of $\operatorname{Sym}^d \mathbb{C}^2 \otimes \operatorname{Sym}^e \mathbb{C}^2$ are bihomogeneous polynomials of bidegree $(d, e)$ in variables $x, y, z, w$, where $x, y$ are of degree $(1, 0)$, while $z, w$ are of degree $(0, 1)$. The $v_{d,e}(\mathbb{P}^1 \times \mathbb{P}^1)$-rank of such a polynomial $F$ is

$$\begin{aligned} \mathrm{r}_{v_{d,e}(\mathbb{P}^1\times\mathbb{P}^1)}(F) = \min\{r \in \mathbb{Z}_{>0} \mid & \exists l_1, \dots, l_r \text{ linear forms in } x, y \\ & \text{and } m_1, \dots, m_r \text{ linear forms in } z, w \\ & \text{such that } F = l_1^d m_1^e + \dots + l_r^d m_r^e\}. \end{aligned}$$

We investigate the case of monomials more closely, let $F = x^k y^l z^m w^n$, where $k \geq l, m \geq n$. Since $\mathrm{r}_{v_{k+l}(\mathbb{P}^1)}(x^k y^l) = k + 1$, and $\mathrm{r}_{v_{m+n}(\mathbb{P}^1)}(z^m w^n) = m + 1$, we get

$$\mathrm{r}_{v_{d,e}(\mathbb{P}^1\times\mathbb{P}^1)}(x^k y^l z^m w^n) \leq (k + 1)(m + 1). \tag{1.1}$$

But the equality in the above equation does not always hold, as pointed out in [34, Remark 16]. We have $\mathrm{r}_{v_{3,3}(\mathbb{P}^1\times\mathbb{P}^1)}(x^2 y u^2 v) \leq 8 < 9$.

In Section 5.1 we prove the following theorem:

**Theorem 1.9.** *The following inequalities hold:*

*(i)* $\mathrm{r}_{v_{k+l,m+n}(\mathbb{P}^1\times\mathbb{P}^1)}(x^k y^l z^m w^n) \leq (k+1)(n+1)+(l+1)(m+1)-(l+1)(n+1)$,

*(ii)* $\mathrm{r}_{v_{k+l,m+n}(\mathbb{P}^1\times\mathbb{P}^1)}(x^k y^l z^m w^n) \geq (k + 1)(n + 1)$ *for* $k > l$,
$\mathrm{r}_{v_{k+l,m+n}(\mathbb{P}^1\times\mathbb{P}^1)}(x^k y^l u^m v^n) \geq (l + 1)(m + 1)$ *for* $m > n$,

*(iii)* $\mathrm{r}_{v_{k+l,m+n}(\mathbb{P}^1\times\mathbb{P}^1)}(x^k y^l z^m w^n) \geq (l + 2)(n + 2) - 1$ *for* $k > l$, *and* $m > n$,

Let us look the cases where the rank is determined by these inequalities. When we set $m = n$ in the first inequality of Point (ii), from Equation (1.1) we get $\mathrm{r}_{v_{k+l,m+n}(\mathbb{P}^1\times\mathbb{P}^1)}(x^k y^l z^m w^n) = (k+1)(m+1)$. Also when we set $k = l+1$ and $m = n + 1$, we get (by Points (i) and (iii)) that $\mathrm{r}_{v_{k+l,m+n}(\mathbb{P}^1\times\mathbb{P}^1)}(x^k y^l z^m w^n) = (l + 2)(n + 2) - 1$.

*Remark* 1.10. The author expects to be able to prove that the bihomogeneous rank of $x^k y z^m w$ over the reals is $2(k + m)$. The case of other bihomogeneous monomials in $\mathbb{R}[x, y, z, w]$ seems to be more difficult.

It is interesting to see what happens when we take $X$ to be some other surface. We provide rank calculations for a few classical surfaces in algebraic geometry. These are the Hirzebruch surface $\mathbb{F}_1$ (a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$), weighted projective space $\mathbb{P}(1,1,4)$ (same as the projective space $\mathbb{P}^2$, but the variables have degrees $1,1,4$), and an example of a fake weighted projective plane. We analyze them in Sections 5.2, 5.3, and 5.4, respectively. Some interesting pathologies occur there which do not show up in the case of the projective plane. We discuss them in Remarks 5.5 and 5.6.

## 1.2 Determinantal methods

We consider the problem of finding equations of closures of sets of points of $X$-rank at most $r$. Hence, let us make the following definition:

**Definition 1.11.** Let $W$ be a complex finite dimensional vector space, and let $X \subseteq \mathbb{P}W$ be an variety not contained in a hyperplane. For a positive integer $r$, let

$$\begin{aligned}\sigma_r(X) &= \overline{\bigcup_{x_1,\ldots,x_r \in X} \langle x_1, \ldots, x_r \rangle} \\ &= \overline{\{[F] \in \mathbb{P}W \mid \mathrm{r}(F) \leq r\}}.\end{aligned}$$

be the $r$-th secant variety of $X$.

For years, one of the main tools of getting equations of secant varieties have been what is called "determinantal methods" in algebraic geometry and "rank methods" in computational complexity. They can be summarized in the following proposition, which is given for instance in [62, beginning of Chapter 7]:

**Proposition 1.12.** *Suppose we have a linear map $j : W \to A \otimes B$, where $A, B$ are finite-dimensional vector spaces. Let $k$ be a positive integer such that for every $p \in \hat{X}$ (the affine cone of $X$) we have*

$$\mathrm{rank}(j(p)) \leq k.$$

*Then the following are true:*

*(i) for any $F \in W$ we have*

$$\mathrm{r}(F) \geq \frac{\mathrm{rank}\, j(F)}{k}.$$

*(ii) for any positive integer $r$ we have*

$$j(\sigma_r(X)) \subseteq \sigma_{kr}(\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B)).$$

*Here* $\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B) \subseteq \mathbb{P}(A \otimes B)$ *is the image of the Segre embedding. Hence the $(kr+1)$-th minors of $j \in A \otimes B \otimes W^*$, interpreted as a matrix with entries in $W^*$, give equations of $\sigma_r(X)$.*

See [45, Proposition 1] for a proof.

The problem is to find linear maps $j : W \to A \otimes B$ which deliver useful equations. In [64] two methods are presented. First we follow the construction in [64, Point 1.3 and Section 5].

**Construction 1.13.** Suppose the embedding $X \subseteq \mathbb{P}W$ is given by the complete linear system $H^0(X, \mathcal{L})$ of a very ample line bundle $\mathcal{L}$. This provides an isomorphism $W \cong H^0(X, \mathcal{L})^*$. Let $\mathcal{E}$ be a coherent sheaf on $X$. Consider the canonical morphism $\mathcal{E} \otimes \mathcal{L} \otimes \mathcal{E}^\vee \to \mathcal{L}$, where $\mathcal{E}^\vee$ denotes the dual sheaf. This yields

$$H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{L} \otimes \mathcal{E}^\vee) \to H^0(X, \mathcal{L}),$$

which, after rearranging the factors, gives the map

$$j = j^{\mathcal{E}} \colon H^0(X, \mathcal{L})^* \to H^0(X, \mathcal{E})^* \otimes H^0(X, \mathcal{L} \otimes \mathcal{E}^\vee)^*. \tag{1.2}$$

In the case when $\mathcal{E}$ is a locally free sheaf of rank $k$, we have $\operatorname{rank} j^{\mathcal{E}}(\hat{x}) \leq k$ for every $[\hat{x}] \in X$, see [75, Proposition 2.20] or [64, Proposition 5.1.1]. Hence Proposition 1.12 provides a lower bound for the rank in this case.

**Construction 1.14.** Let $G$ be a complex semisimple group, let $V$ be an irreducible $G$-module, and $X \subseteq \mathbb{P}V$ the unique closed orbit (so that $X$ is a $G$-homogeneous variety). Suppose we have an inclusion of $G$-modules $V \xrightarrow{j} W_1 \otimes W_2$, where $W_1$ and $W_2$ are two irreducible $G$-modules. The number $j(F)$ is constant for any $[F] \in X$ since $X$ is $G$-homogeneous. Hence Proposition 1.12 gives equations for the secant varieties of $X$.

For some examples of the power of determinantal methods for small cases, see Section 3.5.

## 1.3 Limits to determinantal methods

As J.M. Landsberg writes in [63, End of Section 2], he and Giorgio Ottaviani, while looking for equations of secant varieties, first thought they were not clever enough, but then began to realize that there were some limits to determinantal methods. Indeed, the limits (or barriers) were proven to exists independently in the community of algebraic complexity theory ([41], [49]) and in the community of algebraic geometry ([19], [45]). These show that the best lower bound that can be obtained from a determinantal method for a tensor in $\mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$ is $6n + 2$ ([46, Example 6.2]), [49, Corollary 8.5]), and the best lower bound for a polynomial in $\operatorname{Sym}^d \mathbb{C}^{n+1}$ is $2\binom{n+k}{k}$ when $d = 2k + 1$, and $\binom{n+k}{k} + \binom{n+k+1}{k+1}$ for $d = 2k + 2$ (see [9, Theorem 3], [49, Corollary 8.5]).

In order to understand the barriers from the point of view of algebraic geometry, define the notion of cactus $X$-rank. Recall that $X \subseteq \mathbb{P}W$ is a projective variety not contained in a hyperplane, and $F \in W$ is a non-zero vector. The cactus $X$-rank $\operatorname{cr}_X(F)$ of $F$ is

$$\operatorname{cr}_X(F) = \min\{\operatorname{length} R \,|\, [F] \in \langle R \rangle,\ R \hookrightarrow X \text{ zero-dimensional subscheme}\}.$$

We denote by $\operatorname{length} R$ the length of a zero-dimensional scheme $R$, i.e. the dimension of $H^0(R, \mathcal{O}_R)$ as a vector space. Here $\langle \cdot \rangle$ denotes the (projective) linear span of a scheme. When $X$ is fixed and there is no danger of confusion, we write $\operatorname{cr}(F)$ instead of $\operatorname{cr}_X(F)$ and "cactus rank" instead of "cactus $X$-rank".

The cactus rank was first introduced in [57, Definition 5.1] under the name of "scheme length".

**Definition 1.15.** The $r$-th cactus variety is

$$\begin{aligned} \kappa_r(X) &= \overline{\{[F] \in \mathbb{P}W \,|\, \operatorname{cr}(F) \leq r\}} \\ &= \overline{\bigcup_{R \hookrightarrow X,\ \operatorname{length} R \leq r}} \langle R \rangle. \end{aligned}$$

For reasons to study the cactus variety and the cactus rank, see [46, Subsection 1.3], or [19, Subsections 1.2 and 1.3]. In [19], using the idea of cactus varieties, equations for many secant varieties were found. Another motivation is Corollary 3.19. It is an interesting question when Proposition 1.12 works for the cactus rank and cactus variety, in other words, when the determinantal methods give equations of the cactus variety. Before giving some answers, we note that the two conditions (giving equations of the cactus varieties and lower bounds for the cactus rank) are equivalent.

**Proposition 1.16.** *Suppose $X \subseteq \mathbb{P}W$ is a non-degenerate projective variety, and $j : W \to A \otimes B$ is a linear map. Fix a positive integer $k$. Then the following conditions are equivalent:*

*(1) for any $F \in W$ we have*

$$\operatorname{cr}(F) \geq \frac{\operatorname{rank} j(F)}{k},$$

*(2) for any positive integer $r$ we have*

$$j(\kappa_r(X)) \subseteq \sigma_{kr}(\operatorname{Seg}(\mathbb{P}A \times \mathbb{P}B)).$$

See [45, Proposition 3] for a proof.

The question is: for which $k \geq 1$ the conditions in Proposition 1.16 are true. The most interesting case is when $k$ satisfies

$$\operatorname{rank}(j(p)) \leq k$$

for every $p \in X$, compare Proposition 1.12 and Theorem 1.17. Example 3.20 and Remark 5.6 show that the conditions in Proposition 1.16 are not true for such $k$ in general. However, there are two crucial cases when the conditions are true. When we choose $j^{\mathcal{E}}$ as in Construction 1.13 (where $\mathcal{E}$ is a locally free sheaf on $X$), then the conditions in Proposition 1.16 are true for $k$ equal to the rank of $\mathcal{E}$.

**Theorem 1.17.** *Let $X$ be a projective variety embedded by the complete linear system of a very ample line bundle $\mathcal{L}$. Then for any $F \in H^0(X, \mathcal{L})^*$, and any locally free sheaf $\mathcal{E}$ on $X$ of rank $k$*

$$\operatorname{cr}(F) \geq \frac{\operatorname{rank} j^{\mathcal{E}}(F)}{k}.$$

*In other words, if we fix bases of $H^0(X, \mathcal{E})$ and $H^0(X, \mathcal{L} \otimes \mathcal{E}^{\vee})^*$, then for any $r \geq 0$ the $(kr+1)$-th minors of $j$ (which is a matrix with entries in $H^0(X, \mathcal{L})$) give equations for $\kappa_r(X)$.*

This theorem was proven by the author in [45, Theorem 5]

The next case when the conditions in Proposition 1.16 are true is when $\operatorname{rank} j(\hat{p})$ is the same for any $\hat{p} \in \hat{X} \setminus \{0\}$.

**Theorem 1.18.** *Let $X \subseteq \mathbb{P}W$ be a non-degenerate projective variety, and $j : W \to V_1 \otimes V_2$ be a linear map. Suppose the matrix $j(\hat{p})$ has constant rank equal to $k$ for $\hat{p} \in \hat{X} \setminus 0$. Then for any $F \in W$*

$$\operatorname{cr}(F) \geq \frac{\operatorname{rank} j(F)}{k}.$$

*In other words, if we fix bases of $V_1$ and $V_2$, then for any $r \geq 0$ the $(kr+1)$-th minors of $j$ (which is a matrix with entries in $W^*$) give equations for $\kappa_r(X)$.*

This theorem was proven by Jarosław Buczyński (unpublished). We prove Theorems 1.17 and 1.18 in Section 3.4.

Now that we know that we can describe the barriers to determinantal methods by cactus varieties and cactus rank, we calculate the explicit bounds on the cactus rank that we mentioned implicitly at the beginning of this section. These are best lower bounds on the rank of tensors or polynomials that can be obtained by those methods.

We use the language of the Segre-Veronese embedding, which covers both the case of polynomials and the case of tensors.

Let $h_1, \ldots, h_k$ be the basis dual to the standard basis of $\mathbb{R}^k$. Let $l = c_1h_1 + \cdots + c_kh_k$ be a non-zero linear form on $\mathbb{R}^k$, where $c_1, \ldots, c_k$ are nonnegative real numbers. Let $b$ be a positive real number. Consider the morphism

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \xrightarrow{v_{\mathbf{d}}} \mathbb{P}(\operatorname{Sym}^{d_1} \mathbb{C}^{n_1+1} \otimes \cdots \otimes \operatorname{Sym}^{d_k} \mathbb{C}^{n_k+1})$$

given on points by

$$([l_1], \ldots, [l_k]) \mapsto [l_1^{d_1} \otimes \cdots \otimes l_k^{d_k}].$$

**Proposition 1.19.** *Let $F \in \operatorname{Sym}^{d_1} \mathbb{C}^{n_1+1} \otimes \cdots \otimes \operatorname{Sym}^{d_k} \mathbb{C}^{n_k+1}$ be a non-zero form. Then*

$$\operatorname{cr}(F) \leq \sum_{\substack{(e_1,\ldots,e_k)| \\ l(e_1,\ldots,e_k) \leq b \\ 0 \leq e_i \leq d_i}} \binom{n_1 - 1 + e_1}{e_1} \cdots \binom{n_k - 1 + e_k}{e_k} + \sum_{\substack{(e_1,\ldots,e_k)| \\ l(e_1,\ldots,e_k) > b \\ 0 \leq e_i \leq d_i}} \binom{n_1 - 1 + d_1 - e_1}{d_1 - e_1} \cdots \binom{n_k - 1 + d_k - e_k}{d_k - e_k}.$$

In [7] the authors prove a weaker version of the bound in Proposition 1.19. We prove Proposition 1.19 in Section 3.7.

## 1.4 Ways to overcome the limits

One of the ways to overcome the limits to determinantal methods is to understand the structure of the Hilbert scheme, which is hard. The Hilbert scheme $\mathcal{H}ilb(\mathbb{P}^n)$ was defined by Grothendieck. It parametrizes all closed subschemes of $\mathbb{P}^n$. We know that

$$\mathcal{H}ilb(\mathbb{P}^n) = \bigsqcup_{q \in \mathbb{Q}[t]} \mathcal{H}ilb_q(\mathbb{P}^n). \tag{1.3}$$

The variable $q$ ranges over all polynomials that are Hilbert polynomials of some closed subscheme of $\mathbb{P}^n$. The scheme $\mathcal{H}ilb_q(\mathbb{P}^n)$ parametrizes all closed subschemes of $\mathbb{P}^n$ with Hilbert polynomial $q$. Equation (1.3) is a decomposition into connected componenents ([54]). In this thesis, we only consider the case when the polynomial $q$ is constant. We write $q = p \in \mathbb{N}$. The scheme $\mathcal{H}ilb_p(\mathbb{P}^n)$ is called the Hilbert scheme of $p$ points on $\mathbb{P}^n$, in spite of the fact that it usually contains also classes of nonreduced schemes. If $W \hookrightarrow \mathbb{P}^n$ is a locally closed subscheme, the set of all closed subschemes $Z \hookrightarrow \mathbb{P}^n$ which are contained in $W$ is parametrized by a locally closed subscheme

$$\mathcal{H}ilb_p(W) \hookrightarrow \mathcal{H}ilb_p(\mathbb{P}^n).$$

The reason why we need to study Hilbert schemes is the fact that in the definition of the $X$-cactus rank we go over all subschemes of $X$, both smoothable ones (i.e. contained in the same irreducible component of $\mathcal{H}ilb_p(X)$ as reduced schemes), and non-smoothable ones (contained in some other irreducible components of $\mathcal{H}ilb_p(X)$).

There are two ways to apply Hilbert schemes to investigating secant or cactus varieties. One is by considering the multiplication tensors of smoothable or nonsmoothable algebras. This approach is taken in [10], in [60, Section 9], and in the survey article [63, Section 7]. The other one is by explicitly describing the components of the cactus variety in simple cases, when this is still possible. In this thesis, we focus on the latter technique. We sketch it in the remainder of this introduction. We refer the reader to Chapter 6 for more details.

In [19] Weronika and Jarosław Buczyńscy made an important breakthrough. They realized that the cactus variety depends only on Gorenstein schemes.

**Definition 1.20.** A local finite dimensional algebra $(A, \mathfrak{m})$ is Gorenstein if the socle

$$\{a \in A | a\mathfrak{m} = 0\}$$

is a one-dimensional vector space. Any finite dimensional algebra $A$ is Gorenstein if it is a product of local finite dimensional Gorenstein algebras. A zerodimensional scheme $R$ is Gorenstein if it is the spectrum of a finite dimensional Gorenstein algebra. Let $\mathcal{H}ilb_r^{Gor}(X)$ denote the open subset of the Hilbert scheme of $r$ points on $X$ consisting of Gorenstein subschemes.

Let $X \subseteq \mathbb{P}W$ be a projective variety not contained in a hyperplane, and let $F \in W$ be a nonzero vector. By [19, Proposition 2.2] the cactus rank is given by

$$\operatorname{cr}(F) = \min\{r \in \mathbb{Z}_{>0} \mid \text{ there exists } [R] \in \mathcal{H}ilb_r^{Gor}(X) \text{ such that } [F] \in \langle R \rangle\}.$$

From this, we get several consequences. If the Gorenstein locus $\mathcal{H}ilb_r^{Gor}(X)$ is irreducible, then $\kappa_r(X) = \sigma_r(X)$. In general, the number of irreducible components of the cactus variety $\kappa_r(X)$ is bounded from above by the number of irreducible components of $\mathcal{H}ilb_r^{Gor}(X)$. See Section 4.2 for an explanation. Therefore it becomes crucial to distinguish between different components of $\mathcal{H}ilb_r^{Gor}(X)$. This is too hard to do in general, but the first non-trivial case was investigated by Casnati, Jelisiejew, Notari in [30]. Their article describes $\mathcal{H}ilb_r^{Gor}(\mathbb{A}^n)$ for $r \leq 14$ and all $n \in \mathbb{N}$. We elaborate on it in Section 4.1.

Using the description given by Casnati, Jelisiejew and Notari, we show the following characterization.

**Theorem 1.21.** *For $d \geq 5$, the cactus variety $\kappa_{14}(v_d(\mathbb{P}^6))$ has two irreducible components: the secant variety $\sigma_{14}(v_d(\mathbb{P}^6))$, and the variety $\eta_{14}(v_d(\mathbb{P}^6))$ consisting of degree $d$ forms divisible by the $(d-3)$-rd power of a linear form.*

This theorem was proven by the author, Tomasz Mańdziuk and Filip Rupniewski ([47, Corollary 1.4]). In Theorem 6.1, we generalize this result for $\mathbb{P}^n$ instead of $\mathbb{P}^6$. Moreover, we provide an algorithm which, given a polynomial in the cactus variety, checks if it belongs to the corresponding secant variety. See Theorem 6.6. Both Theorem 6.1 and Theorem 6.6 come from [47].

We do a similar calculation for the Segre-Veronese embedding, when the embedding vector $(d_1, \ldots, d_k)$ satisfies $d_i \geq 7$ for all $1 \leq i \leq k$ (Section 6.3). Somehow, the cases with small $d_i$ seem to behave differently. See [60, Section 9] for an analysis of the case $(d_1, \ldots, d_k) = (1, 1, 1)$.

This gives rise to the following question: does investigating the irreducible components of the standard Hilbert scheme $\mathcal{H}ilb_p(\mathbb{A}^n)$ help us to describe the secant varieties at all? It turns out that it can be useful to understand the Grassmann secant varieties.

The notion of Grassmann secant variety and the connected problem of decomposing many forms simultaneously as sums of powers of the same set of linear forms originates from the work of Terracini [76] and was later studied by Bronowski [15]. The paper [25] investigates the relation between the ranks of tensors in the Segre embedding of 3 copies of projective space and the ranks of subspaces in the Segre embedding of 2 copies of projective space. The problem of defectivity of Grassmann secant varieties is addressed in [6], [35], [43]. Simultaneous decomposition of forms of different degrees is studied among others in [4] and [28].

We define the Grassmann secant and cactus varieties in Section 3.2, and analyze them in Section 6.4. Using the work of Cartwright, Erman, Velasco, Viray [29], who classified the irreducible components of $\mathcal{H}ilb_8(\mathbb{A}^n)$, we give a description of the Grassmann cactus variety $\kappa_{8,3}(v_d(\mathbb{P}^n))$. We provide an algorithm similar to the one for the standard cactus variety, see Theorem 6.7. See Theorem 4.3 for the characterization of $\mathcal{H}ilb_8(\mathbb{A}^n)$ given by Cartwright, Erman, Velasco, Viray, and Remark 6.19 for the reason why we consider the specific Grassmann cactus variety $\kappa_{8,3}(v_d(\mathbb{P}^n))$.

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# Chapter 2

# Toric varieties

A lot of the theory developed in this thesis works in the setting of toric varieties. See [39] for a reference on this subject. One of the main actors here is the Cox ring $T$ of a toric variety $X$. Section 2.1 is devoted to discussing different choises for the ideal $I \subseteq T$ of a subscheme $R \hookrightarrow X$. After a few general properties of toric varieties at which we look in Section 2.2, we start analyzing properties of dehomogenization and homogenization, and we continue it until the end of the chapter. An interlude in Section 2.5 allows us to introduce graded dual rings, and the apolarity action, crucial to understanding secant varieties in Chapter 3.

In this chapter, we work over $\mathbb{C}$.

## 2.1 Saturated ideals

Let $X$ be a normal irreducible variety. For a non-zero rational function $f$ on $X$, and an irreducible subset $Y$ of $X$ of codimension one, we define $\mathrm{mult}_Y(f)$ to be the order of vanishing of $f$ on $Y$, that is the valuation of $f$ in the discrete valuation ring $\mathcal{O}_{X,Y}$. We let

$$\mathrm{div}(f) = \sum_{Y \subseteq X} \mathrm{mult}_Y(f) Y,$$

where the sum goes over all irreducible subsets $Y$ of $X$ of codimension one. This sum is always finite. A divisor which is of the form $\mathrm{div}(f)$ for some non-zero rational function $f$ is called a principal divisor. The class group of $X$ is the abelian group of all divisors modulo the subgroup of principal divisors. We denote it by $\mathrm{Cl}(X)$.

Let $\mathcal{K}^*$ be the sheaf of non-zero rational functions on $X$. For each divisor $D$ on $X$, we have its sheaf of sections $\mathcal{O}_X(D)$. It is defined by

$$\mathcal{O}_X(D)(U) = \{f \in \mathcal{K}^*(U) \mid (\mathrm{div}(f) + D)_{|U} \geq 0\} \cup \{0\}.$$

for each Zariski open set $U \subseteq X$. The map $D \mapsto \mathcal{O}_X(D)$ gives an isomorphism of the class group with the abelian group of reflexive sheaves on $X$ of rank 1 up to isomorphism, where the addition of the sheaves $\mathcal{L}, \mathcal{M}$ is just taking the double dual $(\mathcal{L} \otimes \mathcal{M})^{\vee\vee}$.

An important subgroup of the group of all Weil divisors on $X$ is the group of all locally principal divisors (also known as the group of Cartier divisors). If we form a quotient of it by the group of principal divisors, we get the Picard group, denoted by $\mathrm{Pic}(X) \subseteq \mathrm{Cl}(X)$. The elements of the Picard group correspond to those reflexive sheaves of rank one which are line bundles.

Let us now proceed to the definition of the Cox ring of $X$. The simplest case is when $\mathrm{Cl}(X)$ is finitely generated and free. Then we pick a basis $\eta_1, \ldots, \eta_k$ of $\mathrm{Cl}(X)$ over $\mathbb{Z}$, for each $1 \leq i \leq k$ we take a divisor $D_i$ on $X$ with class $\eta_i$, and then we define the Cox ring of $X$ to be

$$\bigoplus_{l_1,\ldots,l_k \in \mathbb{Z}} H^0(X, \mathcal{O}_X(l_1 D_1 + \cdots + l_k D_k)).$$

In general, when $\mathrm{Cl}(X)$ is not free, one has to use some tricks to define the Cox ring, described for instance in [5, Section 1.4]. In this thesis, we make use of the fact that for toric varieties the class group is always finitely generated, and that it has a canonical system of generators, namely the classes of the torus invariant prime divisors. We focus now on toric varieties.

Let $N$ be a lattice (an abelian group isomorphic to $\mathbb{Z}^k$ for some $k \geq 1$). Let $X_\Sigma$ be the toric variety of a fan $\Sigma \subseteq N_\mathbb{R} \coloneqq N \otimes \mathbb{R}$, as in [39, Chapter 3]. We assume that $\Sigma$ has no torus factors, which means that the linear span of $\Sigma$ in $N_\mathbb{R}$ is the whole space. Let $\Sigma(1)$ denote the set of rays of the fan $\Sigma$. Similarly, $\sigma(1)$ denotes the set of rays in a cone $\sigma$.

Let $r$ be the number of rays in $\Sigma$. Let us fix an ordering of the rays, suppose they are $\rho_1, \ldots, \rho_r$. Consider the polynomial ring $T = \mathbb{C}[\alpha_1, \ldots, \alpha_r]$. The ring $T$ is the Cox ring of $X_\Sigma$. For more details, see [39, Section 5.2], where $T$ is denoted by $S$, and is called the total coordinate ring. This ring is graded by the class group $\operatorname{Cl} X_\Sigma$, where

$$\deg \alpha_i = [D_{\rho_i}],$$

and $D_{\rho_i}$ is the torus-invariant divisor corresponding to $\rho_i$, see [39, Chapter 4]. For any $\eta \in \operatorname{Cl} X_\Sigma$, we denote by $T_\eta$ the graded piece of $T$ of degree $\eta$.

For a cone $\sigma \in \Sigma$, define

$$\alpha^{\widehat{\sigma}} = \prod_{i=1|\rho_i \notin \sigma}^{r} \alpha_i.$$

Define a homogeneous ideal in $T$, let $B$ be

$$B = B(\Sigma) = (\alpha^{\widehat{\sigma}} | \sigma \in \Sigma) \subseteq T, \tag{2.1}$$

the irrelevant ideal.

Take any ideals $I, J \subseteq T$. Let $(I :_T J)$ be the set of all $x \in T$ such that $x \cdot J \subseteq I$; it is an ideal of $T$. It is sometimes called the quotient ideal, or the colon ideal. For any ideals $I, J, K \subseteq T$ we have:

- $I \subseteq (I :_T J)$,
- if $J \subseteq K$, then $(I :_T J) \supseteq (I :_T K)$,
- $(I :_T J \cdot K) = ((I :_T J) :_T K)$.

We also define the $J$-saturation of $I$ as

$$\operatorname{sat}(I, J) = \bigcup_{i \geq 1} (I :_T J^i).$$

Note that this is an increasing union because $B^i \supseteq B^j$ for $i < j$, so $\operatorname{sat}(I, J)$ is an ideal. Since $T$ is Noetherian, the union stabilizes in a finite number of steps. We always have $I \subseteq \operatorname{sat}(I, J)$. If this is an equality, we say that $I$ is $J$-saturated.

*Remark* 2.1. Common notation for $\operatorname{sat}(I, J)$ is $(I :_T J^\infty)$, but we do not use it.

*Remark* 2.2 (Geometric meaning of quotient ideal and saturation). For any ideal $I \subseteq T$, consider the vanishing scheme $V(I) \subseteq \mathbb{A}^r_{\mathbb{C}}$. Then for any two ideals $I, J \subseteq T$, the scheme $V(\mathrm{sat}(I, J))$ is obtained by removing from $V(I)$ all the irreducible components, whose support is contained in $V(J)$, and all the embedded components of $V(I)$, whose support is contained in $V(J)$.

The application $V(I) \mapsto V((I :_T J))$ removes all the reduced irreducible components of $V(I)$ contained in $V(J)$ and makes some changes on the nonreduced components and embedded components of $V(I)$ contained in $V(J)$.

Recall the ideal $B \subseteq T$ from Equation (2.1) Our main example of saturation will be the $B$-saturation of ideals of $T$.

**Example 2.3.** Let us look at the projective space $\mathbb{P}^k_{\mathbb{C}}$. See [39, Example 5.1.7]. Here $T = \mathbb{C}[\alpha_0, \dots, \alpha_k]$, and $B = (\alpha_0, \dots, \alpha_k) = \bigoplus_{i \geq 1} T_i$. In this case

$$\mathrm{sat}(I, B) = \{\theta \in T | \text{ for all } i = 0, 1, \dots, k \text{ there is } n \text{ such that } \alpha_i^n \cdot \theta \in I\}.$$

In this case there is a 1-1 correspondence between closed subschemes of $\mathbb{P}^k_{\mathbb{C}}$ and homogeneous $B$-saturated ideals of $T$. Moreover, for any subscheme $R \hookrightarrow \mathbb{P}^k_{\mathbb{C}}$ with ideal sheaf $\mathcal{I}_R$, the ideal given by $\bigoplus_{i \geq 0} H^0(\mathbb{P}^k_{\mathbb{C}}, \mathcal{I}_R \otimes \mathcal{O}(i))$, is $B$-saturated. For more on this, see [55, II, Corollary 5.16 and Exercise 5.10].

For a toric variety the situation is more complicated. There can be many $B$-saturated ideals defining a subscheme $R$. See [37, Theorem 3.7 and the following discussion] for more details. We will not develop these ideas to full extent, we will satisfy ourselves with only a few propositions. Our main aim is Corollary 2.10, which will enable us to use the theory for secant varieties in Chapter 3.

**Definition 2.4.** Let $I \subseteq T$ be an ideal, and $R \hookrightarrow X_\Sigma$ be a closed subscheme. We say that $I$ defines $R$ if for each $\sigma \in \Sigma$ the ideal

$$(I_{\alpha^{\hat{\sigma}}})_0 \to (T_{\alpha^{\hat{\sigma}}})_0$$

is equal to the ideal defining $R$ on the affine patch $\mathrm{Spec}(T_{\alpha^{\hat{\sigma}}})_0 \to X_\Sigma$. For any homogeneous ideal $I$, the scheme $V(I)$ is obtained by patching the affine schemes

$$\mathrm{Spec}((T/I)_{\alpha^{\hat{\sigma}}})_0.$$

We say that a homogeneous element $\theta \in T$ vanishes on $R$ (or is zero on $R$) if the closed subscheme $R \hookrightarrow X_\Sigma$ sits in the closed subscheme $V(\theta)$.

For any subscheme $R \hookrightarrow X_\Sigma$, there always exists a $B$-saturated ideal defining $R$.

**Definition 2.5.** Let $R \hookrightarrow X_\Sigma$ be a closed subscheme. We define $I_{\mathrm{Cl}}(R) \subseteq T$, the ideal of sections of $R$, to be the ideal generated by all homogeneous elements of $T$ which vanish on $R$.

In [16, Subsection 2.1.2] the ideal $I_{\mathrm{Cl}}(R)$ is called the ideal maximally defining $R$.

**Proposition 2.6.** *The ideal $I_{\mathrm{Cl}}(R)$ has the following properties:*

*(i) it defines $R$,*

*(ii) it is $B$-saturated,*

*(iii) any ideal $J$ defining $R$ is contained in $I_{\mathrm{Cl}}(R)$.*

*Proof.* First we prove Point (i). Let $\mathcal{I}_R$ be the ideal sheaf of $R$. Take any $\sigma \in \Sigma$. We have the following inclusions of ideals in the ring $(T_{\alpha^{\hat{\sigma}}})_0$.

$$\sum_{\theta \text{ homogeneous, } R\subseteq V(\theta)} ((\theta)_{\alpha^{\hat{\sigma}}})_0 \subseteq \mathcal{I}_R(U_\sigma), \tag{2.2}$$

$$\sum_{\theta \text{ homogeneous, } R\subseteq V(\theta)} ((\theta)_{\alpha^{\hat{\sigma}}})_0 \subseteq (I_{\mathrm{Cl}}(R)_{\alpha^{\hat{\sigma}}})_0. \tag{2.3}$$

Inclusion (2.2) follows from the fact that each homogeneous $\theta$ appearing in the sum vanishes on $R$, so it factors through the ideal of $R$ on the affine open set $U_\sigma$. Inclusion (2.3) is a consequence of the definition of $I_{\mathrm{Cl}}(R)$.

We prove that both Inclusions (2.2) and (2.3) are in fact equalities. We start with Inclusion (2.2). Let $\frac{\zeta}{(\alpha^{\hat{\sigma}})^k} \in \mathcal{I}_R(U_\sigma)$ for some $k \in \mathbb{Z}$ and some homogeneous $\zeta \in T$. The section $\zeta$ vanishes on $R \cap U_\sigma$. By [39, Lemma 6.A.2(b)] applied to the sheaf $i_*\mathcal{O}_R$ ($i$ being the inclusion $i : R \to X_\Sigma$), we get that for some $l \in \mathbb{N}$ the global section $(\alpha^{\hat{\sigma}})^l\zeta$ is zero on $R$, so it appears on the left-hand side of Inclusion (2.2). Hence, Inclusion (2.2) is an equality.

We move along to Inclusion (2.3). Let $\theta_1, \ldots, \theta_l$ be a set of homogeneous generators of $I_{\mathrm{Cl}}(R)$. It suffices to prove that

$$((\theta_1)_{\alpha^{\hat{\sigma}}})_0, \ldots, ((\theta_r)_{\alpha^{\hat{\sigma}}})_0$$

generate $(I_{\mathrm{Cl}}(R)_{\alpha^{\hat{\sigma}}})_0$. Suppose we have

$$\frac{\theta}{(\alpha^{\hat{\sigma}})^k} \in (I_{\mathrm{Cl}}(R)_{\alpha^{\hat{\sigma}}})_0$$

for some positive integer $k$ and some homogeneous $\theta \in I_{\mathrm{Cl}}(R)$. Then we get $\theta = \xi_1\theta_1 + \cdots + \xi_l\theta_l$ for some $\xi_i \in T$. We may assume that $\xi_i$ are homogeneous and that $\deg\theta = \deg\xi_i + \deg\theta_i$ for each $i$. Then

$$\frac{\theta}{(\alpha^{\hat{\sigma}})^k} = \sum_{i=1}^{l} \frac{\xi_i\theta_i}{(\alpha^{\hat{\sigma}})^k},$$

as claimed.

Let us proceed to the proof of Point (ii). Let $\theta \in T$ be such that for each $\sigma \in \Sigma$ we have $\alpha^{\hat{\sigma}}\theta \in I_{\mathrm{Cl}}(R)$. To check if $\theta \in I_{\mathrm{Cl}}(R)$, it is enough to check locally, and on $U_\sigma$ we can divide by $\alpha^{\hat{\sigma}}$ which is invertible in the localization. This justifies Point (ii).

Finally, we prove Point (iii). Let $J$ be an ideal defining $R$, and let $\theta \in J$ be a homogeneous element. Then $R = V(J) \subseteq V(\theta)$, so $\theta$ is a section vanishing on $R$, hence $\theta \in I_{\mathrm{Cl}}(R)$. □

**Definition 2.7.** Let $R \hookrightarrow X_\Sigma$ be a closed subscheme. We define $I_{\mathrm{Pic}}(R) \subseteq T$, the ideal of Picard sections of $R$, to be the ideal generated by homogeneous elements of $T$, which are of a degree belonging to the Picard group, and which vanish on $R$.

The behaviour of the ideal $I_{\mathrm{Pic}}(R)$ is, in general, much worse than that of $I_{\mathrm{Cl}}(R)$. On proper toric varieties whose Picard group is trivial, they are just the zero ideals and fail to define the scheme $R$ (except when $R$ is equal to $X_\Sigma$). In order to mend this phenomenon, Kajiwara in [61, Definition 1.5] made the following definition.

**Definition 2.8.** The toric variety $X_\Sigma$ has enough Cartier divisors if for each torus-invariant open affine subset $U$, its complement $X_\Sigma \setminus U$ is the support of an effective torus-invariant Cartier divisor.

**Proposition 2.9.** *Let $R \hookrightarrow X_\Sigma$ be a subscheme. The following conditions are true:*

*(i) If $X_\Sigma$ has enough Cartier divisors, then $I_{\mathrm{Pic}}(R)$ defines $R$.*

*(ii) The ideal $I_{\mathrm{Pic}}(R)$ agrees with its $B$-saturation in the degrees belonging to the Picard group.*

*(iii) $I_{\mathrm{Pic}}(R)$ is contained in any $B$-saturated ideal defining $R$.*

*Proof.* We start with Point (i). We have the following inclusions for each $\sigma \in \Sigma$.

$$\sum_{\substack{\theta \mid R \subseteq V(\theta) \\ \theta \in T_\eta \text{ for some } \eta \in \operatorname{Pic} X_\Sigma}} ((\theta)_{\alpha^{\hat{\sigma}}})_0 \subseteq (I_{\mathrm{Pic}}(R)_{\alpha^{\hat{\sigma}}})_0 \subseteq (I_{\mathrm{Cl}}(R)_{\alpha^{\hat{\sigma}}})_0.$$

Let $\theta_1, \ldots, \theta_l$ be a set of homogeneous generators of $I_{\mathrm{Pic}}(R)$. Since $I_{\mathrm{Cl}}(R)$ defines $R$, it is enough to prove that

$$((\theta_1)_{\alpha^{\hat{\sigma}}})_0, \ldots, ((\theta_r)_{\alpha^{\hat{\sigma}}})_0$$

generate $(I_{\mathrm{Cl}}(R)_{\alpha^{\hat{\sigma}}})_0$. As $X_\Sigma$ has enough Cartier divisors, we have the following equality

$$X_\Sigma \setminus U_\sigma = \operatorname{supp} V(\alpha_1^{b_1} \cdots \alpha_r^{b_r})$$

for $b_1, \ldots, b_r \in \mathbb{N}$ such that $V(\alpha_1^{b_1} \cdots \alpha_r^{b_r})$ is an effective Cartier divisor. We set $\xi = \alpha_1^{b_1} \cdots \alpha_r^{b_r}$. We know that

$$(T_{\alpha^{\hat{\sigma}}})_0 = (T_\xi)_0.$$

Suppose $\frac{\zeta}{\xi^k} \in (I_{\mathrm{Cl}}(R)_\xi)_0$ with $\deg \zeta, \deg \xi \in \operatorname{Pic} X_\Sigma$, and $\zeta \in I_{\mathrm{Cl}}(R)$. But then also $\zeta \in I_{\mathrm{Pic}}(R)$, hence $\zeta = \sum_{i=1}^{l} \zeta_i \theta_i$ for some $\zeta_i \in T$. We may assume that $\zeta_i$ are homogeneous. Then

$$\frac{\zeta}{\xi^k} = \sum_{i=1}^{l} \frac{\zeta_i \theta_i}{\xi^k},$$

as desired.

We prove Point (ii). We want to argue that $I_{\mathrm{Pic}}(R)_\eta = (I_{\mathrm{Pic}}(R) :_T B^k)_\eta$ for each $\eta \in \operatorname{Pic} X_\Sigma$ and each $k \in \mathbb{N}$. Take $\eta \in \operatorname{Pic} X_\Sigma$, and $k \in \mathbb{N}$. Pick any $\theta \in T_\eta$ such that $(\alpha^{\hat{\sigma}})^k \theta \in I_{\mathrm{Pic}}(R)$ for each $\sigma \in \Sigma$. Then

$$\theta(\alpha^{\hat{\sigma}})^k = \sum_{i=1}^{l_\sigma} \zeta_{\sigma,i} \xi_{\sigma,i},$$

where $\zeta_{\sigma,i}$ are homogeneous, and $\xi_{\sigma,i}$ are homogeneous of degree belonging to $\operatorname{Pic} X_\Sigma$ and vanish on $R$. Then, locally on $U_\sigma$, we can write

$$\theta = \sum_{i=1}^{l_\sigma} ((\alpha^{\hat{\sigma}})^{-k}\zeta_{\sigma,i})\xi_{\sigma,i}.$$

The section $\alpha^{\hat{\sigma}}$ is invertible on $U_\sigma$, so $(\alpha^{\hat{\sigma}})^{-k}\zeta_{\sigma,i}$ is defined on $U_\sigma$. Hence, $\theta$ is zero on $R$ on each $U_\sigma$, and thus on all of $X_\Sigma$, and $\deg\theta$ belongs to $\operatorname{Pic} X_\Sigma$. It follows that $\theta \in I_{\mathrm{Pic}}(R)$.

Finally, we proceed to Point (iii). Let $J$ be a $B$-saturated ideal defining $R$. Let $\theta \in I_{\mathrm{Pic}}(R)$ be a homogeneous element of degree $\eta \in \operatorname{Pic} X_\Sigma$. Take any $\sigma \in \Sigma$. By [37, Lemma 3.4] there exist integers $b_{i_1}, \ldots, b_{i_l}$ such that

$$\eta = [b_{i_1} D_{i_1} + \cdots + b_{i_l} D_{i_l}],$$

and that $D_{i_j}$ correspond to rays not in $\sigma$ for all $1 \le j \le l$. Hence,

$$\frac{\theta}{\alpha_{i_1}^{b_{i_1}} \cdots \alpha_{i_l}^{b_{i_l}}} \in (I_{\mathrm{Pic}}(R)_{\alpha^{\hat{\sigma}}})_0 \subseteq (I_{\mathrm{Cl}}(R)_{\alpha^{\hat{\sigma}}})_0 = (J_{\alpha^{\hat{\sigma}}})_0.$$

It follows that $\theta \in J_{\alpha^{\hat{\sigma}}}$ for all $\sigma \in \Sigma$. Therefore, $(\alpha^{\hat{\sigma}})^k\theta \in J$ for some $k \in \mathbb{N}$ and all $\sigma \in \Sigma$. As $J$ is $B$-saturated, it follows that $\theta \in J$, as desired. □

**Corollary 2.10.** *Any two $B$-saturated ideals defining the same scheme agree in degrees belonging to the Picard group.*

## 2.2 Further properties of toric varieties

Let $\eta \in \operatorname{Cl} X_\Sigma$. Recall the isomorphism of $H^0(X_\Sigma, \mathcal{O}(\eta))$ and $\mathbb{C}[\alpha_1, \ldots, \alpha_r]_\eta$ given in [39, Proposition 5.3.7]. The variety $X_\Sigma$ can be obtained as an almost geometric quotient of an action of $G := \operatorname{Hom}(\operatorname{Cl} X_\Sigma, \mathbb{C}^*)$ on $\mathbb{C}^{\Sigma(1)} \setminus Z$, where $Z$ is defined as $V(B)$. See [39, Proposition 5.0.11] for the definition of an almost geometric quotient and [39, Section 5.1] for an explicit construction of this quotient. We denote this map by

$$\mathbb{C}^{\Sigma(1)} \setminus Z \xrightarrow{[\cdot]} X_\Sigma.$$

**Proposition 2.11.** *Suppose $\eta \in \operatorname{Pic} X_\Sigma$. Take any section $s \in H^0(X_\Sigma, \mathcal{O}(\eta))$ and the corresponding polynomial $\theta \in T_\eta$. Also let $p$ be a point in $X_\Sigma$ and take any $(\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$ such that $[\lambda_1, \ldots, \lambda_r] = p$. Then*

$$s(p) = 0 \iff \theta(\lambda_1, \ldots, \lambda_r) = 0.$$

The proof can be found in the author's master thesis [44, Fact 1.3] or in the article [46, Proposition 3.4].

**Corollary 2.12.** *Suppose $\eta \in \operatorname{Pic} X_\Sigma$. Suppose $\theta_1, \theta_2 \in T_\eta$ are polynomials and $s_1, s_2$ are the corresponding sections of $\mathcal{O}(\eta)$. Also fix, as above, $p \in X_\Sigma$ and $(\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$ such that $[\lambda_1, \ldots, \lambda_r] = p$. Then if $\theta_2(\lambda_1, \ldots, \lambda_r)$ and $s_2(p)$ are non-zero, we get*

$$\frac{\theta_1(\lambda_1, \ldots, \lambda_r)}{\theta_2(\lambda_1, \ldots, \lambda_r)} = \frac{s_1(p)}{s_2(p)}.$$

*Proof.* Take $\mu \in \mathbb{C}$ such that $\theta_1(\lambda_1, \ldots, \lambda_r) = \mu\theta_2(\lambda_1, \ldots, \lambda_r)$. Then use Proposition 2.11 for $\theta_1 - \mu\theta_2$ and the corresponding section $s_1 - \mu s_2$. $\square$

We need the following proposition to understand dehomogenization and homogenization better in Section 2.3.

**Proposition 2.13.** *Let $X_\Sigma$ be a smooth complete toric variety. Pick any $\sigma \in \Sigma$ of full dimension. Let $\rho_1, \ldots, \rho_d$ be the rays that are not in $\sigma$. Then the classes $[D_{\rho_1}], \ldots, [D_{\rho_d}]$ are a basis of the class group.*

*Proof.* The fact that the classes $[D_{\rho_1}], \ldots, [D_{\rho_d}]$ generate the class group follows from [37, Lemma 3.4] (recall that for smooth varieties $\operatorname{Pic} X_\Sigma = \operatorname{Cl} X_\Sigma$).

Now consider the exact sequence ([39, Theorem 4.1.3])

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \operatorname{Cl} X_\Sigma \to 0,$$

where $M$ is the lattice dual to $N$. We get that $\operatorname{rank} \operatorname{Cl} X_\Sigma = \#\Sigma(1) - \dim M_\mathbb{R}$, which is equal to the number of rays not in $\sigma$, since the cone $\sigma$ is smooth. But from the fact that the classes $[D_{\rho_1}], \ldots, [D_{\rho_d}]$ generate the class group and from the fact that a subgroup of a lattice is a lattice, we can write the following exact sequence.

$$0 \to \mathbb{Z}^l \to \bigoplus_{\rho \notin \sigma(1)} \mathbb{Z}[D_\rho] \to \operatorname{Cl} X_\Sigma \to 0$$

for some integer $l$. As a consequence, we get that $l = 0$, so

$$\operatorname{Cl} X_\Sigma \cong \bigoplus_{\rho \notin \sigma(1)} \mathbb{Z}[D_\rho],$$

as desired.

$\square$

In Chapter 5 we will encounter monomials on toric varieties whose cactus rank is less than their border rank. To make sense of this phenomenon, it is useful to study the following description of the tangent space at a torus-invariant point.

Let $M$ be the dual lattice of the lattice $N$. Let $X_P$ be the toric variety embedded by a very ample polytope $P$ with vertices in lattice $M$ (see [39, Chapter 2]). Let $v$ be a vertex of the polytope $P$ (which corresponds to a torus fixed point $p \in X_P$).

**Proposition 2.14.** *The projective embedded tangent space at $p$ in the embedding by $P$ is the projective linear span of the monomials corresponding to the lattice points*

$$\{w + v \mid w \text{ is an element of the Hilbert basis}$$
$$\text{of the semigroup } \mathbb{N}(P \cap M - v)\} \cup \{v\}.$$

*Proof.* Let $z_0, \ldots, z_k$ be the projective coordinates corresponding to the monomials in the embedding by $P$, with $z_0$ corresponding to the vertex $v$. Let us look at the affine chart given by setting $z_0 = 1$. The equations of the toric variety in the affine chart come from integral relations between the lattice points of $P \cap M - v$. The equations of the embedded tangent space at $p$ (in the affine chart) are given in the following way: the forms

$$\sum_{i=1}^{k} \frac{\partial f}{\partial z_i}_{|[z_1,\ldots,z_k]=[0,0,\ldots,0]} z_i,$$

where $f$ is an equation of the embedded variety $X_P$ in the affine chart, give all the equations.

Suppose $h_j$ is an element of the Hilbert basis of $\mathbb{N}(P \cap M - v)$, and that the coordinate $z_j$ corresponds to $h_j$. Any relation is of the form

$$lh_j + l_1 h_{j_1} + \cdots + l_m h_{j_m} = l_{m+1} h_{j_{m+1}} + \cdots + l_n h_{j_n}, \tag{2.4}$$

where $h_{j_i} \in P \cap M - v$, $h_{j_i}$ are mutually different, $h_{j_i} \neq h_j$, and $l \geq 0, l_i \geq 1$ for $i = 1, \dots, n$ are positive integers. In this relation the left-hand side is not equal to $h_j$ as $h_j$ is not a sum. Let us look at the polynomial equation coming form this Relation (2.4). It is

$$z_j^l z_{j_1}^{l_1} \cdot \dots \cdot z_{j_m}^{l_m} = z_{j_{m+1}}^{l_{m+1}} \cdot \dots \cdot z_{j_n}^{l_n}, \tag{2.5}$$

where the left-hand side is not equal to $z_j$, and the right-hand side does not contain $z_j$. If we differentiate this equation with respect to $z_j$ and substitute the point p (which has coordinates $(z_1, \dots, z_k) = (0, \dots, 0)$), we get $0 = 0$. Therefore $z_j$ does not appear in the equation of the embedded tangent space at $p$ coming from Equation (2.5). Hence, the point $(z_1, z_2, \dots, z_k) = (0, \dots, 0, 1, 0, \dots, 0)$ (where the 1 is on the $j$-th place) satisfies this equation of embedded tangent space at $p$.

Since the point $(0, \dots, 0, 1, 0, \dots, 0)$ is independent of the chosen Equation (2.5), we get that it satisfies all the equations of the projectivized tangent space at $p$.

We come back the projective coordinates. We proved that for every vector $h_j$ in the Hilbert basis of $\mathbb{N}(P \cap M - v)$ the point $[0, \dots, 0, 1, 0, \dots, 0]$ (where the 1 is on the $j$-th place) is in the embedded tangent space. Also the point $p = [1, 0, \dots, 0]$ is in this space. As this projective space has dimension equal to the cardinality of the Hilbert basis (see [39, Lemma 1.3.10]), we get the desired equality. $\square$

## 2.3 Dehomogenization and homogenization

Let $X_\Sigma$ be a smooth projective toric variety. Denote the rays of the fan by $\rho_1, \dots, \rho_r$. Fix $\sigma \in \Sigma$. We want to restrict $X_\Sigma$ to the affine patch $U_\sigma$. Suppose the rays that are not in $\sigma$ are $\rho_1, \dots, \rho_k$. Let $S = \mathbb{C}[\alpha_{k+1}, \dots, \alpha_r]$. Denote by $(\cdot)^{\mathrm{deh}} : T \to S$ the dehomogenization on $T$ (i.e. setting $\alpha_1, \dots, \alpha_k$ to 1).

**Proposition 2.15.** *For any $\eta \in \mathrm{Cl}\, X_\Sigma = \mathrm{Pic}\, X_\Sigma$ the map*

$$(\cdot)^{\mathrm{deh}} : T_\eta \to S$$

*is injective.*

*Proof.* Suppose $\theta^{\mathrm{deh}} = 0$ for some $0 \neq \theta \in T_\eta$. Then there exist two different monomials $\alpha_1^{c_1} \cdots \alpha_r^{c_r}$ and $\alpha_1^{d_1} \cdots \alpha_r^{d_r}$ of degree $\eta$ such that after applying $(\cdot)^{\mathrm{deh}}$ they are the same. This means that $c_{k+1} = d_{k+1}, \dots, c_r = d_r$, so $\deg \alpha_1^{c_1} \cdots \alpha_k^{c_k} = \deg \alpha_1^{d_1} \cdots \alpha_k^{d_k}$. The tuples $(c_1, \dots, c_k)$ and $(d_1, \dots, d_k)$ are different, so this gives a non-trivial relation between the classes corresponding to $\alpha_1, \dots, \alpha_k$, contradicting Proposition 2.13. $\square$

Let us define the homogenization $\zeta^{\mathrm{hom}}$ of a non-zero polynomial $\zeta \in S$. Suppose

$$\zeta = \sum_{\eta \in \mathrm{Cl}\, X_\Sigma} \zeta_\eta,$$

where each $\zeta_\eta$ is homogeneous of degree $\eta$. Let $D_i$ be the divisor corresponding to $\rho_i$. From Proposition 2.13 we know that the classes $[D_i]$, where $i = 1, \dots, k$, form a basis of the class group. Hence, for each $\eta \in \mathrm{Cl}\, X_\Sigma$ such that $\zeta_\eta \neq 0$ we have

$$\eta = a_{\eta,1}[D_1] + \ldots + a_{\eta,k}[D_k],$$

where $a_{\eta,i} \in \mathbb{Z}$. Let $b_i = \max\{a_{\eta,i} | \eta \in \mathrm{Cl}\, X_\Sigma, \zeta_\eta \neq 0\}$. Then we set

$$\zeta^{\mathrm{hom}} = \sum_{\eta \in \mathrm{Cl}\, X_\Sigma} \alpha_1^{b_1 - a_{\eta,1}} \cdot \ldots \cdot \alpha_k^{b_k - a_{\eta,k}} \zeta_\eta.$$

This is homogeneous of degree $b_1[D_1] + \ldots + b_k[D_k]$.

**Proposition 2.16.** *Suppose $\theta \in T$ is homogeneous and non-zero. Write $\theta = \alpha_1^{e_1} \cdots \alpha_k^{e_k} \hat{\theta}$ in such a way that $\hat{\theta}$ is not divisible by $\alpha_1, \dots, \alpha_k$. Then $(\theta^{\mathrm{deh}})^{\mathrm{hom}} = \hat{\theta}$.*

*Proof.* We know that $\theta^{\mathrm{deh}}$ is non-zero by Proposition 2.15. Let $\theta^{\mathrm{deh}} = \sum_{\eta \in \mathrm{Cl}\, X_\Sigma} \zeta_\eta$, where each $\zeta_\eta$ has degree $\eta$. Then

$$\theta = \alpha_1^{e_1} \cdots \alpha_k^{e_k} \cdot \left( \sum_{\eta \in \mathrm{Cl}\, X_\Sigma} \alpha_1^{d_{\eta,1}} \cdots \alpha_k^{d_{\eta,k}} \zeta_\eta \right)$$

for some natural $d_{\eta,i}$. For any $\eta \in \mathrm{Cl}\, X_\Sigma$ there exist integers $a_{\eta,1}, \dots, a_{\eta,k}$ such that

$$\eta = a_{\eta,1}[D_1] + \ldots + a_{\eta,k}[D_k].$$

It follows that for any $\eta \in \mathrm{Cl}\, X_\Sigma$ such that $\zeta_\eta \neq 0$ we have

$$\deg \theta = (e_1 + d_{\eta,1} + a_{\eta,1})[D_1] + \ldots + (e_k + d_{\eta,k} + a_{\eta,k})[D_k].$$

Therefore, for each $i = 1, \ldots, k$ the number $d_{\eta,i} + a_{\eta,i}$ is independent of $\eta$, so we can set $c_i = d_{\eta,i} + a_{\eta,i}$. We know that $c_i \geq a_{\eta,i}$ for each $\eta \in \mathrm{Cl}\, X_\Sigma$ such that $\zeta_\eta \neq 0$, so

$$c_i \geq \max_{\eta \in \mathrm{Cl}\, X_\Sigma | \zeta_\eta \neq 0} a_{\eta,i}.$$

If this inequality is strict for some $i$, then $\alpha_i$ divides $\hat{\theta}$, which is a contradiction. Hence, for every $i = 1, 2, \ldots, k$ we have

$$c_i = \max_{\eta \in \mathrm{Cl}\, X_\Sigma | \zeta_\eta \neq 0} a_{\eta,i},$$

and therefore

$$d_{\eta,i} = \max_{\eta \in \mathrm{Cl}\, X_\Sigma | \zeta_\eta \neq 0} a_{\eta,i} - a_{\eta,i}.$$

It follows that $\hat{\theta}$ is the homogenization of $\theta^{\mathrm{deh}}$. □

**Definition 2.17.** Suppose $I \subseteq S$ is an ideal. Let

$$I^{\mathrm{hom}} = (\zeta^{\mathrm{hom}} | \zeta \in I \setminus \{0\})$$

be the homogenization of $I$. It is a homogeneous ideal of $T$.

**Proposition 2.18.** *The ideal $I^{\mathrm{hom}}$ is saturated with respect to $\alpha_1 \cdots \alpha_k$.*

*Proof.* Suppose that $\alpha_1 \cdots \alpha_k \theta \in I^{\mathrm{hom}}$ for some non-zero homogeneous $\theta \in T$, then

$$\alpha_1 \cdots \alpha_k \theta = \xi_1 \zeta_1^{\mathrm{hom}} + \ldots + \xi_l \zeta_l^{\mathrm{hom}}$$

for some $\xi_i \in T$ and $\zeta_i \in I$. If we set $\alpha_1, \ldots, \alpha_k$ to 1, we get

$$\theta^{\mathrm{deh}} = \xi_1^{\mathrm{deh}} \zeta_1 + \ldots + \xi_l^{\mathrm{deh}} \zeta_l.$$

This means that $\theta^{\mathrm{deh}} \in I$, and it follows that $(\theta^{\mathrm{deh}})^{\mathrm{hom}} \in I^{\mathrm{hom}}$ from the definition of $I^{\mathrm{hom}}$. But $\theta$ is divisible by $(\theta^{\mathrm{deh}})^{\mathrm{hom}}$ by Proposition 2.16, so $\theta \in I^{\mathrm{hom}}$. □

*Remark* 2.19. Since $I^{\mathrm{hom}}$ is saturated with respect to $\alpha_1 \cdots \alpha_k$, it is saturated with respect to any ideal $B$ containing $\alpha_1 \cdots \alpha_k$.

**Proposition 2.20.** *Suppose $\theta \in T$ is homogeneous, and $I \subseteq S$ is an ideal. Then we have the following equivalence*

$$\theta^{\mathrm{deh}} \in I \iff \theta \in I^{\mathrm{hom}}.$$

*Proof.* Assume that $\theta^{\mathrm{deh}} \in I$. Let us write $\theta = \alpha_1^{e_1} \cdots \alpha_k^{e_k} \hat{\theta}$ with $\hat{\theta}$ not divisible by $\alpha_1, \ldots, \alpha_k$. By Proposition 2.16 we have $\hat{\theta} = (\theta^{\mathrm{deh}})^{\mathrm{hom}}$, so $\hat{\theta} \in I^{\mathrm{hom}}$. It follows that $\theta = \alpha_1^{e_1} \cdots \alpha_k^{e_k} \hat{\theta} \in I^{\mathrm{hom}}$.

Now assume $\theta \in I^{\mathrm{hom}}$, then $\theta = \xi_1 \zeta_1^{\mathrm{hom}} + \cdots + \xi_m \zeta_m^{\mathrm{hom}}$ for some $\zeta_i \in I$ and $\xi_i \in T$. After applying $(\cdot)^{\mathrm{deh}}$, we get

$$\theta^{\mathrm{deh}} = \xi_1^{\mathrm{deh}} \zeta_1 + \cdots + \xi_m^{\mathrm{deh}} \zeta_m \in I,$$

as desired. □

## 2.4 Homogenization of binomial ideals

In general, the homogenization of an ideal on the affine patch $U_\sigma$ can be computed by homogenizing the generators and saturating with respect to $\alpha_1 \cdots \alpha_k$ (where $1, \ldots, k$ are the indices of rays outside $\sigma$). A drawback of this method is that saturation is hard to control. In this section, we show that for unital binomial ideals, saturation can be tamed.

Recall that $S = \mathbb{C}[\alpha_{k+1}, \ldots, \alpha_r]$. In the following propositions, a bold font denotes multi-indices (vectors of integers), indexed by integers $k+1, \ldots, r$. For a multi-index $\mathbf{t} = (t_{k+1}, \ldots, t_r)$ we use the notation $\alpha^{\mathbf{t}} = \alpha_{k+1}^{t_{k+1}} \cdots \alpha_r^{t_r}$.

The following proposition is the main part of this section. It is a part of the author's article [46]. In order to prove it, we need Lemma 2.22.

**Proposition 2.21.** *Suppose $I \subseteq S$ is an ideal generated by binomials of the form $\alpha^{\mathbf{a}} - \alpha^{\mathbf{b}}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{N}^{r-k}$. Then*

$$I^{\mathrm{hom}} = ((\alpha^{\mathbf{a}} - \alpha^{\mathbf{b}})^{\mathrm{hom}} | \alpha^{\mathbf{a}} - \alpha^{\mathbf{b}} \in I \setminus 0).$$

**Lemma 2.22.** *Suppose $I \subseteq S$ is an ideal generated by binomials of the form $\alpha^{\mathbf{a}} - \alpha^{\mathbf{b}}$ and that $\zeta \in I$. There exist binomials $\alpha^{\mathbf{c}_i} - \alpha^{\mathbf{d}_i} \in I$ (where $\mathbf{c}_i, \mathbf{d}_i \in \mathbb{N}^{r-k}$) and $\lambda_i \in \mathbb{C}$, where $i = 1, 2, \ldots, l$, such that*

$$\zeta = \sum_{i=1}^{l} \lambda_i (\alpha^{\mathbf{c}_i} - \alpha^{\mathbf{d}_i})$$

*and every $\alpha^{\mathbf{c}_i}, \alpha^{\mathbf{d}_i}$ appears as a monomial of $\zeta$ with a non-zero coefficient.*

*Proof.* Suppose we have

$$\zeta = \sum_{i=1}^{m} \kappa_i(\alpha^{\mathbf{a}_i} - \alpha^{\mathbf{b}_i}),$$

where $\alpha^{\mathbf{a}_i} - \alpha^{\mathbf{b}_i} \in I$ and $\kappa_i \in \mathbb{C} \setminus \{0\}$ for $i = 1, 2, \ldots, m$. Suppose that some monomial $\alpha^{\mathbf{b}}$ appears in the sum on right-hand side and that it does not appear on the left-hand side. Possibly changing the signs of some $\kappa_i$, we may assume that there are indices $i_1, \ldots, i_n$ such that $\mathbf{b}_{i_1} = \cdots = \mathbf{b}_{i_n} = \mathbf{b}$ and $\sum_{j=1}^{n} \kappa_{i_j} = 0$, and that $\mathbf{b}$ appears nowhere else in the sum on the right-hand side. In this case $\kappa_{i_1} = -\sum_{j=2}^{n} \kappa_{i_j}$ and therefore

$$\zeta = \sum_{i|\mathbf{b}_i \neq \mathbf{b}} \kappa_i(\alpha^{\mathbf{a}_i} - \alpha^{\mathbf{b}_i}) + \sum_{j=2}^{n} \kappa_{i_j}(\alpha^{\mathbf{a}_{i_j}} - \alpha^{\mathbf{a}_{i_1}}),$$

where each $\alpha^{\mathbf{a}_{i_j}} - \alpha^{\mathbf{a}_{i_1}} = (\alpha^{\mathbf{a}_{i_j}} - \alpha^{\mathbf{b}_{i_j}}) - (\alpha^{\mathbf{a}_{i_1}} - \alpha^{\mathbf{b}_{i_1}}) \in I$. We have reduced the number of summands on the right-hand side. Continuing this process, we get to the situation where every monomial on the right-hand side appears on the left-hand side with a non-zero coefficient. □

*Proof of Proposition 2.21.* Let $\zeta \in I$ be a non-zero polynomial. From Lemma 2.22 we get that there are $\lambda_i \in \mathbb{C}$ and $\alpha^{\mathbf{c}_i} - \alpha^{\mathbf{d}_i}$, where $i = 1, \ldots, l$, such that

$$\zeta = \sum_{i=1}^{l} \lambda_i(\alpha^{\mathbf{c}_i} - \alpha^{\mathbf{d}_i})$$

and every $\alpha^{\mathbf{c}_i}, \alpha^{\mathbf{d}_i}$ appears as a monomial $\alpha^{\mathbf{e}_s}$ of $\zeta$ with a non-zero coefficient. Let $s(i), s'(i)$ be such that $\alpha^{\mathbf{c}_i} = \alpha^{\mathbf{e}_{s(i)}}$ and $\alpha^{\mathbf{d}_i} = \alpha^{\mathbf{e}_{s'(i)}}$. We need to get back to the definition of homogenization. Suppose $\alpha^{\mathbf{e}_s}$ is of degree $a_{s,1}[D_1] + \cdots + a_{s,k}[D_k]$. Then $\alpha^{\mathbf{c}_i}$ is of degree $a_{s(i),1}[D_1] + \cdots + a_{s(i),k}[D_k]$, and $\alpha^{\mathbf{d}_i}$ is of degree $a_{s'(i),1}[D_1] + \cdots + a_{s'(i),k}[D_k]$. It follows that

$$(\alpha^{\mathbf{c}_i} - \alpha^{\mathbf{d}_i})^{\mathrm{hom}} = \alpha_1^{b_{i,1} - a_{s(i),1}} \cdots \alpha_k^{b_{i,k} - a_{s(i),k}} \alpha^{\mathbf{c}_i} - \alpha_1^{b_{i,1} - a_{s'(i),1}} \cdots \alpha_k^{b_{i,k} - a_{s'(i),k}} \alpha^{\mathbf{d}_i},$$

where $b_{i,j} = \max(a_{s(i),j}, a_{s'(i),j})$. We write

$$\zeta = \sum_{s=1}^{m} \mu_s \alpha^{\mathbf{e}_s}.$$

Then

$$\zeta^{\text{hom}} = \sum_{s=1}^{m} \mu_s \alpha_1^{\bar{b}_1 - a_{s,1}} \cdots \alpha_k^{\bar{b}_k - a_{s,k}} \alpha^{\mathbf{e}_s}.$$

Here

$$\bar{b}_j = \max\{a_{s,j} | s = 1, \ldots, m\} = \max\{a_{s(i),j}, a_{s'(i),j} | i = 1, \ldots, l\} \\ = \max\{b_{i,j} | i = 1, \ldots, l\},$$

as every $\alpha^{\mathbf{c}_i}, \alpha^{\mathbf{d}_i}$ appears as a monomial of $\zeta$. Hence

$$\begin{aligned}
\zeta^{\text{hom}} &= \sum_{s=1}^{l} \mu_s \alpha_1^{\bar{b}_1 - a_{s,1}} \cdots \alpha_k^{\bar{b}_k - a_{s,k}} \alpha^{\mathbf{e}_s} \\
&= \sum_{i=1}^{l} \lambda_i \Big( \alpha_1^{\bar{b}_1 - a_{s(i),1}} \cdots \alpha_k^{\bar{b}_k - a_{s(i),k}} \alpha^{\mathbf{c}_i} - \alpha_1^{\bar{b}_1 - a_{s'(i),1}} \cdots \alpha_k^{\bar{b}_k - a_{s'(i),k}} \alpha^{\mathbf{d}_i} \Big) \\
&= \sum_{i=1}^{l} \lambda_i \alpha_1^{\bar{b}_1 - b_{i,1}} \cdots \alpha_k^{\bar{b}_k - b_{i,k}} \Big( \alpha_1^{b_{i,1} - a_{s(i),1}} \cdots \alpha_k^{b_{i,k} - a_{s(i),k}} \alpha^{\mathbf{c}_i} \\
&- \alpha_1^{b_{i,1} - a_{s'(i),1}} \cdots \alpha_k^{b_{i,k} - a_{s'(i),k}} \alpha^{\mathbf{d}_i} \Big) \\
&= \sum_{i=1}^{l} \lambda_i \alpha_1^{\bar{b}_1 - b_{i,1}} \cdots \alpha_k^{\bar{b}_k - b_{i,k}} (\alpha^{\mathbf{c}_i} - \alpha^{\mathbf{d}_i})^{\text{hom}}.
\end{aligned}$$

□

## 2.5 Interlude: divided power structures

In order to better understand secant varieties in Chapter 3, we need the notion of the derivation action. In what follows, we define the annihilator with respect to this action, and we investigate its properties in Sections 2.6 and 2.7. We use the divided power language since it allows us to write the derivation action without the coefficients coming from factorials. We start with the following definition, taken from [1, Tag 07GK].

**Definition 2.23.** Let $A$ be a ring. Let $I$ be an ideal of $A$. A collection of maps $(\cdot)^{(n)} : I \to I, n \geq 1$ is a called a *divided power structure* if for all $x, y \in I$, $a \in A$, $n, m \geq 0$ we have:

(a) $x^{(1)} = x$, we also set $x^{(0)} = 1$,

(b) $x^{(n)}x^{(m)} = \binom{n+m}{n} x^{(n+m)}$,

(c) $(ax)^{(n)} = a^n x^{(n)}$,

(d) $(x+y)^{(n)} = \sum_{i=0}^{n} x^{(i)} y^{(n-i)}$,

(e) $(x^{(m)})^{(n)} = \frac{(nm)!}{n!(m!)^n} x^{(nm)}$.

Let $H$ be a finitely generated abelian group, $T = \mathbb{C}[\alpha_1, \ldots, \alpha_r]$ be a polynomial ring. Suppose $T$ is graded by $H$ in such a way that each variable is homogenous and that for each $\eta \in H$ the vector space $T_\eta$ is finite-dimensional. In this section, we will call such gradings proper. This is satisfied when $T$ is the Cox ring of a projective toric variety, and the reader may safely think of it as the main example. The ring $T$ has a canonical coalgebra structure given by the map

$$\begin{aligned} T &\to T \otimes T, \\ \alpha_i &\mapsto \alpha_i \otimes 1 + 1 \otimes \alpha_i, \end{aligned}$$

see [42, Section A2.4]. This induces a canonical algebra structure on the dual vector space

$$T_H^* := \bigoplus_{\eta \in H} \operatorname{Hom}(T_\eta, \mathbb{C}),$$

called the divided power algebra. As the name suggests, there is a canonical divided power structure on $T_H^*$, see [42, Proposition–Definition A2.6].

**Proposition 2.24.** *The ring $T_H^*$ does not depend on $H$, where $H$ defines a proper grading on $T$.*

*Proof.* Let $H_1$, $H_2$ be any two finitely generated abelian groups. Suppose $T$ is graded in a proper way by $H_i$ for $i = 1, 2$. Then $T$ is naturally graded by $H_1 \oplus H_2$ in the following way: for $\eta_1 \in H_1, \eta_2 \in H_2$, we define

$$T_{\eta_1,\eta_2} = T_{\eta_1} \cap T_{\eta_2}.$$

This grading is also proper. We show that $T^*_{H_1}$ is cannonically isomorphic to $T^*_{H_1\oplus H_2}$.

$$T^*_{H_1\oplus H_2} = \bigoplus_{(\eta_1,\eta_2)\in H_1\oplus H_2} \mathrm{Hom}(T_{\eta_1,\eta_2},\mathbb{C}) = \bigoplus_{\eta_1\in H_1}\bigoplus_{\eta_2\in H_2} \mathrm{Hom}(T_{\eta_1,\eta_2},\mathbb{C})$$
$$= \bigoplus_{\eta_1\in H_1} \mathrm{Hom}\left(\bigoplus_{\eta_2\in H_2} T_{\eta_1,\eta_2},\mathbb{C}\right) = \bigoplus_{\eta_1\in H_1} \mathrm{Hom}(T_{\eta_1},\mathbb{C}).$$

Then also $T^*_{H_1} = T^*_{H_1\oplus H_2} = T^*_{H_2}$. $\square$

We write $T^*$ from now on. Let $x_1,\dots,x_r$ denote the basis dual to $\alpha_1,\dots,\alpha_r$. In this thesis, we use another notation for the graded dual ring, we write $T^* = \mathbb{C}_{dp}[x_1,\dots,x_r]$. However, $\mathbb{C}_{dp}[x_1,\dots,x_r]$ is isomorphic to the polynomial ring $\mathbb{C}[x_1,\dots,x_r]$, and $x_i^{(d)} = \frac{x_i}{d!}$ for $1\le i\le r$ and $d\in\mathbb{N}$. We define a grading on $T^*$ by

$$\deg x_i = \deg \alpha_i.$$

For any proper grading $H$ on $T$, and any $\eta\in H$, the basis

$$\{x_1^{(a_1)}\cdots x_r^{(a_r)} \,|\, \deg x_1^{(a_1)}\cdots x_r^{(a_r)} = \eta\}$$

of $T^*_\eta$ is dual to the basis

$$\{\alpha_1^{a_1}\cdots\alpha_r^{a_r} \,|\, \deg \alpha_1^{a_1}\cdots\alpha_r^{a_r} = \eta\}$$

of $T_\eta$.

The ring $T$ acts on $T^*$ in a canonical way. This action is known by different names: derivation, contraction or apolarity, and is defined in coordinates by:

$$\alpha_i \lrcorner\, x_1^{(a_1)}\cdots x_r^{(a_r)} = \begin{cases} x_1^{(a_1)}\cdots x_i^{(a_i-1)}\cdots x_r^{(a_r)} & \text{if } a_i>0,\\ 0 & \text{otherwise.}\end{cases} \tag{2.6}$$

*Remark* 2.25. Notice that when we take $\theta\in T_\eta$ and $F\in T^*_\epsilon$, then $\theta\lrcorner F$ is homogeneous of degree $\epsilon-\eta$ for any $\eta,\epsilon\in H$. That follows from the fact that when we multiply by subsequent $\alpha_i$'s, the degree of $F$ decreases by $\deg\alpha_i$. This means that, although $T^*$ is not a graded $T$-module, it becomes a graded $T$-module if we define the grading by

$$\deg x_i^{(a)} = -\deg\alpha_i^a.$$

The introduction of graded dual rings gives us a way to describe maps from toric varieties to projective spaces. By Corollary 2.12 we get the following proposition.

**Proposition 2.26.** *Let $X_\Sigma$ be a proper normal toric variety. Then for any $\eta \in \operatorname{Pic} X_\Sigma$ such that $\mathcal{O}(\eta)$ is basepoint free, the map*

$$\varphi\colon X_\Sigma \to \mathbb{P}(H^0(X_\Sigma, \mathcal{O}(\eta))^*),$$

*attached to the complete linear system $|\mathcal{O}(\eta)|$, is given by*

$$\varphi([\lambda_1, \ldots, \lambda_r]) = \left[ \sum_{\substack{b_1,\ldots,b_r \in \mathbb{Z}_{\geq 0} | \\ x_1^{(b_1)} \cdots x_r^{(b_r)} \in T_\eta^*}} \lambda_1^{b_1} \cdots \lambda_r^{b_r} \cdot x_1^{(b_1)} \cdots x_r^{(b_r)} \right]. \tag{2.7}$$

The proof is given in the author's master thesis [44, Fact 2.4] and in the article [46, Proposition 4.4].

**Definition 2.27.** If $V \subseteq T^*$ is a finite-dimensional vector subspace, we denote by $\operatorname{Ann}(V) \subseteq T$ its annihilator with respect to the $\lrcorner$ action. The set $\operatorname{Ann}(V)$ is an ideal of $T$.

Note that $\operatorname{Ann}(V)$ is homogeneous if $V$ has a basis consisting of homogeneous elements.

## 2.6 Dehomogenization and homogenization in case of the product of projective spaces

If is often crucial to know the behaviour of the minimal generators of $\operatorname{Ann}(F)$ or of $\operatorname{Ann}(V)$. We will need to apply this in two cases: the case of a product of projective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, and a special case of $\mathbb{P}^n$. In the latter case, we get stronger results.

We start with the product of projective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let $n = n_1 + \cdots + n_k$. In this subsection, we modify the way of indexing variables of $S$ and $T$, we let

$$S = \mathbb{C}[\alpha_{1,1}, \ldots, \alpha_{1,n_1}, \ldots, \alpha_{k,1}, \ldots, \alpha_{k,n_k}]$$
$$T = \mathbb{C}[\alpha_{1,0}, \alpha_{1,1}, \ldots, \alpha_{1,n_1}, \ldots, \alpha_{k,0}, \alpha_{k,1}, \ldots, \alpha_{k,n_k}].$$

There are $k$ variables, with respect to which we homogenize, $\alpha_{1,0}, \ldots, \alpha_{k,0}$, each one corresponding to one copy of projective space. The graded duals are

$$S^* = \mathbb{C}_{dp}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$$
$$T^* = \mathbb{C}_{dp}[x_{1,0}, x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, x_{k,1}, \ldots, x_{k,n_k}].$$

The rings $S$,$T$, $S^*$, $T^*$ are naturally graded by $\mathbb{Z}^k$. We define a partial order on multi-indices of length $k$: we let $\mathbf{d} \leq \mathbf{e}$ if and only if $d_i \leq e_i$ for all $1 \leq i \leq k$. For a multi-index of non-negative integers $\mathbf{d} = (d_1, \ldots, d_k)$, we define $S_{\mathbf{d}}$ $(T_{\mathbf{d}}, S^*_{\mathbf{d}}, T^*_{\mathbf{d}})$ to be the graded piece of $S$ $(T, S^*, T^*)$ of degree $\mathbf{d}$. Similarly, let

$$S_{\leq \mathbf{d}} = \bigoplus_{\mathbf{i} \leq \mathbf{d}} S_{\mathbf{i}}.$$

We use analogous notation for the rings $T, S^*, T^*$.

**Lemma 2.28.** *Let $W \subseteq S^*_{\leq \mathbf{d}}$ be a subspace. Let*

$$E = S_{d_1+1,0,\ldots,0} \cup S_{0,d_2+1,0,\ldots,0} \cup \cdots \cup S_{0,\ldots,0,d_k+1}.$$

*The ideal* $\operatorname{Ann}(W)^{\text{hom}} \subseteq T$ *is generated by*

$$\{\zeta^{\text{hom}} | \zeta \in \operatorname{Ann}(W) \cap S_{\leq \mathbf{d}}\} \cup \{\zeta^{\text{hom}} | \zeta \in E\}.$$

*Proof.* Suppose $\theta \in \operatorname{Ann}(W)^{\text{hom}}$ is a homogenous polynomial. Then $\theta^{\text{deh}} \in \operatorname{Ann}(W)$ by Proposition 2.20. We write

$$\theta^{\text{deh}} = \zeta_{\leq \mathbf{d}} + \zeta_{\text{other}},$$

with $\zeta_{\leq \mathbf{d}} \in S_{\leq \mathbf{d}}$ and

$$\zeta_{\text{other}} \in \bigoplus_{\mathbf{j} \in \mathbb{N}^k, j_i > d_i \text{ for some } i} S_{\mathbf{j}}.$$

Since $\theta$ is homogeneous, the map $(\cdot)^{\text{deh}}$ gives a bijection between the terms of $\theta$ and the terms of $\theta^{\text{deh}}$. It follows that we can write

$$\theta = \theta_{\leq \mathbf{d}} + \theta_{\text{other}},$$

where $\theta_{\leq \mathbf{d}}, \theta_{\text{other}} \in T$ are homogeneous with $(\theta_{\leq \mathbf{d}})^{\text{deh}} = \zeta_{\leq \mathbf{d}}$ and $(\theta_{\text{other}})^{\text{deh}} = \zeta_{\text{other}}$. But then, by Proposition 2.16 applied to $\theta_{\leq \mathbf{d}}$ and $\theta_{\text{other}}$, we get

$$\theta = \alpha_{1,0}^{e_1} \cdots \alpha_{k,0}^{e_k} (\zeta_{\leq \mathbf{d}})^{\text{hom}} + \alpha_{1,0}^{e'_1} \cdots \alpha_{k,0}^{e'_k} (\zeta_{\text{other}})^{\text{hom}} \tag{2.8}$$

for some $e_i, e'_i \in \mathbb{N}$. As $\zeta_{\mathrm{other}} \in \operatorname{Ann}(W)$ because it has a large degree, we get that $\zeta_{\leq \mathbf{d}} \in \operatorname{Ann}(W)$. Moreover, $\zeta_{\mathrm{other}}$ is a sum of monomials, each one divisible by an element of the set $E$. Therefore, Equation (2.8) is a presentation of $\theta$ as a combination of something from $\{\zeta^{\mathrm{hom}} \mid \zeta \in \operatorname{Ann}(W) \cap S_{\leq \mathbf{d}}\}$ and something from $\{\zeta^{\mathrm{hom}} \mid \zeta \in E\}$. $\square$

If $A$ is an algebra graded in $\mathbb{Z}^k$, and $\mathbf{e} \in \mathbb{Z}^k$, then

$$H(A, \mathbf{e}) \coloneqq \dim_{\mathbb{C}} A_{\mathbf{e}}$$

is called the *Hilbert function* of $A$ at $\mathbf{e}$. We use this notation from now on.

**Lemma 2.29.** *If $W \subseteq S^*_{\leq \mathbf{d}}$ is a linear subspace, and $\mathbf{e} \geq \mathbf{d}$ is a multi-index of length $k$, then*

$$H(T/\operatorname{Ann}(W)^{\mathrm{hom}}, \mathbf{e}) = \dim_{\mathbb{C}} S/\operatorname{Ann}(W).$$

*Proof.* We claim that for any $\mathbf{e} \geq \mathbf{d}$ the map $(\cdot)^{\mathrm{deh}}$ induces a linear isomorphism

$$(T/\operatorname{Ann}(W)^{\mathrm{hom}})_{\mathbf{e}} \to S/\operatorname{Ann}(W). \tag{2.9}$$

To see this, consider the composition of dehomogenization and projection onto $\operatorname{Ann}(W)$

$$\chi : T_{\mathbf{e}} \xrightarrow{(\cdot)^{\mathrm{deh}}} S \to S/\operatorname{Ann}(W).$$

By Proposition 2.20,

$$\ker \chi = (\operatorname{Ann}(W)^{\mathrm{hom}})_{\mathbf{e}}.$$

It is enough to prove that $\chi$ is surjective. For this, take $\zeta \in S$ and write

$$\zeta = \zeta_{\leq \mathbf{d}} + \zeta_{\mathrm{other}},$$

with $\zeta_{\leq \mathbf{d}} \in S_{\leq \mathbf{d}}$ and

$$\zeta_{\mathrm{other}} \in \bigoplus_{\mathbf{j} \in \mathbb{N}^k, j_i > d_i \text{ for some } i} S_{\mathbf{j}}.$$

Since $S_{\leq \mathbf{d}} \subseteq S_{\leq \mathbf{e}}$, and the map $T_{\mathbf{e}} \xrightarrow{(\cdot)^{\mathrm{deh}}} S_{\leq \mathbf{e}}$ is surjective, there exists $\theta \in T_{\mathbf{e}}$ such that $\theta^{\mathrm{deh}} = \zeta_{\leq \mathbf{d}}$. Then $\chi(\theta) = [\zeta_{\leq \mathbf{d}}] = [\zeta]$. The claim that the map in Equation (2.9) is an isomorphism follows, and so does the statement of the proposition. $\square$

Up to this point, we have only used dehomogenization and homogenization between the rings $S$ and $T$. However, we also need another type of homogenization, on the graded dual rings.

**Definition 2.30.** If $\mathbf{e}$ is a multi-index with $\mathbf{e}$, and $f \in S^*$, then

$$f^{\mathrm{hom},\mathbf{e}} = \sum_{\mathbf{i} \leq \mathbf{e}} x^{(\mathbf{e}-\mathbf{i})} f_{\mathbf{i}},$$

where $x^{(\mathbf{j})} = x_{1,0}^{(j_1)} \cdots x_{k,0}^{(j_k)}$ for any $\mathbf{j} \in \mathbb{N}^k$.

Notice that we do not define any dehomogenization on the graded dual ring $T^*$. The following lemma shows the interplay between homogenization, dehomogenization and the apolarity action.

**Lemma 2.31.** *Let $\mathbf{m}, \mathbf{d}$, and $\mathbf{e}$ be multi-indices of length $k$. Let $\theta \in T_{\mathbf{m}}$, and $g \in S^*_{\leq \mathbf{d}}$. Suppose at least one of the following conditions holds:*

*(i)* $\mathbf{m} + \mathbf{d} \leq \mathbf{e}$, *or*

*(ii)* $k = 1, e = 2d - 1, m \geq d$, *and* $\theta \in (S_1)^{m-d+1} T_{d-1}$.

*Then we have the following equality.*

$$(\theta^{\mathrm{deh}} \lrcorner\, g)^{\mathrm{hom},\mathbf{e}-\mathbf{m}} = \theta \lrcorner\, g^{\mathrm{hom},\mathbf{e}}. \tag{2.10}$$

*Proof.* Note that both Point (i) and Point (ii) imply that for every term $u$ of $\theta^{\mathrm{deh}} \lrcorner\, g$ we have

$$\deg u \leq \mathbf{e} - \mathbf{m}. \tag{2.11}$$

Both sides of Equation (2.10) are bilinear, hence we may assume that $\theta$ and $g$ are monomials. Suppose

$$\theta = \left( \prod_{i=1}^{k} \alpha_{i,0}^{b_i} \right) \prod_{i=1}^{k} \prod_{j=1}^{n_i} \alpha_{i,j}^{a_{i,j}}, \text{ and}$$

$$g = \prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{i,j}^{(c_{i,j})}.$$

If $\deg g = \mathbf{l} = (l_1, \ldots, l_k)$, then

$$g^{\mathrm{hom},\mathbf{e}} = \left( \prod_{i=1}^{k} x_{i,0}^{(e_i - l_i)} \right) \prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{i,j}^{(c_{i,j})}.$$

We calculate the apolarity action

$$\theta^{\mathrm{deh}} \lrcorner\, g = \begin{cases} \prod_{i=1}^{k}\prod_{j=1}^{n_k} x_{i,j}^{(c_{i,j}-a_{i,j})} & \text{if } c_{i,j} \geq a_{i,j} \text{ for } 1 \leq i \leq k,\ 1 \leq j \leq n_i, \\ 0 & \text{otherwise.} \end{cases}$$

$$\theta \lrcorner\, g^{\mathrm{hom},\mathbf{e}} = \begin{cases} \left(\prod_{i=1}^{k} x_{i,0}^{(e_i-l_i-b_i)}\right)\prod_{i=1}^{k}\prod_{j=1}^{n_i} x_{i,j}^{(c_{i,j}-a_{i,j})} & \text{if } \begin{cases} c_{i,j} \geq a_{i,j} \\ e_i \geq l_i + b_i \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $e_i \geq l_i + b_i$ for $1 \leq i \leq k$. To see this, let us calculate $\deg(\theta^{\mathrm{deh}} \lrcorner\, g)$, which is well-defined, since we assumed that $\theta$ and $g$ are monomials. We know that $\deg g = \mathbf{l}$, $\deg\theta = \mathbf{m}$, and $\deg\theta^{\mathrm{deh}} = \mathbf{m} - \mathbf{b}$, hence

$$\deg(\theta^{\mathrm{deh}} \lrcorner\, g) = \mathbf{l} - \mathbf{m} + \mathbf{b}.$$

It follows that

$$\mathbf{e} - \mathbf{b} - \mathbf{l} = \mathbf{e} - \mathbf{m} - \deg(\theta^{\mathrm{deh}} \lrcorner\, g) \tag{2.12}$$

By Equation (2.11) we obtain that $\mathbf{e}-\mathbf{m}-\deg(\theta^{\mathrm{deh}} \lrcorner\, g) \geq \mathbf{0}$, so also $e_i \geq l_i+b_i$ for $1 \leq i \leq k$. But this means that the conditions

$$\begin{cases} c_{i,j} \geq a_{i,j} & \text{for } 1 \leq i \leq k \text{ and } 1 \leq j \leq n_i. \\ e_j \geq l_j + b_j & \text{for } 1 \leq j \leq k \end{cases}$$

and

$$\begin{cases} c_{i,j} \geq a_{i,j} & \text{for } 1 \leq i \leq k \text{ and } 1 \leq j \leq n_i \end{cases}$$

are equivalent. We want to homogenize $\theta^{\mathrm{deh}} \lrcorner\, g$ to degree $\mathbf{e} - \mathbf{m}$. Therefore, by Equation (2.12), the difference in degrees is equal to $\mathbf{e} - \mathbf{b} - \mathbf{l}$, hence, in order to homogenize $\theta^{\mathrm{deh}} \lrcorner\, g$, we need to multiply it by $\prod_{i=1}^{k} x_{i,0}^{(e_i-b_i-l_i)}$. This shows that Equation (2.10) holds for monomials, and thus for all polynomials. $\square$

In Definition 2.30 we defined homogenization on the dual ring $S^*$. We extend this for finite dimensional subspaces of $S^*$.

**Definition 2.32.** Assume $\mathbf{d}, \mathbf{e}$ are vectors of positive integers with $\mathbf{e} \geq \mathbf{d}$ and $W \subseteq S^*_{\leq \mathbf{d}}$ is a linear subspace. We let

$$W^{\mathrm{hom},\mathbf{e}} = \{f^{\mathrm{hom},\mathbf{e}} | f \in W\}$$

be the homogenization of a subspace.

**Lemma 2.33.** *Fix a multi-index* $\mathbf{e}$ *with* $\mathbf{e} \geq \mathbf{d}$*. Let* $W \subseteq S^*_{\leq \mathbf{d}}$ *be a subspace. Then we have:*

*(i)*
$$\mathrm{Ann}(W)^{\mathrm{hom}} \subseteq \mathrm{Ann}(W^{\mathrm{hom},\mathbf{e}}).$$

*(ii)*
$$(\mathrm{Ann}(W)^{\mathrm{hom}})_{\leq \mathbf{e}-\mathbf{d}} = \mathrm{Ann}(W^{\mathrm{hom},\mathbf{e}})_{\leq \mathbf{e}-\mathbf{d}}.$$

The proof is an adaptation of [8, Lemma 2] to the multigraded setting.

*Proof.* We begin with proving Point (ii). Let $\theta$ be a homogeneous polynomial in $T^*_{\leq \mathbf{e}-\mathbf{d}}$. We have

$$\begin{aligned} \theta \in \mathrm{Ann}(W)^{\mathrm{hom}} &\overset{\text{by Proposition 2.20}}{\iff} \theta^{\mathrm{deh}} \in \mathrm{Ann}(W) \\ &\iff \forall_{g \in W}(\theta^{\mathrm{deh}} \lrcorner\, g = 0) \\ &\overset{\text{by Lemma 2.31}}{\iff} \forall_{g \in W}(\theta \lrcorner\, g^{\mathrm{hom},\mathbf{e}} = 0) \\ &\iff \theta \in \mathrm{Ann}(W^{\mathrm{hom},\mathbf{e}}), \end{aligned}$$

which proves Point (ii).

Now we prove the first statement. Let $\tilde{\mathbf{e}} = \max(\mathbf{m}+\mathbf{d}, \mathbf{e})$. Suppose $\theta \in \mathrm{Ann}(W)^{\mathrm{hom}}$ is homogenous of degree $\mathbf{m}$ and that $g \in W$. Then by Proposition 2.20 we get $\theta^{\mathrm{deh}} \in \mathrm{Ann}(W)$, and therefore $\theta^{\mathrm{deh}} \lrcorner\, g = 0$, which implies that $\theta \lrcorner\, g^{\mathrm{hom},\tilde{\mathbf{e}}} = 0$ by Proposition 2.31. But

$$\begin{aligned} \theta \lrcorner\, g^{\mathrm{hom},\mathbf{e}} &= \theta \lrcorner\, ((\alpha_{1,0}^{\tilde{e}_1 - e_1} \cdots \alpha_{k,0}^{\tilde{e}_k - e_k}) \lrcorner\, g^{\mathrm{hom},\tilde{\mathbf{e}}}) \\ &= \alpha_{1,0}^{\tilde{e}_1 - e_1} \cdots \alpha_{k,0}^{\tilde{e}_k - e_k} \lrcorner\, (\theta \lrcorner\, g^{hom,\tilde{\mathbf{e}}}) \\ &= 0. \end{aligned}$$

□

## 2.7 Dehomogenization and homogenization in the Veronese case

We go to the simple case when $k = 1$. In order to make the results in Sections 3.6 and 6.2 stronger than in the case of the Segre-Veronese, we need to analyze the Veronese case in more detail. This part of the thesis comes from [47, Section 3].

Here the coordinate rings are

$$\begin{aligned} S &= \mathbb{C}[\alpha_1, \ldots, \alpha_n], \\ T &= \mathbb{C}[\alpha_0, \alpha_1, \ldots, \alpha_n]. \end{aligned}$$

We dehomogenize with respect to the variable $\alpha_0$. The graded duals are

$$\begin{aligned} S^* &= \mathbb{C}_{dp}[x_1, \ldots, x_n], \\ T^* &= \mathbb{C}_{dp}[x_0, x_1, \ldots, x_n]. \end{aligned}$$

**Lemma 2.34.** *Let $f = f_d + f_{d-1} + \cdots + f_0$ be a polynomial of degree $d \geq 2$ in $S^*$ where $f_i \in S_i^*$. Assume that $f_d$ is not a divided power of a linear form. Then $\operatorname{Ann}(f)^{\mathrm{hom}} \subseteq T$ has a set of minimal generators of degrees not greater than $d$.*

*Proof.* We have $S_{d+1} \subseteq \operatorname{Ann}(f)$ so we may choose a set of generators of $\operatorname{Ann}(f)$ of the form

$$\operatorname{Ann}(f) = (\{\alpha^{\mathbf{u}} \mid \mathbf{u} \in \mathbb{Z}_{\geq 0}^n \text{ s.t. } |\mathbf{u}| = d+1\}) + (\zeta_1, \ldots, \zeta_k) \text{ with } \deg(\zeta_i) \leq d.$$

Using Buchberger's algorithm for this set of generators and grevlex monomial order, we obtain a Gröbner basis of $\operatorname{Ann}(f)$ of the form

$$\{\alpha^{\mathbf{u}} \mid \mathbf{u} \in \mathbb{Z}_{\geq 0}^n \text{ s.t. } |\mathbf{u}| = d+1\} \cup \{\zeta_1, \ldots, \zeta_k\} \cup \{\xi_1, \ldots, \xi_l\}. \tag{2.13}$$

We claim that $\deg \xi_i \leq d$. Let $\mathcal{G} = \{\alpha^{\mathbf{u}} \mid \mathbf{u} \in \mathbb{Z}_{\geq 0}^n \text{ s.t. } |\mathbf{u}| = d+1\}$. Note that each $S$-polynomial considered in the Buchberger's algorithm is divided with remainder by a set of polynomials containing $\mathcal{G}$. Therefore, $S$-polynomials of degree at least $d+1$ do not give new elements of the Gröbner basis.

The ideal $\operatorname{Ann}(f)^{\mathrm{hom}}$ is generated by the homogenizations of the elements in Equation (2.13) ([38, Theorem 8.4.4]). It is enough to show that we can replace the monomial generators of degree $d+1$ written above by some generators of degree not greater than $d$. Let $\mathbf{u} \in \mathbb{Z}_{\geq 0}^n$ with $|\mathbf{u}| = d+1$. Then in $S$, we can write $\alpha^{\mathbf{u}} = \sum_{i=1}^m \theta_i \gamma_i$ for some $\theta_i \in \operatorname{Ann}(f_d)_d$ and $\gamma_i \in S_1$ ([23, Prop. 1.6]). We have $\theta_i \in \operatorname{Ann}(f)$ for degree reasons. Therefore $\alpha^{\mathbf{u}} \in ((\operatorname{Ann}(f)^{\mathrm{hom}})_{\leq d})$ as an element of $T$. $\square$

**Lemma 2.35.** *Let $f = f_d + f_{d-1} + \ldots + f_0$ be a degree $d \geq 1$ polynomial in $S^*$ and $r = \dim_{\mathbb{C}} S/\operatorname{Ann}(f)$. Let $e = 2d - 1$. We have $H(T/\operatorname{Ann}(f^{\mathrm{hom},e}), d) = r$ or $H(T/\operatorname{Ann}(f^{\mathrm{hom},e}), d) = r - 1$. Moreover, in the latter case we get $\operatorname{Ann}(f^{\mathrm{hom},e}) = (\alpha_0^d + \rho) + \operatorname{Ann}(f)^{\mathrm{hom}}$, where $\rho \in T_d$ has degree smaller than $d$ with respect to $\alpha_0$.*

*Proof.* We start with the following
**Observation.** Assume that $m \geq 0$. For $\Gamma = \alpha_0^d \theta_{1+m} + \alpha_0^{d-1}\theta_{2+m} + \ldots + \theta_{d+1+m}$ we have

$$\Gamma \lrcorner f^{\mathrm{hom},e} = 0 \Rightarrow \Gamma \in \operatorname{Ann}(f)^{\mathrm{hom}}.$$

Indeed, by Lemma 2.31, Point (ii), we have

$$\Gamma \lrcorner f^{\mathrm{hom},e} = (\Gamma^{\mathrm{deh}} \lrcorner f)^{\mathrm{hom},e-m} = 0,$$

which implies that $\Gamma^{\mathrm{deh}} \lrcorner f = 0$. Then $\Gamma \in \operatorname{Ann}(f)^{\mathrm{hom}}$.

We claim that $\operatorname{Ann}(f^{\mathrm{hom},e})$ has at most one minimal homogeneous generator of degree $d$ modulo the generators of $(\operatorname{Ann}(f)^{\mathrm{hom}})_d$. Indeed, by the above observation with $m = 0$, any such generator is (up to a scalar) of the form $\alpha_0^d + \rho$, where $\alpha_0^d$ does not divide any monomial in $\rho$. Given two such generators, say $\alpha_0^d + \rho$ and $\alpha_0^d + \rho'$, we have $\alpha_0^d + \rho = (\alpha_0^d + \rho') + (\rho - \rho')$. From the above observation for $m = 0$, it follows that $\rho - \rho'$ is in $(\operatorname{Ann}(f)^{\mathrm{hom}})_d$, so the second new generator is not needed. Therefore, either

$$H(T/\operatorname{Ann}(f^{\mathrm{hom},e}), d) = H(T/\operatorname{Ann}(f)^{\mathrm{hom}}, d) = r, \text{ or}$$
$$H(T/\operatorname{Ann}(f^{\mathrm{hom},e}), d) = H(T/\operatorname{Ann}(f)^{\mathrm{hom}}, d) - 1 = r - 1.$$

Now we assume $H(T/\operatorname{Ann}(f^{\mathrm{hom},e}), d) = r - 1$. Then there exists a homogeneous generator of $\operatorname{Ann}(f^{\mathrm{hom},e})$ of the form $\alpha_0^d + \rho$, where $\alpha_0^d$ does not divide any monomial in $\rho$. It is enough to show that for any $m \geq 0$, if $\Gamma = \alpha_0^{d-1}\theta_{1+m} + \alpha_0^{d-2}\theta_{2+m} + \ldots + \theta_{d+m}$ annihilates $f^{\mathrm{hom},e}$, then $\Gamma \in \operatorname{Ann}(f)^{\mathrm{hom}}$. This is the observation from the beginning. □

The following result is similar to Lemma 2.29. It compares the Hilbert functions of two related quotient algebras, one of $S$ and one of $T$. We use it in the proof of Part (iii) of Theorem 3.31.

**Lemma 2.36.** *Let $J \subseteq T$ be a homogeneous ideal and $\theta = \alpha_0^d + \rho$ be an element of $J_d$ with $\rho$ of degree smaller than $d$ with respect to $\alpha_0$. Consider the elimination ideal $J^c = J \cap S$. Then for any integer $e$ we have*

$$H(T/J, e) \leq H(S/J^c, e) + H(S/J^c, e-1) + \ldots + H(S/J^c, e-d+1).$$

*Proof.* Let $\prec$ be a graded lexicographic order on $T$ with $\alpha_n \prec \alpha_{n-1} \prec \ldots \prec \alpha_0$ and consider its restriction $\prec'$ to $S$. It follows from [42, Theorem 15.3]) that $H(T/J, e)$ is the number of monomials of degree $e$, not in $\mathrm{LT}_{\prec}(J)$. Observe that every monomial divisible by $\alpha_0^d$ is in $\mathrm{LT}_{\prec}(J)$. Therefore, we have

$$H(T/J, e) = \sum_{i=0}^{d-1} \#\{\mu \in S_{e-i} \mid \mu \text{ is a monomial and } \alpha_0^i \mu \notin \mathrm{LT}_{\prec}(J)\}.$$

Fix $0 \leq i \leq d-1$ and let $\mu$ be a monomial of degree $e-i$ from $S$. If $\mu \in \mathrm{LT}_{\prec'}(J^c)$, then there is a homogeneous $\zeta \in J^c$ such that $\mathrm{LM}_{\prec'}(\zeta) = \mu$. Therefore, $\alpha_0^i \zeta \in J$ and $\mathrm{LM}_{\prec}(\alpha_0^i \zeta) = \alpha_0^i \mu$. Thus for $i \in \{0, \ldots, d-1\}$ we have

$$\#\{\mu \in S_{e-i} \mid \mu \text{ is a monomial and } \alpha_0^i \mu \notin \mathrm{LT}_{\prec}(J)\} \leq H(S/J^c, e-i).$$

□

# Chapter 3

# Secant varieties

This chapter is concerned with secant varieties. Section 3.1 is devoted to some of their basic properties. Sections 3.2 and 3.3 introduce apolarity in its standard and border version. In Sections 3.4 and 3.5 we discuss equations of secant varieties. Section 3.6 is about bounding the cactus rank of special forms. The goals of Section 3.7 are proving Proposition 1.19 and discussing some applications.

Throughout the chapter, we work over $\mathbb{C}$.

## 3.1 Basic properties

In this section we review the basic properties of secant varieties. Assume $W$ is a vector space over $\mathbb{C}$, and that $X \subseteq \mathbb{P}W$ is a variety not contained in any hyperplane. Recall the definition of the secant varieties of $X$ given in Definition 1.11.

**Proposition 3.1.** *The algebraic set $\sigma_r(X)$ is irreducible for any $r \geq 1$.*

*Proof.* For any projective algebraic set $Y \subseteq \mathbb{P}W$, let $\hat{Y} \subseteq W$ denote the affine cone of $Y$. Consider the map

$$\varphi : \underbrace{\hat{X} \times \cdots \times \hat{X}}_{r \text{ times}} \to W, \text{ given by}$$
$$(\hat{x}_1, \ldots, \hat{x}_r) \mapsto \hat{x}_1 + \cdots + \hat{x}_r.$$

Since $X$ is irreducible, so is $\hat{X}$. It follows that the product $(\hat{X})^r$ is irreducible, and hence the algebraic set

$$\overline{\varphi((\hat{X})^r)} = \widehat{\sigma_r(X)}$$

is irreducible. $\square$

Let $\mathbb{P}\hat{T}_q X$ denote the projective tangent space of $X$ embedded in $\mathbb{P}W$ at a point $q$, i.e. the projectivization of the affine tangent space to the affine cone of $X$. The following Lemma is an important tool.

**Proposition 3.2** (Terracini's Lemma)**.** *Assume that $X \subseteq \mathbb{P}W$ is a non-degenerate projective variety. Let $r$ be a positive integer. Then for $r$ general points $p_1, \ldots, p_r \in X$ and a general point $q \in \langle p_1, \ldots, p_r \rangle$ we have*

$$\mathbb{P}\hat{T}_q \sigma_r(X) = \langle \mathbb{P}\hat{T}_{p_1} X, \ldots, \mathbb{P}\hat{T}_{p_r} X \rangle.$$

For a proof, see [62, Section 5.3] or [79, Chapter V, Proposition 1.4].

**Corollary 3.3.** *The dimension of $\sigma_r(X)$ is not greater than $r(\dim X + 1) - 1$.*

**Corollary 3.4.** *Assume $X \subseteq \mathbb{P}W$ is a non-degenerate projective variety. If for some $r$ the variety $\sigma_r(X)$ is strictly contained in $\mathbb{P}W$, then the inequality $\dim \sigma_{r+1}(X) > \dim \sigma_r(X)$ holds.*

*Proof.* Suppose we have $\dim \sigma_{r+1}(X) = \dim \sigma_r(X)$. It follows that for general points $p_1, \ldots, p_r, q \in X$ we get

$$\mathbb{P}\hat{T}_q X \subseteq \langle \mathbb{P}\hat{T}_{p_1} X, \ldots, \mathbb{P}\hat{T}_{p_r} X \rangle,$$

which means that all the subsequent secant varieties will be equal to $\sigma_r(X)$. This is a contradiction, as $X$ is not contained in any hyperplane. $\square$

**Definition 3.5.** When we have $\dim \sigma_r(X) = \min(\dim \mathbb{P}W, r(\dim X + 1) - 1)$, we say that $\sigma_r(X)$ is of *expected dimension*.

## 3.2 Classical Apolarity Lemma

In this section, we formulate the Apolarity Lemma. To our knowledge, it first appeared in [57, Lemma 1.15] for $\mathbb{P}^n$. Later it was also stated in the "if and only if" form—the point $F$ has rank at most $r$ if and only if there exists a radical ideal of $r$ points contained in $\mathrm{Ann}(F)$. Since its formulation, it was clear to the experts that it can be generalized in two ways: first by allowing the embedded variety to be something other than $\mathbb{P}^n$, second by demanding that we calculate the simultaneous rank of many forms of the same degree.

The former generalization was done independently in the author's master thesis [44, Chapter 3] and in [48, Lemma 1.3].

Since we sometimes need the Apolarity Lemma to calculate the simulteneous rank of many forms, and this is done neither in [44] nor in [48], we provide the proofs. Still, they are not much different than the ones in the cited sources.

Before we state the most common version of the Apolarity Lemma, we give a simpler version, which has the advantage that it works for any projective variety $X \subseteq \mathbb{P}W$ embedded by the complete linear system of a very ample line bundle $\mathcal{L}$. Then we have $W = H^0(X, \mathcal{L})^*$.

Note that for any closed subscheme $R \hookrightarrow X$ with ideal sheaf $\mathcal{I}_R$, and for any line bundle $\mathcal{L}$ on $X$, the vector subspace $H^0(X, \mathcal{I}_R \otimes \mathcal{L}) \subseteq H^0(X, \mathcal{L})$ consists of sections which pull back to zero on $R$. Let

$$(\cdot \lrcorner \cdot) : H^0(X, \mathcal{L}) \otimes H^0(X, \mathcal{L})^* \to \mathbb{C}$$

denote the natural pairing. The notation agrees with the one in Section 2.5, in particular with Equation (2.6) when $X$ is a toric variety. Now we are ready to formulate the Apolarity Lemma, the simple version:

**Proposition 3.6** (Apolarity Lemma, first version)**.** *Let $V \subseteq H^0(X, \mathcal{L})^*$ be a non-zero linear subspace. Let $R \hookrightarrow X$ be a closed subscheme with ideal sheaf $\mathcal{I}_R \subseteq \mathcal{O}_X$. Then we have the following equivalence*

$$\mathbb{P}(V) \subseteq \langle R \rangle \iff H^0(X, \mathcal{I}_R \otimes \mathcal{L}) \lrcorner V = 0.$$

*Proof.* Take any $\theta \in H^0(X, \mathcal{L})$, let $H_\theta$ be the corresponding hyperplane in $\mathbb{P}(H^0(X, \mathcal{L})^*)$. The following equivalence holds

$$\langle R \rangle \subseteq H_\theta \iff \theta \in H^0(X, \mathcal{I}_R \otimes \mathcal{L}).$$

For any $R$ we have

$$\begin{aligned}
\mathbb{P}(V) \subseteq \langle R \rangle &\iff \forall_{\theta \in H^0(X,\mathcal{L})} (\langle R \rangle \subseteq H_\theta \implies \mathbb{P}(V) \subseteq H_\theta) \\
&\iff \forall_{\theta \in H^0(X,\mathcal{L})} (\theta \in H^0(X, \mathcal{I}_R \otimes \mathcal{L}) \implies \mathbb{P}(V) \subseteq H_\theta) \\
&\iff \forall_{\theta \in H^0(X,\mathcal{L})} (\theta \in H^0(X, \mathcal{I}_R \otimes \mathcal{L}) \implies \theta \lrcorner V = 0) \\
&\iff H^0(X, \mathcal{I}_R \otimes \mathcal{L}) \lrcorner V = 0.
\end{aligned}$$

□

Let $X$ be a projective normal toric variety. Let $T$ be the Cox ring of $X$ (see the beginning of Section 2.1), and $T^*$ be graded dual (see Section 2.5). Suppose $X$ is embedded by the complete linear system of a very ample line bundle $\mathcal{O}(\eta)$, where $\eta \in \operatorname{Pic} X$. We denote this embedding by

$$\varphi : X \to \mathbb{P}T^*_\eta.$$

Before we give the common version of the Apolarity Lemma for toric varieties, we generalize the basic of definitions of ranks and cactus ranks to linear spaces of forms of the same degree.

**Definition 3.7.** For a positive integer $k$, the rank of a $k$-dimensional linear subspace $V$ of $T^*_\eta$ is

$$\mathrm{r}(V) = \min\{r \in \mathbb{Z}_{>0} | \mathbb{P}V \subseteq \langle \varphi(p_1), \dots, \varphi(p_r)\rangle \text{ for some } p_1, \dots, p_r \in X\}.$$

For positive integers $r, k$, the $(r,k)$-th Grassmann secant variety of $X \xrightarrow{\varphi} \mathbb{P}(H^0(X, T^*_\eta))$ is

$$\sigma_{r,k}(\varphi(X)) = \overline{\{[V] \in \mathrm{Gr}(k, T^*_\eta) |\, \mathrm{r}(V) \leq r\}}.$$

The cactus rank of a $k$-dimensional linear subspace $V$ of $T^*_\eta$ is

$$\begin{aligned}\mathrm{cr}(V) = \min\{r \in \mathbb{Z}_{>0} \,|&\text{there exists } R \hookrightarrow X \text{ a finite scheme} \\ &\text{of length } r \text{ such that } \mathbb{P}V \subseteq \langle \varphi(R)\rangle\}.\end{aligned}$$

The $(r,k)$-th Grassmann cactus variety of $X \xrightarrow{\varphi} \mathbb{P}(H^0(X, \mathcal{L})^*)$ is

$$\kappa_{r,k}(\varphi(X)) = \overline{\{[V] \in \mathrm{Gr}(k, T_\eta) |\, \mathrm{cr}(V) \leq r\}}.$$

For the next lemma, recall that for a linear subspace $V \subseteq T^*$, the set $\mathrm{Ann}(V) \subseteq T$ is the annihilator of $V$ with respect to the $\lrcorner$ action. It is an ideal of $T$. Moreover, it is homogeneous, if $V$ is contained in a graded piece of $T^*$.

**Lemma 3.8.** *Let $\eta \in \operatorname{Pic} X$. Let $J \subseteq T$ be a homogenous ideal, and $V \subseteq T^*_\eta$ be a subspace. We have*

$$J_\eta \lrcorner V = 0 \iff J \subseteq \mathrm{Ann}(V).$$

*Proof.* It suffices to prove that $J_\eta \subseteq \operatorname{Ann}(V)_\eta$ implies $J \subseteq \operatorname{Ann}(V)$. Suppose $J_\eta \subseteq \operatorname{Ann}(V)_\eta$. Take any $\theta \in J_\epsilon$ for some $\epsilon \in \operatorname{Cl} X_\Sigma$. We want to show that $\theta \lrcorner V = 0$. We have $T_{\eta-\epsilon} \cdot \theta \subseteq J_\eta$, because $\theta$ is in the ideal. This means that $(T_{\eta-\epsilon} \cdot \theta) \lrcorner V = 0$, i.e. $T_{\eta-\epsilon} \lrcorner (\theta \lrcorner V) = 0$. Now, for each $F \in V$ we know that $\theta \lrcorner F$ is an element of $T^*_{\eta-\epsilon}$, which is zero when multiplied by anything from $T_{\eta-\epsilon}$. It follows that $\theta \lrcorner F$ is zero. Hence, $\theta \lrcorner V = 0$. □

Finally, we are ready to give the most common version of the Apolarity Lemma. See Definition 2.4 for what it means for an ideal $I \subseteq T$ to define a subscheme $R \hookrightarrow X$.

**Theorem 3.9** (Apolarity Lemma, second version)**.** *Let $R \hookrightarrow X$ be closed subscheme, and $I \subseteq T$ be a $B$-saturated ideal defining it. For a linear subspace $V \subseteq T^*_\eta$ we have*

$$V \subseteq \langle R \rangle \iff I \subseteq \operatorname{Ann}(V).$$

*Proof.* From Proposition 3.6 we know that $V \subseteq \langle R \rangle$ if and only if $I_{\mathrm{Cl}}(R)_\eta \subseteq \operatorname{Ann}(V)_\eta$. However, by Corollary 2.10, $I$ agrees with $I_{\mathrm{Cl}}(R)$ in degree $\eta$. It remains to prove that $I_\eta \subseteq \operatorname{Ann}(V)_\eta$ implies $I \subseteq \operatorname{Ann}(V)$. But this follows from Lemma 3.8. □

## 3.3 Border rank and border apolarity

For the following definition, we work in the same setting as in Section 3.2. Let $X$ be a projective normal toric variety. Let $T$ be the Cox ring of $X$ (see the beginning of Section 2.1), and $T^*$ be graded dual (see Section 2.5). Suppose $X$ is embedded by the complete linear system of a very ample line bundle $\mathcal{O}(\eta)$, where $\eta \in \operatorname{Pic} X$. We denote this embedding by

$$\varphi : X \to \mathbb{P}T^*_\eta.$$

**Definition 3.10.** The border rank of a $k$-dimensional linear space $V \subseteq T^*_\eta$ is

$$\operatorname{br}(V) = \min\{r \in \mathbb{Z}_{>0} | [V] \in \sigma_{r,k}(\varphi(X))\}.$$

The border cactus rank of $V$ is

$$\operatorname{bcr}(V) = \min\{r \in \mathbb{Z}_{>0} | [V] \in \kappa_{r,k}(\varphi(X)\}.$$

If $k = 1$, i.e. if $V = \langle F \rangle$ for an element $F \in T^*_\eta$ we obtain the classical notion of border rank of $F$, as in [62, Chapter 3].

Note that the border rank is more natural from the point of view of algebraic geometry than the rank, since it provides a condition for $[F]$ to be a point of a Zariski closed subset of $\mathbb{P}T^*_\eta$. However, the variant of apolarity for the border rank has been stated only very recently (see [22] for the case $V = \langle F \rangle$ and [21] for the general case). Nevertheless, it has already been applied in [36],[56], and [66].

However, we only need border apolarity for the Segre-Veronese case. We define the coordinate ring and the graded dual ring to be

$$T = \mathbb{C}[\alpha_{1,0}, \dots, \alpha_{1,n_1}, \dots, \alpha_{k,0}, \dots, \alpha_{k,n_k}],$$
$$T^* = \mathbb{C}_{dp}[x_{1,0}, \dots, x_{1,n_1}, \dots, x_{k,0}, \dots, x_{k,n_k}].$$

The rings $T$, and $T^*$ are naturally graded by $\mathbb{Z}^k$. As before, for a vector of non-negative integers $\mathbf{d} = (d_1, \dots, d_k)$, we define $T_{\mathbf{d}}$ $(T^*_{\mathbf{d}})$ to be the graded piece of $T(T^*)$ of degree $\mathbf{d}$.

In Theorem 3.12, we consider the cactus and border cactus rank with respect to the Segre-Veronese embedding $v_{\mathbf{d}}$ of degree $\mathbf{d}$ for a multi-index $\mathbf{d}$. This is the map attached to the linear system $|\mathcal{O}(\mathbf{d})|$ or, equivalently, it is given on points by

$$\begin{aligned} v_{\mathbf{d}} \colon \mathbb{P}T^*_{1,0,\dots,0} \times \cdots \times \mathbb{P}T^*_{0,\dots,0,1} &\to \mathbb{P}T^*_{\mathbf{d}} \\ ([l_1], \dots, [l_k]) &\mapsto [l_1^{(d_1)} \cdots l_k^{(d_k)}]. \end{aligned}$$

For simplicity, let us write $X = v_{\mathbf{d}}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$.

**Definition 3.11.** For a positive integer $s$, let $h_s \colon \mathbb{N}^k \to \mathbb{N}$ be given by

$$h_s(\mathbf{d}) = \min\{\dim_{\mathbb{C}} T_{\mathbf{d}}, s\}.$$

Notice that $h_s$ depends on the numbers $n_1, \dots, n_k$.

We have to consider multigraded Hilbert schemes, see [52]. Denote by $\mathrm{Hilb}^{h_r}_T$ the multigraded Hilbert scheme associated with the polynomial ring $T$ (with the natural grading by $\mathbb{Z}^k$, and Hilbert function $h_r$). Let $\mathrm{Slip}_{r,X}$ be the closure in $\mathrm{Hilb}^{h_r}_T$ of points corresponding to $B$-saturated ideals of $r$ points.

**Theorem 3.12** (Border Apolarity Lemma)**.** *Let $r$ be a positive integer, $\mathbf{d}$ be a vector of positive integers of length $k$, and $V \subseteq T^*_{\mathbf{d}}$ be a linear subspace. Then* $\mathrm{br}(V) \leq r$ *holds if and only if there exists an ideal $[I] \in \mathrm{Slip}_{r,X}$ such that*

$$I \subseteq \mathrm{Ann}(V).$$

The proof is just a version of [22, Lemma 3.16], but for subspaces. Since we need it in the more general version, we give it here. Theorem 3.12 follows directly from Lemma 3.13.

**Lemma 3.13.** *Let $r$ be a positive integer, set $k = \dim_{\mathbb{C}} V$, $r' = h_r(\mathbf{d})$, and consider the $(r,k)$-th Grassmann secant variety $\sigma_{r,k}(X)$. The natural map $p : \mathrm{Slip}_{r,X} \to \mathrm{Gr}(r', T^*_{\mathbf{d}})$ taking a homogeneous ideal $I$ to $I^{\perp}_{\mathbf{d}}$ exists and is regular. Define $\mathcal{U} \subseteq \mathrm{Slip}_{r,X} \times \mathrm{Gr}(k, T^*_{\mathbf{d}})$ to be the pullback via $p$ of an indicence subbundle on the Grassmannian $\mathrm{Gr}(k, T^*_{\mathbf{d}})$, i.e.*

$$\mathcal{U} = \{(I,V) \mid I \in \mathrm{Slip}_{r,X}, V \in \mathrm{Gr}(k, T^*_{\mathbf{d}}), V \subseteq p(I)\}.$$

*Then the Grassmann secant variety $\sigma_{r,k}(X)$ is equal to the image of $\mathcal{U}$ under the projection $\mathcal{U} \to \mathrm{Gr}(k, T^*_{\mathbf{d}})$:*

$$\sigma_{r,k}(X) = \{V \in \mathrm{Gr}(k, T^*_{\mathbf{d}}) \mid \exists I \in \mathrm{Slip}_{r,X} \text{ such that } V \subseteq p(I)\}.$$

*Proof.* The natural map $p$ exists and is regular by the universal properties of the Grassmannian $\mathrm{Gr}(r', T^*_{\mathbf{d}})$ and of the multigraded Hilbert scheme. To prove the second assertion, note that $\mathrm{Slip}_{r,X}$ is projective by [52, Corollary 1.2]; thus $\mathcal{U}$ is projective and therefore the image of $\mathcal{U}$ under the projection is also closed in $\mathrm{Gr}(k, T^*_{\mathbf{d}})$. Moreover, by [22, Proposition 3.13] a very general ideal $I \in \mathrm{Slip}_{r,X}$ is the saturated ideal of $r$ distinct points $p_1, \ldots, p_r \in X$. The fiber $\mathrm{Gr}(k, I^{\perp}_{\mathbf{d}}) = \mathcal{U}_I \subseteq \{I\} \times \mathrm{Gr}(k, T^*_{\mathbf{d}})$ is the set of $k$-dimensional subspaces of $T^*_{\mathbf{d}}$ that are contained in the linear span $\langle p_1, \ldots, p_r \rangle$, hence the fiber is contained in $\sigma_{r,k}(X)$. On the other hand, reversing the above argument, we pick a very general point $V \in \sigma_{r,k}(X)$. Then we have $V \subseteq \langle p_1, \ldots, p_r \rangle$ for some points $p_1, \ldots, p_r$ in a very general position, and $I = I_{\mathrm{Cl}}(p_1, \ldots, p_r)$ is a $B$-saturated ideal with Hilbert function $h_r$ ([22, Lemma 3.6]). Thus $I \in \mathrm{Slip}_{r,X}$ and $V \in \mathrm{Gr}(k, I^{\perp}_{\mathbf{d}})$ by Theorem 3.9, and therefore $V$ is in the image of the second projection $\mathcal{U} \to \mathrm{Gr}(k, T^*_{\mathbf{d}})$. Hence, the image of $\mathcal{U} \to \mathrm{Gr}(k, T^*_{\mathbf{d}})$ is dense in $\sigma_{r,k}(X)$. $\square$

In order to formulate a version of apolarity for the border cactus rank, we need to consider more general Hilbert functions that just $h_r$. However, we only need it for the Veronese case, i.e. we assume that $k = 1$.

**Definition 3.14.** A function $h\colon \mathbb{N} \to \mathbb{N}$ satisfying the following conditions will be called an $s$-standard Hilbert function:

(a) $h(d) \leq h(d+1)$ for all nonnegative integers $d$,

(b) if $h(d) = h(d+1)$ for some nonnegative integer $d$, then $h(e) = s$ for all integers $e$ with $e \geq d$,

(c) $0 \leq h(d) \leq h_s(d)$ for all nonnegative integers $d$.

**Theorem 3.15** (Weak Border Cactus Apolarity Lemma)**.** *Let $d, r$ be positive integers, and $V \subseteq T_d^*$ be a non-zero subspace. If $\operatorname{bcr}(V) \leq r$, then there exists a homogeneous ideal $I \subseteq \operatorname{Ann}(V)$ such that $T/I$ has an $r$-standard Hilbert function for $\mathbb{P}^n$.*

For a proof see [21, Theorem 1.1].

## 3.4 Proofs of Theorems 1.17 and 1.18

Some parts of the following proofs are based on [45, Section 2]. However, some of them had to be modified so that they could help us to justify Proposition 3.21 in the next section.

Let $X$ be a complex projective variety embedded by the complete linear system of a very ample line bundle $\mathcal{L}$. For a closed subscheme $i\colon R \hookrightarrow X$, let $\mathcal{I}_R$ denote its ideal sheaf on $X$, and $\langle R \rangle$ denotes its linear span in $\mathbb{P}(H^0(X, \mathcal{L})^*)$ (as in Section 3.2). Recall that for any locally free sheaf $\mathcal{E}$ on $X$, the vector subspace $H^0(X, \mathcal{I}_R \otimes \mathcal{E}) \subseteq H^0(X, \mathcal{E})$ consists of the sections which pull back to zero on $R$.

Take a coherent sheaf $\mathcal{E}$ on $X$. Let $\lrcorner$ be the map

$$H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{L})^* \xrightarrow{\cdot \lrcorner \cdot} H^0(X, \mathcal{L} \otimes \mathcal{E}^\vee)^*$$

given by rearranging terms of Equation (1.2). The notation agrees with the one in Section 2.5, in particular with Equation (2.6) when $X$ is a toric variety and $\mathcal{E}$ is a line bundle. If we fix $F \in H^0(X, \mathcal{L})^*$, we get the so-called catalecticant homomorphism $C_F^{\mathcal{E}} = \cdot \lrcorner F$

$$C_F^{\mathcal{E}} : H^0(X, \mathcal{E}) \to H^0(X, \mathcal{L} \otimes \mathcal{E}^\vee)^*$$

corresponding to the tensor

$$j^{\mathcal{E}}(F) \in H^0(X, \mathcal{E})^* \otimes H^0(X, \mathcal{L} \otimes \mathcal{E}^\vee)^*.$$

Notice that $\operatorname{rank} C_F^{\mathcal{E}} = \operatorname{rank} j^{\mathcal{E}}(F)$.

**Proposition 3.16.** *Suppose $R \hookrightarrow X$ is a subscheme of $X$. Suppose $\mathcal{E}$ is a coherent sheaf on $X$. Then for any $F \in \langle R \rangle$ the image of the linear map*

$$H^0(X, \mathcal{I}_R \otimes \mathcal{E}) \to H^0(X, \mathcal{E}),$$

*induced by the map of sheaves $\mathcal{I}_R \otimes \mathcal{E} \to \mathcal{E}$, is contained in the kernel of $C_F^{\mathcal{E}}$.*

*Remark* 3.17. Proposition 3.16 can be thought of as a version of the apolarity lemma for coherent sheaves on $X$.

*Proof of Proposition 3.16.* We have the following commutative diagram of sheaves on $X$:

$$\begin{tikzcd} \mathcal{I}_R \otimes \mathcal{E} \otimes \mathcal{E}^\vee \otimes \mathcal{L} \arrow[r] \arrow[d] & \mathcal{I}_R \otimes \mathcal{L} \arrow[d] \\ \mathcal{E} \otimes \mathcal{E}^\vee \otimes \mathcal{L} \arrow[r] & \mathcal{L}. \end{tikzcd}$$

This gives rise to a commutative diagram of global sections:

$$\begin{tikzcd} H^0(X, \mathcal{I}_R \otimes \mathcal{E}) \otimes H^0(X, \mathcal{E}^\vee \otimes \mathcal{L}) \arrow[r] \arrow[d, dashed] & H^0(X, \mathcal{I}_R \otimes \mathcal{L}) \arrow[d] \arrow[dr, dotted, bend left] & \\ H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{E}^\vee \otimes \mathcal{L}) \arrow[r] \arrow[drr, dashed, bend right] & H^0(X, \mathcal{L}) \arrow[dr, "\cdot \lrcorner F"'] & \\ & & \mathbb{C}. \end{tikzcd}$$

The composition

$$H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{E}^\vee \otimes \mathcal{L}) \to H^0(X, \mathcal{L}) \xrightarrow{\cdot \lrcorner F} \mathbb{C}$$

is equal to the tensor corresponding to $C_F^{\mathcal{E}}$. Therefore, we want to show that the composition along the dashed path in the diagram is zero. It suffices to show that the dotted arrow is zero. But it is true by Proposition 3.6. □

For a complex variety $X$, a coherent sheaf $\mathcal{E}$ on $X$, and an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, we denote by $\mathcal{I} \cdot \mathcal{E}$ the image of the canonical map

$$\mathcal{I} \otimes \mathcal{E} \to \mathcal{E}.$$

**Proposition 3.18.** *Let $X$ be a complex proper variety, and let $U$ be a non-empty open subset of $X$. Let $R$ be a zero-dimensional subscheme of $X$ with ideal sheaf $\mathcal{I}_R$. Let $\mathcal{E}$ be a coherent sheaf on $X$ of rank $k$. If the support of $R$ is contained in $U$, and $\mathcal{E}$ is locally free on $U$, then the length of $R$ is greater or equal to*

$$\frac{h^0(X, \mathcal{E}) - h^0(X, \mathcal{I}_R \cdot \mathcal{E})}{k}.$$

*Proof.* We have an exact sequence

$$0 \to \mathcal{I}_R \to \mathcal{O}_X \to \mathcal{O}_R \to 0.$$

We tensor it with $\mathcal{E}$:

$$\mathcal{I}_R \otimes \mathcal{E} \to \mathcal{E} \to \mathcal{E}_{|R} \to 0. \tag{3.1}$$

The Sequence (3.1) does not have to be exact on the left, so let us replace the first term by the image of the map $\mathcal{I}_R \otimes \mathcal{E} \to \mathcal{E}$, getting the exact sequence

$$0 \to \mathcal{I}_R \cdot \mathcal{E} \to \mathcal{E} \to \mathcal{E}_{|R} \to 0.$$

After taking global sections (which are left-exact), we get an exact sequence

$$0 \to H^0(X, \mathcal{I}_R \cdot \mathcal{E}) \to H^0(X, \mathcal{E}) \to H^0(R, \mathcal{E}_{|R}).$$

It follows that

$$h^0(R, \mathcal{E}_{|R}) \geq h^0(X, \mathcal{E}) - h^0(X, \mathcal{I}_R \cdot \mathcal{E}).$$

But on a zero-dimensional scheme, every locally free sheaf trivializes. This means $h^0(R, \mathcal{E}_{|R}) = h^0(R, \mathcal{O}_R^{\oplus k})$, which is equal to $k$ times the length of $R$. □

*Proof of Theorem 1.17.* Take any zero-dimensional scheme $R \hookrightarrow X$ such that $F \in \langle R \rangle$ and $\mathrm{cr}(F) = \operatorname{length} R$. Since we assume that $\mathcal{E}$ is locally free on $X$, we know that $\mathcal{I}_R \cdot \mathcal{E} = \mathcal{I}_R \otimes \mathcal{E}$. We have

$$\begin{aligned} \operatorname{length} R \geq \frac{h^0(X,\mathcal{E}) - h^0(X, \mathcal{I}_R \otimes \mathcal{E})}{k} \geq \frac{h^0(X,\mathcal{E}) - \dim\ker C_F^{\mathcal{E}}}{k} &= \frac{\dim\operatorname{im} C_F^{\mathcal{E}}}{k} \\ &= \frac{\operatorname{rank} j(F)}{k}, \end{aligned}$$

where the first inequality follows from Proposition 3.18 applied for $U = X$, and the second from Proposition 3.16. □

*Proof of Theorem 1.18.* Consider the canonical map $\mathcal{O}_X \otimes V_1^* \otimes V_2^* \to \mathcal{O}_X(1)$, which comes from the map $j : V_1^* \otimes V_2^* \to W^*$. After rearranging the factors, it becomes

$$f : \mathcal{O}_X \otimes V_1^* \to \mathcal{O}_X(1) \otimes V_2.$$

For every point $x \in X$, the linear map $f|_x \colon V_1^* \to V_2$ equals $j(x)$, so $\dim\operatorname{im} f|_x$ is constant. Thus $\mathcal{E} = \operatorname{im} f$ is a rank $k$ vector bundle. Pick any finite $R \subseteq X$ of length at most $r$. Then $\dim_{\mathbb{C}} H^0(R, \mathcal{E}|_R) \leq kr$. Consider the $\mathbb{C}$-linear map $V_1^* \to H^0(R, \mathcal{E}|_R), \psi \mapsto f(1 \otimes \psi)$. Its kernel $Z \subseteq V_1^*$ is of codimension at most $kr$. Then we get $\dim_{\mathbb{C}} Z^{\perp} \leq kr$. Moreover, we claim

$$j(R) \subseteq \mathbb{P}(Z^{\perp} \otimes V_2).$$

Indeed, $Z$ is also the kernel of the map $p \colon V_1^* \to H^0(R, \mathcal{O}_R(1) \otimes V_2) \cong H^0(R, \mathcal{O}_R(1)) \otimes V_2$. Thus, if we look at the corresponding linear map

$$q \colon V_1^* \otimes V_2^* \to W^* \to H^0(R, \mathcal{O}_R(1)),$$

we get that $q(Z \otimes V_2^*) = 0$. But this means that $j(R) \subseteq \mathbb{P}(Z^{\perp} \otimes V_2)$, as claimed.

Every map in $Z^{\perp} \otimes W$ has rank at most $\dim_{\mathbb{C}} Z^{\perp} \leq kr$, hence we have

$$\langle j(R) \rangle \subseteq \mathbb{P}(Z^{\perp} \otimes V_2) \subseteq \sigma_{kr}(\mathrm{Seg}(\mathbb{P}Z^{\perp} \times \mathbb{P}V_2)).$$

□

As a corollary of Theorem 1.17 and main theorems of [19], we get that equations coming from vector bundles are not enough to define secant varieties of the Veronese embedding in general. The slogan is that cactus varieties fill up the ambient space much quicker than the secant varieties.

**Corollary 3.19.** *Let $v_d : \mathbb{P}V \to \mathbb{P}S^dV$ be the $d$-th Veronese embedding of the $n$-dimensional complex projective space $\mathbb{P}V$. Let $r$ be a positive integer. For each vector bundle $\mathcal{E}$ on $\mathbb{P}V$ of rank $k$, let $Z_{r,\mathcal{E}} \subseteq \mathbb{P}S^dV$ be the vanishing set of equations coming from $(kr+1)$-th minors of $\mathcal{E}$, and let*

$$Z_r = \bigcap_{\mathcal{E}} Z_{r,\mathcal{E}}.$$

*Suppose either*

- *$n \geq 6$ and $r \geq 14$ or*
- *$n = 5$ and $r \geq 42$ or*
- *$n = 4$ and $r \geq 140$.*

*Suppose $d \geq 2r - 1$. Then $\sigma_r(v_d(\mathbb{P}V)) \subsetneq Z_r$.*

*Proof.* From [19, Proposition 6.2 and Theorem 1.4] for such $n, d, r$ we have $\sigma_r(v_d(\mathbb{P}V)) \subsetneq \kappa_r(v_d(\mathbb{P}V))$. Since from Theorem 1.17 we know that

$$\kappa_r(v_d(\mathbb{P}V)) \subseteq Z_r,$$

we get the desired result. □

## 3.5 Examples of equations

The following example is a private communication from Jarosław Buczyński.

**Example 3.20.** Suppose $A, B$ are 5-dimensional vector spaces. Let $\mathbb{P}A \times \mathbb{P}B \to \mathbb{P}(A \otimes B)$ be the Segre embedding, and consider

$$X = \sigma_2(\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B)) \hookrightarrow \mathbb{P}(A \otimes B),$$

so that $X$ is the projectivization of the set of matrices of rank at most 2. We study the second secant variety of $X$ (not of $\mathbb{P}A \times \mathbb{P}B$), which is the projectivization of the set of matrices of rank at most 4. The singular points of $X$ are the matrices of rank 1. Take $j : A \otimes B \to A \otimes B$ to be identity. The highest rank of $j(F)$ for $[F] \in X$ is 2. Hence Proposition 1.12 for $k = 2$ and $r = 2$ says that $\sigma_2(X) \subseteq \sigma_4(\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B))$, so that the determinant of $A \otimes B$ gives a non-trivial equation of $\sigma_2(X)$. But $\kappa_2(X)$ is $\mathbb{P}(A \otimes B)$, and therefore $\kappa_2(X) \nsubseteq \sigma_4(\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B))$. This follows from the fact that for any projective variety $Y$ the cactus variety $\kappa_2(Y)$ contains all projectivized tangent spaces at the points of $Y$ and that the projectivized tangent space at any singular point of $X$ is $\mathbb{P}(A \otimes B)$.

### 3.5.1 Equations given by coherent sheaves that are not vector bundles

The following proposition says that vector bundles are not the only sheaves that can give useful equations. In fact, equations given by other coherent sheaves have the advantage that they do not necessarily vanish on cactus varieties (see Remark 5.6). However, such equations are not so widely studied as the ones given by vector bundles.

**Proposition 3.21.** *Let $X$ be a complex projective variety embedded by the complete linear system of a line bundle $\mathcal{L}$. Suppose $\mathcal{F}$ is a coherent sheaf on $X$. Then for every point in $\hat{x} \in \hat{X}$ we have*

$$\operatorname{rank} j^{\mathcal{F}}(\hat{x}) \leq \operatorname{rank} \mathcal{F}.$$

*Therefore the conditions in Proposition 1.12 hold for $k = \operatorname{rank} \mathcal{F}$.*

*Proof.* Let $U$ be a dense open subset of $X$ such that $\mathcal{F}|_U$ is a locally free sheaf. We claim that $\operatorname{rank} j^{\mathcal{F}}(\hat{x}) \leq \operatorname{rank} \mathcal{F}$ for points $[\hat{x}] \in U$. Indeed, by Proposition 3.18 applied to the scheme $R = [\hat{x}] \in U$, we get that

$$\operatorname{rank} \mathcal{F} \geq \operatorname{codim}_{H^0(X,\mathcal{F})} H^0(X, \mathcal{I}_{[\hat{x}]} \cdot \mathcal{F}).$$

But then Proposition 3.16 tells us that this codimension is greater or equal to $\operatorname{rank} C^{\mathcal{F}}_{\hat{x}} = \operatorname{rank} j^{\mathcal{F}}(\hat{x})$, proving the claim.

The set of points $\hat{x} \in H^0(X, \mathcal{L})^*$ such that $\operatorname{rank} j^{\mathcal{F}}(\hat{x}) \leq \operatorname{rank} \mathcal{F}$ is closed, as it is the inverse image of the affine cone of the variety

$$\sigma_{\operatorname{rank} \mathcal{F}}(\operatorname{Seg}(\mathbb{P}H^0(X, \mathcal{F})^* \times \mathbb{P}H^0(X, \mathcal{L} \otimes \mathcal{F}^{\vee})^*))$$

under the map $j^{\mathcal{F}}$. Hence $\operatorname{rank} j^{\mathcal{F}}(\hat{x}) \leq \operatorname{rank} \mathcal{F}$ for all $\hat{x} \in \hat{X}$. $\square$

### 3.5.2 Equations given by reflexive sheaves of rank one

The main examples of determinantal equations are catalecticant equations. These can be either given by line bundles or by other reflexive sheaves of rank one. In the former case, they give equations of the cactus variety, and in the latter case, they can give equations of secant varieties which are not equations of the corresponding cactus variety (i.e. they can overcome the barriers, see Sections 1.3 and 1.4). Unfortunately, on smooth varieties all reflexive sheaves of rank one are line bundles, so in this way we do not get anything new.

Catalecticant equations can be written down in a simple way. Suppose $X_\Sigma$ is a toric variety embedded by the complete linear system of a very ample line bundle $\mathcal{O}(\eta)$. Suppose $T$ is the Cox ring of $X_\Sigma$. Let $\epsilon \in \operatorname{Cl} X_\Sigma$. Consider the linear map

$$\lrcorner : T_\epsilon \otimes T^*_\eta \to T^*_{\eta-\epsilon}$$

(see Equation (2.6)). We fix bases of $T_\epsilon$ and $T^*_{\eta-\epsilon}$, and interpret this map as a matrix with entries in $T_\eta$. Then the equations coming from the reflexive sheaf $\mathcal{O}(\epsilon)$ are given by the minors of this matrix. Therefore to understand the equations coming from the reflexive sheaves of rank one is the same as to understand the apolarity map $\lrcorner$.

The following proposition will be useful in Section 3.6 and Chapter 5.

**Proposition 3.22.** *For any class $\epsilon \in \operatorname{Cl} X_\Sigma$, and any $F \in T^*_\eta$ we have*

$$H(T/\operatorname{Ann}(F), \epsilon) \leq \operatorname{br}(F) \leq \operatorname{r}(F).$$

*Moreover, if $\epsilon \in \operatorname{Pic} X_\Sigma$, then for any $F \in T^*_\eta$*

$$H(T/\operatorname{Ann}(F), \epsilon) \leq \operatorname{bcr}(F) \leq \operatorname{cr}(F).$$

*Proof.* The first assertion follows from Proposition 3.21. The second assertion for the cactus rank follows from Theorem 1.17. But Proposition 1.16 implies that it also works for the border cactus rank. □

**Example 3.23.** If $d \geq 4$, then size 4 minors given by $\mathcal{O}(2)$ and $\mathcal{O}(1)$ cut out $\sigma_3(v_d(\mathbb{P}^n))$ scheme-theoretically. See [64, Chart on page 572].

**Example 3.24.** If $j \leq \binom{\delta+n-1}{n}$, then size $j+1$ minors of $\mathcal{O}(\delta)$ cut out a scheme, one of whose irreducible components is $\sigma_j(v_{2\delta}(\mathbb{P}^n)$. This is [57, Theorem 4.10A].

### 3.5.3 Equations given by Young flattenings

**Example 3.25** (Aronhold invariant)**.** Let $V$ be a 3-dimensional vector space over $\mathbb{C}$. We consider the third Veronese embedding $v_3 : \mathbb{P}V \to \mathbb{P}S^3V$. Take the linear map $\phi : S^3V \to (V \otimes \Lambda^2 V) \otimes (V^* \otimes V)$ given by first embedding $S^3V \subseteq V \otimes V \otimes V$, then tensoring with $\mathrm{id}_V \in V^* \otimes V$, then skew-symmetrizing. Then the principal Pfaffians of $\phi$ are (up to scale) equal to the classical Aronhold invariant, that is the equation of $\sigma_3(v_3(\mathbb{P}V)) \subseteq \mathbb{P}S^3V$, which is a hypersurface. See [64, Example 1.2.1] for more details.

We can put Example 3.25 into a more general setting of Young flattenings. By a partition we mean a non-increasing finite sequence of positive integers. We denote partitions by Greek letters $\mu, \nu, \lambda$. The group $GL(V)$ acts on the Young module $S_\mu V$. Let $X \subseteq S_\mu V$ be the unique closed orbit of this action. If we embed $S_\mu V$ into a tensor product of two representations

$$S_\mu V \subseteq S_\nu V \otimes S_\lambda V,$$

we get equations for the cactus varieties $\kappa_r(X)$ by Theorem 1.18.

If we go back to Example 3.25, it can be verified that the Aronhold invariant is given by the inclusion

$$S^3 V \subseteq S_{2,1} V \otimes S_{2,1} V.$$

Using this method, G. Ottaviani and J.M. Landsberg obtain equations of secant varieties of the Veronese surface (see [64, Section 4]).

### 3.5.4 Equations given by locally free sheaves of rank at least two

An interesting family of equations is given in the master thesis of Adam Michalik [67, Section 3.3]. He needs the following definition.

**Definition 3.26.** A presentation of a locally free sheaf $\mathcal{E}$ on $\mathbb{P}^n$ is a morphism of vector bundles

$$p_{\mathcal{E}} : \mathcal{L}_1 \to \mathcal{L}_0$$

such that $\mathcal{L}_0, \mathcal{L}_1$ are direct sums of line bundles on $\mathbb{P}^n$, and

$$\operatorname{im} p_{\mathcal{E}} = \mathcal{E}.$$

Notice that for each $d \in \mathbb{Z}$, the map $p_{\mathcal{E}}$ induces a morphism of sheaves on $\mathbb{P}^n$

$$\mathcal{O}(d) \to \mathcal{O}(d) \otimes \mathcal{H}om(\mathcal{L}_1, \mathcal{L}_0)$$

given by

$$\mathcal{O}(d)(U) \ni s \mapsto s \otimes p_{\mathcal{E}}|_U.$$

After rearranging the factors, and taking global sections, this gives rise to a linear map

$$P_{\mathcal{E}} : H^0(\mathbb{P}^n, \mathcal{L}_1) \otimes H^0(\mathbb{P}^n, \mathcal{L}_0^\vee \otimes \mathcal{O}(d)) \to H^0(\mathbb{P}^n, \mathcal{O}(d)).$$

The reason why this definition is useful is the following theorem. We consider the secant varieties to $d$-th Veronese embedding of $\mathbb{P}^n$. The following Theorem is a combination of [64, Proposition 5.1.1] and [67, Theorem 3.3.1]

**Theorem 3.27.** *Suppose that the maps*

$$H^0(\mathbb{P}^n, \mathcal{L}_1) \to H^0(\mathbb{P}^n, \mathcal{E}), \text{ and } H^0(\mathbb{P}^n, \mathcal{L}_0^\vee \otimes \mathcal{L}) \to H^0(\mathbb{P}^n, \mathcal{E}^\vee \otimes \mathcal{L})$$

*are surjective. Then for any positive integer $r$ the $(r \operatorname{rank} \mathcal{E} + 1)$-th minors of $P_{\mathcal{E}}$ vanish on $\sigma_r(v_d(\mathbb{P}^n))$.*

Michalik considers a specific vector bundle $\mathcal{E}$ on $\mathbb{P}^3$ of rank 2, called a null-correlation bundle, which can be given the following presentation $p_{\mathcal{E}} : \mathcal{O}^5 \to \mathcal{O}(2)^5$

$$\begin{bmatrix} 0 & -x_0^2 & x_2^2 & x_0x_1 + x_2x_3 & -x_0x_2 \\ x_0^2 & 0 & x_2x_3 - x_0x_1 & x_3^2 & -x_0x_3 \\ -x_2^2 & x_0x_1 - x_2x_3 & 0 & -x_1^2 & x_1x_2 \\ -x_0x_1 - x_2x_3 & -x_3^2 & x_1^2 & 0 & x_1x_3 \\ x_0x_2 & x_0x_3 & -x_1x_2 & -x_1x_3 & 0 \end{bmatrix}.$$

Using this, he proves the following theorem ([67, Theorem 3.3.2]).

**Theorem 3.28.** *Let $k, d, r$ be integers with $d, r > 0$. Then the $(2r+1)$-th minors of*

$$P_{\mathcal{E}} : H^0(\mathbb{P}^3, \mathcal{O}(k))^5) \to H^0(\mathbb{P}^3, \mathcal{H}om(\mathcal{O}(k+2)^5, \mathcal{O}(d)))$$

*vanish on $\sigma_r(v_d(\mathbb{P}^3))$.*

## 3.6 Bounding the cactus rank of special forms or spaces

In this section, we state some results concerning upper bounds on the cactus rank of (divided power) homogenization of forms. Notice that we continue to use the language of divided powers, as it makes the statements and proofs cleaner.

Let $S, T, S^*, T^*, S_{\mathbf{d}}, S_{\leq \mathbf{d}}$, etc. and variables $\alpha_{i,j}$ be as in Section 2.6. Recall the definition of Hilbert function $h_r$ given in Definition 3.11.

The following simple lemma will make the proofs easier.

**Lemma 3.29.** *Let $J, K \subseteq T$ be homogeneous ideals such that $J_{\mathbf{s}} = K_{\mathbf{s}}$ for a multi-index $\mathbf{s}$. If $K$ is generated in degrees at most $\mathbf{s}$, then $J_{\mathbf{t}} \supseteq K_{\mathbf{t}}$ for all multi-indices $\mathbf{t}$ with $\mathbf{t} \geq \mathbf{s}$.*

*Proof.* We have $J_{\mathbf{t}} \supseteq (J_{\mathbf{s}})_{\mathbf{t}} = (K_{\mathbf{s}})_{\mathbf{t}} = K_{\mathbf{t}}$. $\square$

We consider the cactus rank and border cactus rank with respect to the Segre-Veronese embedding of degree $\mathbf{d}$ for a multi-index $\mathbf{d}$. This is the map attached to the linear system $|\mathcal{O}_{\mathbb{P}T^*_{1,0,\ldots,0}\times\cdots\times\mathbb{P}T^*_{0,\ldots,0,1}}(\mathbf{d})|$ or, equivalently, it is given on points by

$$\begin{aligned} v_{\mathbf{d}} : \mathbb{P}T^*_{1,0,\ldots,0} \times \cdots \times \mathbb{P}T^*_{0,\ldots,0,1} &\to \mathbb{P}T^*_{\mathbf{d}} \\ ([l_1], \ldots, [l_r]) \qquad &\mapsto [l_1^{(d_1)} \cdots l_k^{(d_k)}]. \end{aligned}$$

**Theorem 3.30** (Multigrading)**.** *Let $\mathbf{e}, \mathbf{d}$ be multi-indices with $\mathbf{e} \geq \mathbf{d}$. Let $W \subseteq S^*_{\leq \mathbf{d}}$ be a linear subspace and $r = \dim_{\mathbb{C}} S/\operatorname{Ann}(W)$.*

*(i) The cactus rank $\operatorname{cr}(W^{\mathrm{hom},\mathbf{e}})$ of $W^{\mathrm{hom},\mathbf{e}}$ is not greater than $r$.*

*(ii) If we have $e_i \geq 2d_i$ for $1 \leq i \leq k$, and $W = \langle f \rangle$, the border cactus rank $\operatorname{bcr}(f^{\mathrm{hom},\mathbf{e}})$ of $f^{\mathrm{hom},\mathbf{e}}$ equals $r$.*

*(ii') If $k = 1$ (i.e. we are in the case of the Veronese embedding), and we have $e_1 \geq 2d_1$, then the border cactus rank $\operatorname{bcr}(W^{\mathrm{hom},e_1})$ of $W^{\mathrm{hom},e_1}$ equals $r$.*

*(iii) If we have $e_i \geq 2d_i + 1$ for $1 \leq i \leq k$, and $J \subseteq \operatorname{Ann}(W^{\mathrm{hom},\mathbf{e}})$ is a homogeneous ideal such that $T/J$ has Hilbert function $h_r$, then $\operatorname{sat}(J, B) = \operatorname{Ann}(W)^{\mathrm{hom}}$.*

*Proof.*

(i) We have $\operatorname{Ann}(W)^{\mathrm{hom}} \subseteq \operatorname{Ann}(W^{\mathrm{hom},\mathbf{e}})$ by Lemma 2.33(i). Let $R$ be the scheme defined by the ideal $\operatorname{Ann}(W)^{\mathrm{hom}}$. Since the Hilbert function of $T/\operatorname{Ann}(W)^{\mathrm{hom}}$ is $r$ in large enough degrees (Lemma 2.29), the length of $R$ is equal to $r$. The ideal $\operatorname{Ann}(W)^{\mathrm{hom}}$ is saturated (Remark 2.19), and it defines $R$. By Theorem 3.9, we have $W^{\mathrm{hom},\mathbf{e}} \subseteq \langle v_{\mathbf{e}}(R) \rangle$. Hence, the cactus rank of $W^{\mathrm{hom},\mathbf{e}}$ is at most $r$.

(ii) We have $H(T/\operatorname{Ann}(f)^{\text{hom}}, \mathbf{m}) = r$ for $\mathbf{m} \geq \mathbf{d}$ by Lemma 2.29. Therefore, by Lemma 2.33(ii) we have

$$H(T/\operatorname{Ann}(f^{\text{hom},\mathbf{e}}), \mathbf{e} - \mathbf{d}) = r.$$

By Proposition 3.22 we get $\operatorname{bcr}(f^{\text{hom},\mathbf{e}}) \geq r$, which together with Point (i) (and the standard inequality $\operatorname{bcr}(G) \leq \operatorname{cr}(G)$ for any $G \in T^*_{\mathbf{e}}$) implies that

$$\operatorname{bcr}(f^{\text{hom},\mathbf{e}}) = r.$$

(ii') We have $H(T/\operatorname{Ann}(W)^{\text{hom}}, m) = r$ for $m \geq d_1$ by Lemma 2.29. Therefore, by Lemma 2.33(ii) we have

$$H(T/\operatorname{Ann}(W^{\text{hom},e_1}), e_1 - d_1) = r.$$

Thus there exists no ideal $J \subseteq \operatorname{Ann}(W^{hom,e_1})$ such that $T/J$ has an $(r-1)$-standard Hilbert function for $\mathbb{P}^n$ (see Definition 3.14). By Theorem 3.15 we get $\operatorname{bcr}(W^{hom,e_1}) \geq r$, which together with Point (i) implies that $\operatorname{bcr}(W^{hom,e_1}) = r$.

(iii) Assume that $J \subseteq \operatorname{Ann}(W^{\text{hom},\mathbf{e}})$ is such that $T/J$ has Hilbert function $h_r$. By Lemmas 2.33(ii) and 2.29

$$H(T/\operatorname{Ann}(W^{\text{hom},\mathbf{e}}), \mathbf{e} - \mathbf{d}) = H(T/\operatorname{Ann}(W)^{\text{hom}}, \mathbf{e} - \mathbf{d}) = r.$$

We have $H(T/J, \mathbf{e} - \mathbf{d}) \leq r$, since the Hilbert function of $T/J$ is $h_r$, but we also have

$$H(T/J, \mathbf{e} - \mathbf{d}) \geq H(T/\operatorname{Ann}(W^{\text{hom},\mathbf{e}}), \mathbf{e} - \mathbf{d}) = r.$$

Thus the quotients $T/J$ and $T/\operatorname{Ann}(W^{\text{hom},\mathbf{e}})$ have the same Hilbert function $r$ at $\mathbf{e} - \mathbf{d}$ and $J \subseteq \operatorname{Ann}(W^{\text{hom},\mathbf{e}})$, hence

$$J_{\mathbf{e}-\mathbf{d}} = \operatorname{Ann}(W^{\text{hom},\mathbf{e}})_{\mathbf{e}-\mathbf{d}}.$$

In particular, we have

$$J_{\mathbf{e}-\mathbf{d}} = \operatorname{Ann}(W^{\text{hom},\mathbf{e}})_{\mathbf{e}-\mathbf{d}} = (\operatorname{Ann}(W)^{\text{hom}})_{\mathbf{e}-\mathbf{d}}.$$

Since by Lemma 2.28 the ideal $\operatorname{Ann}(W)^{\text{hom}}$ is generated in degrees smaller or equal to $(d_1 + 1, \ldots, d_k + 1)$, it follows from Lemma 3.29 that $J_{\mathbf{m}} \supseteq (\operatorname{Ann}(W)^{\text{hom}})_{\mathbf{m}}$ for every $\mathbf{m} \geq \mathbf{e} - \mathbf{d}$.

But ideals $J$ and $\operatorname{Ann}(W)^{\text{hom}}$ have the same Hilbert function for large enough degree, so they agree in large enough degree. Hence we have $\operatorname{sat}(J, B) = \operatorname{sat}(\operatorname{Ann}(W)^{\text{hom}}, B) = \operatorname{Ann}(W)^{\text{hom}}$.

$\square$

In the case when $W = \langle f \rangle$, and $k = 1$ (so that we consider the Veronese embedding), we get stronger results. They originally appeared in [47, Section 4]. We write $T = \mathbb{C}[\alpha_0, \dots, \alpha_n]$ and $T^* = \mathbb{C}_{dp}[x_0, \dots, x_n]$. Now $\alpha_0$ is the additional variable, with respect to which we homogenize.

**Theorem 3.31** (Standard grading, polynomial). *Let $f = f_d + \ldots + f_0 \in S^* = \mathbb{C}_{dp}[x_1, ..., x_n]$ be a degree $d \geq 2$ polynomial, $r = \dim_{\mathbb{C}} S/\operatorname{Ann}(f)$. For an integer $e \geq d$, we have the following:*

*(i) The cactus rank $\operatorname{cr}(f^{\text{hom},e})$ of $f^{\text{hom},e}$ is not greater than $r$.*

*(ii) If $e \geq 2d$, the border cactus rank $\operatorname{bcr}(f^{\text{hom},e})$ of $f^{\text{hom},e}$ equals $r$. Moreover, the same is true for $e = 2d - 1$ if we assume further that $f_d$ is not a divided power of a linear form.*

*(iii) Assume that $f_d$ is not a divided power of a linear form. If $e \geq 2d$ and $J \subseteq \operatorname{Ann}(f^{\text{hom},e})$ is a homogeneous ideal such that $T/J$ has Hilbert function $h_r$, then $\operatorname{sat}(J, B) = \operatorname{Ann}(f)^{\text{hom}}$. Moreover, the same is true for $e = 2d - 1$ if we assume further that $r > 2d$.*

*Proof.*

(i) It follows directly from Theorem 3.30(i).

(ii) If $e \geq 2d$, then the claim follows from Theorem 3.30(ii).

Suppose that $e = 2d - 1$ and $f_d$ is not a power of a linear form. If $H(T/\operatorname{Ann}(f^{\text{hom},e}), d) = r$, then we get the inequality

$$\operatorname{bcr}(f^{\text{hom},e}) \geq r$$

by Proposition 3.22. Together with Point (i), we get the desired equality. Suppose that

$$H(T/\operatorname{Ann}(f^{\text{hom},e}), d) \neq r. \tag{3.2}$$

From Lemma 2.34 it follows that $\operatorname{Ann}(f)^{\text{hom}}$ is generated in degrees at most $d$. Then Equation (3.2) and Lemma 2.35 together imply that

$$H(T/\operatorname{Ann}(f^{\text{hom},e}), d) = r - 1$$

and $\operatorname{Ann}(f^{\text{hom},e})$ has no minimal generator of degree greater than $d$.

We prove that there is no homogeneous ideal $J \subseteq \operatorname{Ann}(f^{\mathrm{hom},e})$ such that $T/J$ has an $(r-1)$-standard Hilbert function (see Definition 3.14). Assume that $J$ is such an ideal. Then we have $J_d = \operatorname{Ann}(f^{\mathrm{hom},e})_d$ since $H(T/\operatorname{Ann}(f^{\mathrm{hom},e}), d) = r-1 = H(T/J, d)$. Therefore for every $m \geq d$ we have $J_m \supseteq \operatorname{Ann}(f^{\mathrm{hom},e})_m$, by Lemma 3.29. This gives a contradiction since $H(T/\operatorname{Ann}(f^{\mathrm{hom},e}), t) = 0$ for $t \gg 0$.

From Theorem 3.15 it follows that $\operatorname{bcr}(f^{\mathrm{hom},e}) \geq r$ and from Point (i) we have an equality.

(iii) Suppose that $J \subseteq \operatorname{Ann}(f^{\mathrm{hom},e})$ is such that $T/J$ has Hilbert function $h_r$. We consider the following four cases:

(I) $e \geq 2d$;

(II) $e = 2d-1$ and $H(T/\operatorname{Ann}(f^{\mathrm{hom},e}), d) = r$;

(III) $e = 2d-1$, $H(T/\operatorname{Ann}(f^{\mathrm{hom},e}), d) = r-1$ and $H(T/J, d) = r-1$;

(IV) $e = 2d-1$, $H(T/\operatorname{Ann}(f^{\mathrm{hom},e}), d) = r-1$, $H(T/J, d) = r$.

We explain that these are the only possible cases. Suppose that $e = 2d-1$ and

$$H(T/\operatorname{Ann}(f^{\mathrm{hom},e}), d) \neq r.$$

Then

$$H(T/\operatorname{Ann}(f^{\mathrm{hom},e}), d) = r-1$$

by Lemma 2.35. It suffices to show that if $H(T/\operatorname{Ann}(f^{\mathrm{hom},e}), d) = r-1$, then $H(T/J, d) \in \{r-1, r\}$. This holds since $T/J$ has Hilbert function $h_r$ and $J \subseteq \operatorname{Ann}(f^{\mathrm{hom},e})$.

We prove that $\operatorname{sat}(J, B) = \operatorname{Ann}(f)^{\mathrm{hom}}$ in each case.

(I) By Lemma 2.33(ii) and Lemma 2.29

$$H(T/\operatorname{Ann}(f^{\mathrm{hom},e}), e-d) = H(T/\operatorname{Ann}(f)^{\mathrm{hom}}, e-d) = r.$$

We have $H(T/J, e-d) \leq r$, since the Hilbert function of $T/J$ is $h_r$, but we also have

$$H(T/J, e-d) \geq H(T/\operatorname{Ann}(f^{\mathrm{hom},e}), e-d) = r.$$

Thus the quotients $T/J$ and $T/\operatorname{Ann}(f^{\text{hom},e})$ have the same Hilbert function $r$ at $e-d$ and $J \subseteq \operatorname{Ann}(f^{\text{hom},e})$, hence

$$J_{e-d} = \operatorname{Ann}(f^{\text{hom},e})_{e-d} = (\operatorname{Ann}(f)^{\text{hom}})_{e-d}.$$

Since $\operatorname{Ann}(f)^{\text{hom}}$ is generated in degrees smaller or equal $d \leq e-d$, by Lemma 2.34, it follows from Lemma 3.29 that $J_m \supseteq (\operatorname{Ann}(f)^{\text{hom}})_m$ for $m \geq e-d$. The quotients $T/J$ and $T/\operatorname{Ann}(f)^{\text{hom}}$ have the same Hilbert polynomial. Hence we have $\operatorname{sat}(J, B) = \operatorname{sat}(\operatorname{Ann}(f)^{\text{hom}}, B) = \operatorname{Ann}(f)^{\text{hom}}$.

(II) We know that both ideals $J$ and $\operatorname{Ann}(f)^{hom}$ are contained in $\operatorname{Ann}(f^{\text{hom},e})$, and the Hilbert function of the quotients $T/J$ and $T/\operatorname{Ann}(f)^{\text{hom}}$ is bounded from above by $r$. Hence, we have

$$J_d = \operatorname{Ann}(f^{\text{hom},e})_d = (\operatorname{Ann}(f)^{\text{hom}})_d.$$

The ideal $\operatorname{Ann}(f)^{\text{hom}}$ is generated in degrees at most $d$ by Lemma 2.34. Thus from Lemma 3.29 we have $J_m \supseteq (\operatorname{Ann}(f)^{\text{hom}})_m$ for $m \geq d$. Ideals $J$ and $\operatorname{Ann}(f)^{\text{hom}}$ have the same Hilbert polynomial, therefore $\operatorname{sat}(J, B) = \operatorname{sat}(\operatorname{Ann}(f)^{\text{hom}}, B) = \operatorname{Ann}(f)^{\text{hom}}$.

(III) We have $J_d = \operatorname{Ann}(f^{\text{hom},e})_d$ and the ideal $\operatorname{Ann}(f^{\text{hom},e})$ is generated in degrees at most $d$ by Lemmas 2.35 and 2.34. Thus for $m \geq d$ we get $J_m \supseteq \operatorname{Ann}(f^{\text{hom},e})_m$, by Lemma 3.29. This is a contradiction, as the Hilbert polynomial of $T/J$ is nonzero.

(IV) The ideal $J$ has a generator of the form $\alpha_0^d + \rho$ where $\rho \in T_d$ has degree smaller than $d$ with respect to $\alpha_0$ (again by Lemma 2.35). Since $\operatorname{codim}_{\operatorname{Ann}(f^{\text{hom},e})_d} J_d = 1$ we have

$$\operatorname{codim}_{(\operatorname{Ann}(f^{\text{hom},e})^c)_d}(J^c)_d \leq 1.$$

Here $K^c$ denotes $K \cap S$ for any ideal $K \subseteq T$. We shall consider $I = \operatorname{Ann}(f_d)$. We claim that $I_d \subseteq (\operatorname{Ann}(f^{\text{hom},e})^c)_d$. To justify the claim, it suffices to show that for every $\theta \in S_d$ such that $\theta \lrcorner f_d = 0$, we have $\theta \lrcorner f^{\text{hom},e} = 0$. But the form $\theta$ annihilates the form $x_0^{(d-1)} f_d$, since $\theta \lrcorner f_d = 0$. It also annihilates any form $x_0^{(2d-1-i)} f_i$ for $1 \leq i \leq d-1$, as $\deg \theta > \deg f_i$. This finishes the proof of the claim $I_d \subseteq (\operatorname{Ann}(f^{\text{hom},e})^c)_d$.

We also have $H(S/I, d) = 1$. Therefore we get

$$H(S/J^c, d) \le H(S/\operatorname{Ann}(f^{\mathrm{hom},e})^c, d) + 1 \le H(S/I, d) + 1 = 2.$$

Since $d \ge 2$, it follows from the Macaulay's bound ([17], Theorem 4.2.10) that for $m \ge d$ we have $H(S/J^c, m) \le 2$. Hence

$$H(T/J, m) \le H(S/J^c, m) + \ldots + H(S/J^c, m - (d-1)) \le 2d < r$$

for $m \ge 2d - 1$. We used here Lemma 2.36. This gives a contradiction since the Hilbert polynomial of $T/J$ is equal to $r$.

□

The following examples show that the assumptions of Theorem 3.31 are in general as sharp as possible.

**Example 3.32.** Let $S = \mathbb{C}_{dp}[x_1, x_2]$, $f = x_1^{(2)} + x_1x_2$ and assume $e = 2d - 2 = 2$. Then $r = 4$ and $\operatorname{Ann}(f^{\mathrm{hom},2}) = (\alpha_0, \alpha_1^2 - \alpha_1\alpha_2, \alpha_2^2)$. Consider the ideal $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2 - \alpha_1\alpha_2)$. Then $\mathbb{C}[\alpha_0, \alpha_1, \alpha_2]/J$ has Hilbert function $h_3$. Therefore, the assumption $e \ge 2d - 1$ in Theorem 3.31 (ii) cannot be weakened in general.

**Example 3.33.** As in Example 3.32, let $S = \mathbb{C}_{dp}[x_1, x_2]$ and $f = x_1^{(2)} + x_1x_2$. Then $r = 4 = 2d$. If $e = 3$, then $\operatorname{Ann}(f^{\mathrm{hom},3}) = (\alpha_0^2, \alpha_1^2 - \alpha_1\alpha_2, \alpha_2^2)$. The ideal $J = (\alpha_0^2, \alpha_1^2 - \alpha_1\alpha_2)$ is saturated and $\mathbb{C}[\alpha_0, \alpha_1, \alpha_2]/J$ has Hilbert function $h_4$. However, $J$ does not contain $\alpha_2^2 \in \operatorname{Ann}(f)^{\mathrm{hom}}$. Therefore, the assumption $r > 2d$ in Theorem 3.31 (iii) cannot be skipped.

**Example 3.34.** Let $S = \mathbb{C}_{dp}[x_1, x_2, x_3]$ and $f = x_1x_2x_3$. Then $r = 8 > 6 = 2d1$. If $e = 2d - 2 = 4$, then $\operatorname{Ann}(f^{\mathrm{hom},4}) = (\alpha_0^2, \alpha_1^2, \alpha_2^2, \alpha_3^2)$. Consider the ideal $J = (\alpha_0^3, \alpha_0^2\alpha_1, \alpha_1^2, \alpha_0^2\alpha_2, \alpha_2^2, \alpha_0^2\alpha_3)$. Then $\mathbb{C}[\alpha_0, \alpha_1, \alpha_2, \alpha_3]/J$ has Hilbert function $h_8$ and $\operatorname{sat}(J, B) = (\alpha_0^2, \alpha_1^2, \alpha_2^2) \ne \operatorname{Ann}(f)^{\mathrm{hom}}$. Therefore, the assumption $e \ge 2d - 1$ in Theorem 3.31 (iii) cannot be weakened in general.

We recall some notation from Section 3.3 which will be used in the proof of the following lemma. Let $\mathrm{Hilb}_T^{h_r}$ denote the multigraded Hilbert scheme associated with the polynomial ring $T$ (with the standard $\mathbb{Z}$-grading) and the function $h_r$, as defined in Definition 3.11. Let $\mathrm{Slip}_{r,\mathbb{P}T_1}$ be the closure in $\mathrm{Hilb}_T^{h_r}$ of points corresponding to saturated ideals of $r$ points. Let $\mathcal{H}ilb_r(\mathbb{P}^n)$ denote the Hilbert scheme of $r$ points on $\mathbb{P}^n$ and $\mathcal{H}ilb_r^{sm}(\mathbb{P}^n)$ denote the closure of the set of smooth schemes.

**Lemma 3.35.** *Let $d, e$ be positive integers with $e \geq d$. Let $W \subseteq S^*_{\leq d}$ be a linear subspace. Let $r = \dim_{\mathbb{C}} S/\operatorname{Ann}(W)$. If $S/\operatorname{Ann}(W)$ is smoothable, then the border rank of $W^{\mathrm{hom},e}$ is at most $r$.*

*Proof.* Observe that $\mathrm{Slip}_{r,\mathbb{P}T_1}$ surjects onto $\mathcal{H}ilb_r^{sm}(\mathbb{P}^n)$ under the natural map

$$\mathrm{Hilb}_T^{h_r} \to \mathcal{H}ilb_r(\mathbb{P}^n)$$

given on closed points by $[I] \mapsto [\operatorname{Proj} T/I]$. Thus there is an ideal $[J] \in \mathrm{Slip}_{r,\mathbb{P}T_1}$ with $\mathrm{sat}(J, B) = \operatorname{Ann}(W)^{\mathrm{hom}}$. Since $\operatorname{Ann}(W)^{\mathrm{hom}} \subseteq \operatorname{Ann}(W^{\mathrm{hom},e})$ by Lemma 2.33(i). We have $J \subseteq \operatorname{Ann}(W^{\mathrm{hom},e})$ and hence

$$[W^{\mathrm{hom},e}] \in \sigma_{r,\dim(W^{\mathrm{hom},e})}(v_d(\mathbb{P}T_1))$$

by Theorem 3.12. □

## 3.7 Bounding the cactus rank of any forms

In this section, originating from [46], we use apolarity and we improve the bound for the cactus rank given in [7]. We prove Proposition 1.19 from the Introduction. However, first we restate it in the language from Chapters 2 and 3.

Let $h_1, \ldots, h_k$ be the basis dual to the standard basis of $\mathbb{R}^k$. Let $l = c_1 h_1 + \cdots + c_k h_k$ be a non-zero linear form on $\mathbb{R}^k$, where $c_1, \ldots, c_k$ are nonnegative real numbers. Let $b$ be a positive real number.

**Proposition 3.36.** *Let*

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \xrightarrow{v_{\mathbf{d}}} \mathbb{P}(T^*_{\mathbf{d}})$$

*be the Segre-Veronese embedding. Let $F \in T^*_{\mathbf{d}}$ be a non-zero form. Then*

$$\begin{aligned}\mathrm{cr}(F) \leq & \sum_{\substack{(e_1,\ldots,e_k)| \\ l(e_1,\ldots,e_k)\leq b \\ 0\leq e_i\leq d_i}} \binom{n_1 - 1 + e_1}{e_1} \cdots \binom{n_k - 1 + e_k}{e_k} \\ & + \sum_{\substack{(e_1,\ldots,e_k)| \\ l(e_1,\ldots,e_k)> b \\ 0\leq e_i\leq d_i}} \binom{n_1 - 1 + d_1 - e_1}{d_1 - e_1} \cdots \binom{n_k - 1 + d_k - e_k}{d_k - e_k}.\end{aligned}$$

*Proof.* Let $f$ be the unique polynomial in $S^*_{\leq \mathbf{d}}$ such that $f^{\mathrm{hom},\mathbf{d}} = F$. Such an $f$ exists, because we work over $\mathbb{C}$, which has characteristic 0. We get that $f$ is a polynomial in $n_1$ variables of degree $(1, 0, \ldots, 0)$, $n_2$ variables of degree $(0, 1, 0, \ldots, 0)$, $\ldots$, and $n_k$ variables of degree $(0, \ldots, 0, 1)$. By Lemma 2.33(i), we get

$$\mathrm{cr}(F) \leq \dim_{\mathbb{C}} S/\operatorname{Ann}(f).$$

Therefore, we need to bound from above

$$\dim_{\mathbb{C}} S/\operatorname{Ann}(f).$$

This is equal to the dimension of the space of all partial derivatives of $f$. We do this just as in [9, Proof of Theorem 3]. The space of all multihomogeneous polynomials of degree $(e_1, \ldots, e_k)$ in $n_1+n_2+\cdots+n_k$ variables has dimension

$$\binom{n_1 - 1 + e_1}{e_1} \cdots \binom{n_k - 1 + e_k}{e_k}.$$

We bound the linear span of partials $\theta \lrcorner f$, where $\theta$ is a form in $S$ of degree $\mathbf{e}$ satisfying $l(\mathbf{d} - \mathbf{e}) > b$, by

$$\dim_{\mathbb{C}} \bigoplus_{\substack{\mathbf{e}: l(\mathbf{d}-\mathbf{e})>b \\ 0\leq e_i \leq d_i}} S_{\mathbf{e}} = \dim_{\mathbb{C}} \bigoplus_{\substack{\mathbf{e}: l(\mathbf{e})>b \\ 0\leq e_i \leq d_i}} S_{\mathbf{d}-\mathbf{e}}.$$

We bound the space of the other partials by the dimension of the space of all polynomials inside

$$\bigoplus_{\substack{\mathbf{e}: l(\mathbf{e})\leq b \\ 0\leq e_i \leq d_i}} S_{\mathbf{e}}.$$

Hence,

$$\begin{aligned}
\mathrm{cr}(F) \leq & \sum_{\substack{(e_1,\ldots,e_k)| \\ l(e_1,\ldots,e_k)\leq b \\ 0\leq e_i\leq d_i}} \binom{n_1 - 1 + e_1}{e_1} \cdots \binom{n_k - 1 + e_k}{e_k} \\
& + \sum_{\substack{(e_1,\ldots,e_k)| \\ l(e_1,\ldots,e_k)> b \\ 0\leq e_i\leq d_i}} \binom{n_1 - 1 + d_1 - e_1}{d_1 - e_1} \cdots \binom{n_k - 1 + d_k - e_k}{d_k - e_k}.
\end{aligned}$$

□

This is stronger than the bound in [7], since we take into account the grading by $\mathbb{Z}^k$.

**Example 3.37.** Consider the Segre embedding

$$\underbrace{\mathbb{P}^n \times \cdots \times \mathbb{P}^n}_{k \text{ times}} \hookrightarrow \mathbb{P}(\mathbb{C}^{n+1} \otimes \cdots \otimes \mathbb{C}^{n+1}),$$

which is given by the line bundle $\mathcal{O}(1, \ldots, 1)$. Then for any $F$ the linear form $h_1 + \cdots + h_k$ and $b = \frac{k}{2}$ give the bound

$$\begin{aligned} \operatorname{cr}(F) \leq 1 + kn + \binom{k}{2} n^2 + \cdots + \binom{k}{k/2} n^{k/2} \\ + \binom{k}{k/2-1} n^{k/2-1} + \cdots + \binom{k}{2} n^2 + kn + 1 \end{aligned}$$

for $k$ even, and

$$\begin{aligned} \operatorname{cr}(F) \leq 1 + kn + \binom{k}{2} n^2 + \cdots + \binom{k}{\lfloor k/2 \rfloor} n^{\lfloor k/2 \rfloor} \\ + \binom{k}{\lfloor k/2 \rfloor} n^{\lfloor k/2 \rfloor} + \cdots + \binom{k}{2} n^2 + kn + 1 \end{aligned}$$

for $k$ odd.

**Example 3.38.** Consider the Segre-Veronese embedding

$$v_{d,d} \colon \mathbb{P}^n \times \mathbb{P}^1 \to \mathbb{P}(\operatorname{Sym}^d \mathbb{C}^{n+1} \otimes \operatorname{Sym}^d \mathbb{C}^2).$$

For any $F$ the linear form $h_1$ and $b = d/2$ give the bound

$$\operatorname{cr}(F) \leq 2(d+1) \binom{n + \frac{d-1}{2}}{n}$$

for $d$ odd, and

$$\operatorname{cr}(F) \leq (d+1) \left( \binom{n + \frac{d-2}{2}}{n} + \binom{n + \frac{d}{2}}{n} \right) \tag{3.3}$$

for $d$ even.

**Example 3.39.** Consider the Segre-Veronese embedding

$$v_{3,3,2} \colon \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}(\operatorname{Sym}^3 \mathbb{C}^4 \otimes \operatorname{Sym}^3 \mathbb{C}^4 \otimes \operatorname{Sym}^2 \mathbb{C}^4).$$

For any $F$ the linear form $2h_1 + 2h_2 + 3h_3$ and $b = 9$ give the bound

$$\operatorname{cr}(F) \leq 780,$$

which is better than the bound 810 given by $h_1 + h_2 + h_3$ and $b = 4$.

# Chapter 4

# Applications of Hilbert schemes

In this chapter we state some properties of the Hilbert scheme of points $\mathcal{H}ilb_p(\mathbb{A}^n)$, and then look at a few applications of Hilbert schemes. These include bounding the number of components of the corresponding cactus varieties (Section 4.2), proving that mapping a polynomial to its apolar algebra is a morphism of schemes under certain assumptions (Section 4.3), and analyzing sets of cubics or subspaces with a given Hilbert function (Sections 4.4, 4.5).

A simple intuition what a Hilbert scheme of points is, is given in the beginning of Section 1.4. For precise definitions and a construction, see [69].

Throughout the chapter, we work over $\mathbb{C}$, except in Section 4.3, where we only need an algebraically closed field $\Bbbk$ of any characteristic.

## 4.1 Description of reducible Hilbert schemes

First we give a theorem of Casnati, Jelisiejew and Notari describing when $\mathcal{H}ilb_r^{Gor}(\mathbb{A}^n)$ is reducible for $r \leq 14$. Recall that $\mathcal{H}ilb_r^{Gor}(\mathbb{A}^n) \subseteq \mathcal{H}ilb_r(\mathbb{A}^n)$ is the open subset of Gorenstein subschemes (see Definition 1.20).

*Remark* 4.1. The Hilbert scheme is defined as a scheme representing a functor (see [69]), so it has a scheme structure. In particular, we consider it with Zariski topology. That is why in Theorems 4.2 and 4.3 we can take the closure and talk about irreducibility.

**Theorem 4.2** (Casnati, Jelisiejew, Notari)**.**

*(i) the scheme $\mathcal{H}ilb_r^{Gor}(\mathbb{A}^n)$ is irreducible for $r < 14$ and any $n \in \mathbb{N}$,*

*(ii) the scheme* $\mathcal{H}ilb_{14}^{Gor}(\mathbb{A}^n)$ *is reducible if and only if* $n \geq 6$*,*

*(iii) if the scheme* $\mathcal{H}ilb_{14}^{Gor}(\mathbb{A}^n)$ *is reducible, it has two irreducible components:* $\mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n)$*, the closure of the set of smooth schemes, and* $\mathcal{H}_{1661}(\mathbb{A}^n)$*, the closure of the set of local algebras with local Hilbert function* $(1, 6, 6, 1)$*.*

*Proof.* See [30, Theorems A and B] for Parts (i) and (ii). Part (iii) follows from [30, Theorem 6.17 and Lemma 6.19]. For a precise proof, see [30, p. 1567]. □

This allows us to distinguish between secant and cactus varieties, in other words, to overcome the barriers of determinantal methods. We do this in Chapter 6, in particular in Theorem 6.1.

There is a similar classification for the whole Hilbert scheme.

**Theorem 4.3** ([29, Theorem 1.1])**.**

*(i) the scheme* $\mathcal{H}ilb_r(\mathbb{A}^n)$ *is irreducible for* $r < 8$ *and any* $n \in \mathbb{N}$*,*

*(ii) the scheme* $\mathcal{H}ilb_8(\mathbb{A}^n)$ *is reducible if and only if* $n \geq 4$*,*

*(iii) if the scheme* $\mathcal{H}ilb_8(\mathbb{A}^n)$ *is reducible, it has two irreducible components* $\mathcal{H}ilb_8^{sm}(\mathbb{A}^n)$*, the closure of the set of smooth schemes, and* $\mathcal{H}_{143}(\mathbb{A}^n)$ *the closure of the set of local algebras with local Hilbert function* $(1, 4, 3)$*.*

This result allows us to distinguish between the Grassmann secant and cactus varieties, in other words, to overcome the barriers of determinantal methods. We do this in Chapter 6, in particular in Theorem 6.3. See Remark 6.19 for the reason why this result concerns Grassmann secant varieties, not usual secant varieties.

Let $\mathbf{n} = (n_1, \ldots, n_k)$ be a multi-index of positive integers, and define $n = n_1 + \cdots + n_k$. We show that the number of components of $\mathcal{H}ilb_r^{Gor}(\mathbb{A}^n)$ and $\mathcal{H}ilb_r^{Gor}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$ is the same and that the corresponding components have the same dimension. We also prove an analogous statement for $\mathcal{H}ilb_r(\mathbb{A}^n)$ and $\mathcal{H}ilb_r(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$. Since the proofs of these statements are the same, we do two proofs in one. We introduce the notation $\mathcal{H}ilb_r^*(X)$, where $* \in \{\emptyset, Gor\}$. The scheme $\mathcal{H}ilb_r^{\emptyset}(X) \coloneqq \mathcal{H}ilb_r(X)$.

The standard inclusion $\mathbb{A}^n \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ of an open subset induces an inclusion of an open subset

$$i : \mathcal{H}ilb_r^*(\mathbb{A}^n) \hookrightarrow \mathcal{H}ilb_r^*(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$$

for $* \in \{\emptyset, Gor\}$.

**Proposition 4.4.** *For any positive integers $r, n$, and $* \in \{\emptyset, Gor\}$ the inclusion $i$ induces a bijection between irreducible components of $\mathcal{H}ilb_r^*(\mathbb{A}^n)$ and $\mathcal{H}ilb_r^*(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$. In particular, the corresponding irreducible components have the same dimension.*

*Proof.* Let $G = GL(n_1+1) \times \cdots \times GL(n_k+1)$, and consider the morphism

$$p : G \times \mathcal{H}ilb_r^*(\mathbb{A}^n) \to \mathcal{H}ilb_r^*(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}),$$

induced by the standard action of $G$ on $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. We claim that $p$ is surjective. The claim follows from the fact that for each set of points $x_1, \ldots, x_r$ on $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, and for every $i$ with $1 \leq i \leq k$, there exists a hyperplane in $\mathbb{P}^{n_i}$ that does not pass through any of the projections of $x_1, \ldots, x_r$ on the $i$-th factor (this follows from the dual statement which says that for any arrangement of hyperplanes there exists a point not passing through any of them). The action of $G$ allows us to move those hyperplanes so that their complements become the desired distinguished open affines.

Take an irreducible component $Z$ of $\mathcal{H}ilb_r^*(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$. Then, since $p$ is surjective, there exists an irreducible component $W$ of $G \times \mathcal{H}ilb_r^*(\mathbb{A}^n)$ such that $p(W) = Z$. But all irreducible components of $G \times \mathcal{H}ilb_r^*(\mathbb{A}^n)$ are of the form $G \times W'$, where $W'$ is an irreducible component of $\mathcal{H}ilb_r^*(\mathbb{A}^n)$. Hence, $W = G \times W'$ for some irreducible component $W'$ of $\mathcal{H}ilb_r^*(\mathbb{A}^n)$. Since

$$\mathcal{H}ilb_r^*(\mathbb{A}^n) \subseteq \mathcal{H}ilb_r^*(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$$

is an open subset, the set $\overline{W'}$ is an irreducible component of $\mathcal{H}ilb_r^*(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$. But $W' \subseteq p(W) = Z$, so $Z = \overline{W'}$ is an irreducible component coming from $\mathcal{H}ilb_r^*(\mathbb{A}^n)$.

Therefore, we proved that $\mathcal{H}ilb_r^*(\mathbb{A}^n)$ is an open dense subset of the scheme $\mathcal{H}ilb_r^*(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$. The claim on dimension follows. $\square$

Proposition 4.4 allows us to use the notation $\mathcal{H}_{1661}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$ and $\mathcal{H}_{143}(\mathbb{P}^n)$ for the irreducible components of $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$ and $\mathcal{H}ilb_8(\mathbb{P}^n)$, respectively, whose general element is a non-smoothable scheme.

In order to perform the last step of the algorithms in Theorems 6.6 and 6.7 we need to know the dimension of the tangent space to $\mathcal{H}_{1661}(\mathbb{P}^n)$ and $\mathcal{H}_{143}(\mathbb{P}^n)$ at a general point.

**Lemma 4.5.** *Let $[R] \in \mathcal{H}_{1661} \subseteq \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n)$ be a non-smoothable subscheme. Then the dimension of the tangent space $\dim_{\mathbb{C}} T_{[R]}\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n)$ equals $14n-8$.*

*Proof.* Let $R' \subseteq \mathbb{P}^6$ be a subscheme abstractly isomorphic to $R$. From [31, Lem. 2.3] we have

$$\dim_{\mathbb{C}} T_{[R]}\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^n) = 14n + T_{[R']}\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6) - 84.$$

From [24, Theorem 1.1] $R'$ is non-smoothable, hence $\dim T_{[R']}\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6) = 76$ by [59, Claim 3]. □

**Lemma 4.6.** *Let $n \geq 4$ and $[R] \in \mathcal{H}_{143} \subseteq \mathcal{H}ilb_8(\mathbb{P}^n)$ be a non-smoothable subscheme. Then the dimension of the tangent space $\dim_{\mathbb{C}} T_{[R]}\mathcal{H}ilb_8(\mathbb{P}^n)$ equals $8n-7$.*

*Proof.* Let $R' \subseteq \mathbb{P}^4$ be a subscheme abstractly isomorphic to $R$. From [31, Lem. 2.3] we have

$$\dim_{\mathbb{C}} T_{[R]}\mathcal{H}ilb_8(\mathbb{P}^n) = 8n + T_{[R']}\mathcal{H}ilb_8(\mathbb{P}^4) - 32.$$

From [24, Theorem 1.1] $R'$ is non-smoothable, hence $\dim T_{[R']}\mathcal{H}ilb_8(\mathbb{P}^4) = 25$ by [29, Theorem 1.3 and the comment above]. □

## 4.2 Components of cactus varieties

We examine the fourteenth cactus variety of the Segre-Veronese embedding. As we will see, it has at most two components.

Let $\mathbf{n} = (n_1, \ldots, n_k)$ be a multi-index of positive integers and define $n = n_1 + \cdots + n_k$. Let $\mathbf{d} = (d_1, \ldots, d_k)$ be a multi-index of positive integers. For the sake of simplicity, in this section we use the notation

$$\begin{aligned} \operatorname{Sym}^{\mathbf{d}} &= \operatorname{Sym}^{d_1} \mathbb{C}^{n_1+1} \otimes \cdots \otimes \operatorname{Sym}^{d_k} \mathbb{C}^{n_k+1}, \\ \mathbb{P}^{\mathbf{n}} &= \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}. \end{aligned}$$

Recall that the Segre-Veronese embedding is the map attached to the linear system $|\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{d})|$. Equivalently, it is given on points by

$$\begin{aligned} v_{\mathbf{d}} : \mathbb{P}^{\mathbf{n}} &\to \mathbb{P}\operatorname{Sym}^{\mathbf{d}}, \\ ([l_1], \ldots, [l_k]) &\mapsto [l_1^{d_1} \cdots l_k^{d_k}]. \end{aligned}$$

Consider the following rational map $\varphi$, which assigns to a scheme $R$ its projective linear span $\langle v_{\mathbf{d}}(R)\rangle$

$$\varphi : \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^{\mathbf{n}}) \dashrightarrow \mathrm{Gr}(14, \mathrm{Sym}^{\mathbf{d}}).$$

Assume that each $d_i \geq 2$, and $n \geq 6$, so that $\dim \mathrm{Sym}^{\mathbf{d}} \geq 14$. Let $U \subseteq \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^{\mathbf{n}})$ be a dense open subset on which $\varphi$ is regular.

Consider the projectivized universal bundle $\mathbb{P}\mathcal{S}$ over $\mathrm{Gr}(14, \mathrm{Sym}^{\mathbf{d}})$, given as a set by

$$\mathbb{P}\mathcal{S} = \{([P],[p]) \in \mathrm{Gr}(14, \mathrm{Sym}^{\mathbf{d}}) \times \mathbb{P}(\mathrm{Sym}^{\mathbf{d}})) \mid p \in P\},$$

together with the inclusion $i : \mathbb{P}\mathcal{S} \hookrightarrow \mathrm{Gr}(14, \mathrm{Sym}^{\mathbf{d}}) \times \mathbb{P}(\mathrm{Sym}^{\mathbf{d}})$. We pull the commutative diagram

$$\begin{tikzcd}
\mathbb{P}\mathcal{S} \arrow[rr, hook, "i"] \arrow[dr, "\pi"'] & & \mathrm{Gr}(14, \mathrm{Sym}^{\mathbf{d}}) \times \mathbb{P}(\mathrm{Sym}^{\mathbf{d}}) \arrow[dl, "\mathrm{pr}_1"] \\
& \mathrm{Gr}(14, \mathrm{Sym}^{\mathbf{d}}) &
\end{tikzcd}$$

back along $\varphi$ to $U$, getting the commutative diagram

$$\begin{tikzcd}
\varphi^*(\mathbb{P}\mathcal{S}) \arrow[rr, hook, "\varphi^* i"] \arrow[dr, "\varphi^*\pi"] & & U \times \mathbb{P}(\mathrm{Sym}^{\mathbf{d}}) \arrow[dl, "\mathrm{pr}_1"] \\
& U. &
\end{tikzcd}$$

Let $Y$ be the closure of $\varphi^*(\mathbb{P}\mathcal{S})$ inside $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^{\mathbf{n}}) \times \mathbb{P}(\mathrm{Sym}^{\mathbf{d}})$. The scheme $Y$ has the following decomposition into irreducible components

$$Y = Y_1 \cup Y_2, \tag{4.1}$$

corresponding to two irreducible components of $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^{\mathbf{n}})$, the schemes $\mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{P}^{\mathbf{n}})$ and $\mathcal{H}_{1661}(\mathbb{P}^{\mathbf{n}})$, respectively. For the description of irreducible components of $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^{\mathbf{n}})$, see Theorem 4.2 and Proposition 4.4.

Then

$$\kappa_{14}(v_{\mathbf{d}}(\mathbb{P}^{\mathbf{n}})) = \mathrm{pr}_2(Y) \tag{4.2}$$
$$\sigma_{14}(v_{\mathbf{d}}(\mathbb{P}^{\mathbf{n}})) = \mathrm{pr}_2(Y_1), \text{ and we define} \tag{4.3}$$
$$\eta_{14}(v_{\mathbf{d}}(\mathbb{P}^{\mathbf{n}})) := \mathrm{pr}_2(Y_2). \tag{4.4}$$

In Proposition 4.7 we bound from above the dimension of the irreducible subset $\eta_{14}(v_{\mathbf{d}}(\mathbb{P}^{\mathbf{n}}))$ by $14n+5$. Later, in the proof of Theorem 6.2, we will identify a $(14n+5)$-dimensional subset of $\kappa_{14}(v_{\mathbf{d}}(\mathbb{P}^{\mathbf{n}})) \setminus \sigma_{14}(v_{\mathbf{d}}(\mathbb{P}^{\mathbf{n}}))$. It will allow us to conclude that the closure of this subset is $\eta_{14}(v_{\mathbf{d}}(\mathbb{P}^{\mathbf{n}}))$.

**Proposition 4.7.** *Dimension of $\eta_{14}(v_{\mathbf{d}}(\mathbb{P}^{\mathbf{n}}))$ is less or equal $14n+5$.*

*Proof.* We have the following commutative diagram

$$\begin{tikzcd}
\mathbb{P}(\mathrm{Sym}^{\mathbf{d}}) \supseteq \sigma \cup \eta & Y_1 \cup Y_2 \arrow[l] \arrow[r, dashed] \arrow[d, "\chi"] & \mathbb{P}\mathcal{S} \arrow[d] \\
\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^{\mathbf{n}}) \arrow[r, equal] & \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{P}^{\mathbf{n}}) \cup \mathcal{H}_{1661}(\mathbb{P}^{\mathbf{n}}) \arrow[r, dashed] & \mathrm{Gr}(14, \mathrm{Sym}^{\mathbf{d}})
\end{tikzcd}$$

where $\sigma$ and $\eta$ denote $\sigma_{14}(v_{\mathbf{d}}(\mathbb{P}^{\mathbf{n}}))$ and $\eta_{14}(v_{\mathbf{d}}(\mathbb{P}^{\mathbf{n}}))$ respectively, and $\chi : Y_1 \cup Y_2 \to \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^{\mathbf{n}})$ is the projection. Then $\dim \eta_{14}(v_{\mathbf{d}}(\mathbb{P}^{\mathbf{n}})) \leq \dim(Y_2) = m + 13$, where $m = \dim \mathcal{H}_{1661}(\mathbb{P}^{\mathbf{n}})$ and 13 is the dimension of the general fiber of the map $\chi|_{Y_2} : Y_2 \to \mathcal{H}_{1661}(\mathbb{P}^{\mathbf{n}})$. It follows from Theorem 4.2 and Proposition 4.4, that $m = 14n - 8$ and therefore $\dim \eta_{14}(v_{\mathbf{d}}(\mathbb{P}^{\mathbf{n}})) \leq 14n + 5$. □

Now we do a similar thing for the Grassmann secant and cactus varieties. We assume that $k = 1$ and we consider the $d$-th Veronese embedding $v_d$.

For $d \geq 2$ and $n \geq 4$ we will define a subset $\eta_{8,3}(v_d(\mathbb{P}^n))$ of the Grassmann cactus variety $\kappa_{8,3}(v_d(\mathbb{P}^n))$. Later, in Theorem 6.3, it will be shown that for $d \geq 5$

$$\kappa_{8,3}(v_d(\mathbb{P}^n)) = \sigma_{8,3}(v_d(\mathbb{P}^n)) \cup \eta_{8,3}(v_d(\mathbb{P}^n))$$

is the decomposition into irreducible components.

Assume that $d \geq 2$ and $n \geq 4$, so that $\dim \mathrm{Sym}^d \mathbb{C}^{n+1} > 8$. Consider the following rational map $\varphi$, which assigns to a scheme $R$ its projective linear span $\langle v_d(R) \rangle$

$$\varphi : \mathcal{H}ilb_8(\mathbb{P}^n) \dashrightarrow \mathrm{Gr}(8, \mathrm{Sym}^d \mathbb{C}^{n+1}).$$

Let $U \subseteq \mathcal{H}ilb_8(\mathbb{P}^n)$ be a dense open subset on which $\varphi$ is regular. Consider the projectivized incidence bundle $\mathbb{P}\mathcal{S}$ over the Grassmannian $\mathrm{Gr}(8, \mathrm{Sym}^d \mathbb{C}^{n+1})$, given as a set by

$$\mathbb{P}\mathcal{S} = \{([V_1], [V_2]) \in \mathrm{Gr}(8, \mathrm{Sym}^d \mathbb{C}^{n+1}) \times \mathrm{Gr}(3, \mathrm{Sym}^d \mathbb{C}^{n+1}) | V_2 \subseteq V_1\},$$

together with the inclusion $i : \mathbb{P}\mathcal{S} \hookrightarrow \mathrm{Gr}(8, \mathrm{Sym}^d \mathbb{C}^{n+1}) \times \mathrm{Gr}(3, \mathrm{Sym}^d \mathbb{C}^{n+1})$. We pull the commutative diagram

$$\begin{tikzcd} \mathbb{P}\mathcal{S} \arrow[rr, hook, "i"] \arrow[dr, "\pi"] & & \mathrm{Gr}(8, \mathrm{Sym}^d \mathbb{C}^{n+1}) \times \mathrm{Gr}(3, \mathrm{Sym}^d \mathbb{C}^{n+1}) \arrow[dl, "\mathrm{pr}_1"] \\ & \mathrm{Gr}(8, \mathrm{Sym}^d \mathbb{C}^{n+1}) & \end{tikzcd}$$

back along $\varphi$ to $U$, getting the commutative diagram

$$\begin{tikzcd} \varphi^*(\mathbb{P}\mathcal{S}) \arrow[rr, hook, "\varphi^* i"] \arrow[dr, "\varphi^*\pi"] & & U \times \mathrm{Gr}(3, \mathrm{Sym}^d \mathbb{C}^{n+1}) \arrow[dl, "\mathrm{pr}_1"] \\ & U. & \end{tikzcd}$$

Let $Y$ be the closure of $\varphi^*(\mathbb{P}\mathcal{S})$ inside $\mathcal{H}ilb_8(\mathbb{P}^n) \times \mathrm{Gr}(3, \mathrm{Sym}^d \mathbb{C}^{n+1})$. The scheme $Y$ has two irreducible components, $Y_1$ and $Y_2$, corresponding to two irreducible components of $\mathcal{H}ilb_8(\mathbb{P}^n)$, the schemes $\mathcal{H}ilb_8^{sm}(\mathbb{P}^n)$ and $\mathcal{H}_{143}(\mathbb{P}^n)$, respectively, see Theorem 4.3 and Proposition 4.4. Then for $d \geq 2$

$$\kappa_{8,3}(v_d(\mathbb{P}^n)) = \mathrm{pr}_2(Y) \tag{4.5}$$
$$\sigma_{8,3}(v_d(\mathbb{P}^n)) = \mathrm{pr}_2(Y_1), \text{ and we define} \tag{4.6}$$
$$\eta_{8,3}(v_d(\mathbb{P}^n)) := \mathrm{pr}_2(Y_2). \tag{4.7}$$

In the following proposition we bound from above the dimension of the irreducible subset $\eta_{8,3}(v_d(\mathbb{P}^n))$ by $8n + 8$. Later, in Theorem 6.3, we will identify a $(8n + 8)$-dimensional subset of $\kappa_{8,3}(v_d(\mathbb{P}^n)) \setminus \sigma_{8,3}(v_d(\mathbb{P}^n))$. We will be able to conclude that the closure of this subset is $\eta_{8,3}(v_d(\mathbb{P}^n))$.

**Proposition 4.8.** *Dimension of $\eta_{8,3}(v_d(\mathbb{P}^n))$ is less or equal $8n + 8$.*

*Proof.* We have the following commutative diagram

$$\begin{tikzcd} \mathrm{Gr}(3, \mathrm{Sym}^d \mathbb{C}^{n+1}) \supseteq \sigma \cup \eta & Y_1 \cup Y_2 \arrow[l] \arrow[r, dashed] \arrow[d, "\chi"] & \mathbb{P}\mathcal{S} \arrow[d] \\ \mathcal{H}ilb_8(\mathbb{P}^n) \arrow[r, equal] & \mathcal{H}ilb_8^{sm}(\mathbb{P}^n) \cup \mathcal{H}_{143}(\mathbb{P}^n) \arrow[r, dashed] & \mathrm{Gr}(8, \mathrm{Sym}^d \mathbb{C}^{n+1}), \end{tikzcd}$$

where $\sigma$ and $\eta$ denote $\sigma_{8,3}(v_d(\mathbb{P}^n))$ and $\eta_{8,3}(v_d(\mathbb{P}^n))$ respectively, and $\chi : Y_1 \cup Y_2 \to \mathcal{H}ilb_8(\mathbb{P}^n)$ is the projection. Then $\dim \eta_{8,3}(v_d(\mathbb{P}^n)) \leq \dim(Y_2) = m+15$, where $m = \dim \mathcal{H}_{143}$ and 15 is the dimension of the general fiber of the map $\chi|_{Y_2} : Y_2 \to \mathcal{H}_{143}$. It follows from Theorem 4.3 and Proposition 4.4, that $m = 8n - 7$ and therefore $\dim \eta_{8,3}(v_d(\mathbb{P}^n)) \leq 8n + 8$. ∎

## 4.3 Construction of a morphism to the Hilbert scheme

Let $\Bbbk$ be an algebraically closed field, $S = \Bbbk[\alpha_1, \alpha_2, ..., \alpha_n]$ be a polynomial ring and consider its graded dual $S^* = \Bbbk_{dp}[x_1, x_2, ..., x_n]$. In this section (coming from [47, Appendix A]) we prove the following theorem, which is used in Chapter 6.

**Theorem 4.9.** *Let $l, m, r$ be positive integers. Consider a locally closed reduced subscheme $E$ of $\mathrm{Gr}(l, S^*_{\leq m})$ whose closed points satisfy*

$$E(\Bbbk) \subseteq \{[W] \in \mathrm{Gr}(l, S^*_{\leq m}) \mid \operatorname{Spec} S/\operatorname{Ann}(W) \textit{ has length } r\}.$$

*The natural map from $E$ to the Hilbert scheme of $r$ points in $\mathbb{A}^n$, given on closed points by $[W] \mapsto [\operatorname{Spec} S/\operatorname{Ann}(W)]$, is a morphism of $\Bbbk$-schemes.*

For a $\Bbbk$-algebra $A$ we define $S^*_A := S^* \otimes_\Bbbk A$ and $S_A = S \otimes_\Bbbk A$. Given a $\Bbbk$-algebra homomorphism $\varphi : A \to B$ we will denote by the same letter the induced homomorphisms $S_A \to S_B$ and $S^*_A \to S^*_B$. The action $\lrcorner : S \times S^* \to S^*$ as defined in Equation (2.6) extends $A$-linearly to an action $\lrcorner : S_A \times S^*_A \to S^*_A$. Moreover, any homomorphism of algebras $\varphi : A \to B$ is compatible with the $\lrcorner$ action (see [13, Page III.158]). For any $A$-submodule $W$ of $S^*_A$, by $S_A \lrcorner W$ we denote the $S_A$-submodule of $S^*_A$ generated by $W$. Given $t$ in $\operatorname{Spec} A$ we denote by $k(t)$ the residue field of $t$ on $\operatorname{Spec} A$ and we denote by $\iota_t$ the natural morphism from $A$ to $k(t)$.

**Lemma 4.10.** *Let $\varphi\colon A \to B$ be a morphism of $\Bbbk$-algebras and $W \subseteq S^*_A$ be an $A$-submodule. Then the natural map*

$$(S_A \lrcorner W) \otimes_A B \to S^*_B$$

*surjects onto $S_B \lrcorner \varphi(W)$.*

*Proof.* Let $\theta \in S_A$, $f \in W$ and $b \in B$. Then $(\theta \lrcorner f) \otimes_A b \mapsto b(\varphi(\theta) \lrcorner \varphi(f))$ so the image of $(S_A \lrcorner W) \otimes_A B$ is contained in $S_B \lrcorner \varphi(W)$. Let $\eta \lrcorner \varphi(f) \in S_B \lrcorner \varphi(W)$ with $\eta = \sum_{\mathbf{u}} b_{\mathbf{u}} \alpha^{\mathbf{u}}$ for some $f \in W$ and $b_{\mathbf{u}} \in B$. Then

$$\sum_{\mathbf{u}} (\alpha^{\mathbf{u}} \lrcorner f) \otimes b_{\mathbf{u}} \mapsto (\eta \lrcorner \varphi(f)).$$

□

In some special cases, the surjection from Lemma 4.10 is in fact an isomorphism.

**Corollary 4.11.** *If $S_A^*/(S_A \lrcorner W)$ is a flat $A$-module or if $B$ is a flat $A$-module (for instance if $B = A_{\mathfrak{p}}$), then the map*

$$(S_A \lrcorner W) \otimes_A B \to S_B \lrcorner \varphi(W)$$

*from Lemma 4.10 is an isomorphism.*

*Proof.* By Lemma 4.10 it is enough to show that the natural map

$$(S_A \lrcorner W) \otimes_A B \to S_B^*$$

is injective. This follows from the Tor-exact sequence given by application of the functor $- \otimes_A B$ to the short exact sequence

$$0 \to (S_A \lrcorner W) \to S_A^* \to S_A^*/(S_A \lrcorner W) \to 0.$$

□

**Lemma 4.12.** *Let $A$ be a $\Bbbk$-algebra and $W$ be a finite $A$-submodule of $S_A^*$. Then $\operatorname{Hom}_A(S_A \lrcorner W, A) \simeq S_A/\operatorname{Ann}(W)$.*

*Proof.* Let $N = S_A \lrcorner W$ and define a homomorphism $\psi_A$ by

$$\begin{aligned} \psi_A \colon S_A &\to \operatorname{Hom}_A(N, A) \\ (\psi_A(\theta))(f) &= (\theta \lrcorner f)_0. \end{aligned}$$

We have a factorization of $\psi_A$ through $S_A/\operatorname{Ann}(W)$.

We show that $\ker(\psi_A) \subseteq \operatorname{Ann}(W)$. Let $\theta \in \ker(\psi_A)$ and $f \in W$. Let $\theta \lrcorner f = h_d + \ldots + h_0$ for a positive integer $d$. Then for every $j \in \{0, \ldots, d\}$ and $\eta \in (S_A)_j$ we have $0 = (\theta \lrcorner (\eta \lrcorner f))_0 = \eta \lrcorner h_j$. Thus $h_j = 0$ and hence $\theta \in \operatorname{Ann}(f)$. Since $f \in W$ was arbitrary, we have $\theta \in \operatorname{Ann}(W)$.

We first assume that $(A, \mathfrak{m})$ is a local ring. Assume that $g_1, \ldots, g_s$ is a minimal set of generators of the $A$-module $N = S_A \lrcorner W$. Let $\mathcal{M}$ be the set of divided power monomials in $S_A^*$ of degree at most $n_0 = \max\{\deg(f) | f \in W\}$. Form a matrix $M$ over $A$ with rows corresponding to $g_1, \ldots, g_s$ and entries equal to coordinates of $g_i$ in the basis $\mathcal{M}$. Then there exists an invertible $s \times s$ minor of $M$. Indeed, otherwise all minors are in the maximal ideal of

$A$ and therefore $\overline{g_1}, \ldots, \overline{g_s} \in N/\mathfrak{m}N$ are $A/\mathfrak{m}$-linearly dependent. Thus, by Nakayama's lemma, $g_1, \ldots, g_s$ is not a minimal set of generators.

We proceed to showing that $\psi_A$ is surjective (for the special case when $A$ is local). Let $\varphi \in \operatorname{Hom}_A(N, A)$ and let $a_i = \varphi(g_i)$ for $i = 1, \ldots, s$. If we write $\theta \in S_A$ as a vector $\mathbf{v}$ in the basis dual to $\mathcal{M}$, then $\psi_A(\theta)(g_i)$ is the $i$-th coordinate of the vector $M \cdot \mathbf{v}$. Therefore, there exists $\theta \in S_A$ with $\psi_A(\theta) = \varphi$, as long as there exists $\mathbf{v}$ with $M \cdot \mathbf{v} = [a_1, \ldots, a_s]^T$. Therefore, it is enough to show that $M$ gives a surjective morphism $A^{\#\mathcal{M}} \to A^s$. Let $M'$ be a $s \times s$ submatrix of $M$ with invertible determinant. We will show that $M'$ defines a surjective morphism $A^s \to A^s$. Let $M'^D$ be the adjugate matrix of $M'$. Given $\mathbf{w} \in A^s$ we have $\mathbf{w} = M' \cdot \mathbf{v}$ for $\mathbf{v} = \frac{1}{\det M'} M'^D \cdot \mathbf{w}$.

Let $A$ be an arbitrary $\Bbbk$-algebra and $Q$ be the cokernel of $\psi_A$. We claim that $Q = 0$. It is enough to show that $Q_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} A$. Let $l \colon A \to A_{\mathfrak{p}}$ be the localization. Then $N_{\mathfrak{p}} \simeq S_{A_{\mathfrak{p}}} \lrcorner\, l(W) \simeq S_{A_{\mathfrak{p}}} \lrcorner\, W_{\mathfrak{p}}$ (the first isomorphism follows from Corollary 4.11). Therefore, by the local case considered before, it is enough to show that $(\psi_A)_{\mathfrak{p}} = \psi_{A_{\mathfrak{p}}}$.

Using isomorphisms $(S_A)_{\mathfrak{p}} \simeq S_{A_{\mathfrak{p}}}$, $S_{A_{\mathfrak{p}}} \lrcorner\, W_{\mathfrak{p}} \simeq N_{\mathfrak{p}}$ and $(\operatorname{Hom}_A(N, A))_{\mathfrak{p}} \simeq \operatorname{Hom}_{A_{\mathfrak{p}}}(N_{\mathfrak{p}}, A_{\mathfrak{p}})$ we can write for $\theta \in S_A$, $f \in N$, $a, b \in A \setminus \mathfrak{p}$:

$$\left(\psi_{A_{\mathfrak{p}}}\Big(\frac{\theta}{a}\Big)\right)\Big(\frac{f}{b}\Big) = \Big(\frac{\theta}{a} \lrcorner \frac{f}{b}\Big)_0$$

and

$$\left((\psi_A)_{\mathfrak{p}}\Big(\frac{\theta}{a}\Big)\right)\Big(\frac{f}{b}\Big) = \frac{\psi_A(\theta)}{a}\Big(\frac{f}{b}\Big) = \frac{\big(\psi_A(\theta)\big)(f)}{ab} = \frac{(\theta \lrcorner f)_0}{ab}.$$

□

The following lemma is a slight modification of [59, Proposition 2.12]. Recall for $t$ a point of $\operatorname{Spec} A$, we denote by $\iota_t$ the natural map $S_A^* \to S_{k(t)}^*$.

**Lemma 4.13.** *Let $l, m \in \mathbb{Z}_{\geq 1}$, let $A$ be a Noetherian $\Bbbk$-algebra, and let $[W]$ be a $(\operatorname{Spec} A)$-point of $\operatorname{Gr}(l, S_{\leq m}^*)$, i.e. $W$ is an $A$-submodule of $(S_A^*)_{\leq m}$ such that the quotient module is locally free of rank $\dim_{\Bbbk} S_{\leq m}^* - l$. Define $a_W : \operatorname{Spec}(S_A/\operatorname{Ann}(W)) \to \operatorname{Spec} A$ to be the natural map. Then the following holds:*

*(i) If $A$ is a reduced finitely generated $\Bbbk$-algebra and*

$$\operatorname{length} S_{k(t)}/\operatorname{Ann}(\iota_t(W))$$

*is independent of the choice of a closed point $t \in \operatorname{Spec} A$ then*

$$S_A^*/(S_A \lrcorner W) \text{ and } S_A/\operatorname{Ann}(W)$$

*are flat $A$-modules.*

*(ii) If $a_W$ is such that $S_A^*/(S_A \lrcorner W)$ is a flat $A$-module, then its base change via any homomorphism between Noetherian rings $\varphi : A \to B$ is equal to*

$$\operatorname{Spec}(S_B/\operatorname{Ann}(\varphi(W))) \to \operatorname{Spec} B.$$

*In particular, the fiber of $a_W$ over $t \in \operatorname{Spec} A$ is naturally*

$$\operatorname{Spec} S_{k(t)}/\operatorname{Ann}(\iota_t(W)).$$

*Proof.* (i) First we prove that $S_A^*/(S_A \lrcorner W)$ is a flat $A$-module. We know that

$$S_A^* \cong (S_A^*)_{\leq m} \oplus (S_A^*)_{>m}.$$

Since $(S_A^*)_{>m}$ is a free $A$-module, it suffices to show that

$$(S_A^*)_{\leq m}/(S_A \lrcorner W)$$

is $A$-flat. Denote this module by $P$.

The module $P$ is finitely generated, hence it is flat if and only if it is locally free. Now $A$ is reduced and finitely generated, so $P$ is $A$-flat if and only if it has locally constant rank: $\dim_{k(t)}(P \otimes k(t))$ is independent of the choice of a closed point $t \in \operatorname{Spec} A$.

We have an exact sequence

$$0 \to S_A \lrcorner W \to (S_A^*)_{\leq m} \to P \to 0.$$

We tensor it by $k(t)$, getting the exact sequence

$$(S \lrcorner W) \otimes_A k(t) \xrightarrow{u} (S_A^*)_{\leq m} \otimes_A k(t) \to P \otimes_A k(t) \to 0.$$

Then

$$\begin{aligned} \dim_{k(t)}(P \otimes_A k(t)) &= \dim_{k(t)}((S_A^*)_{\leq m} \otimes_A k(t)) - \dim_{k(t)} \operatorname{im} u \\ &\overset{\text{by Lemma 4.10}}{=} \dim_{\Bbbk} S_{\leq m}^* - \dim_{k(t)} S_{k(t)} \lrcorner \iota_t(W) \\ &\overset{\text{by Lemma 4.12}}{=} \dim_{\Bbbk} S_{\leq m}^* - \dim_{k(t)} S_{k(t)}/\operatorname{Ann}(\iota_t(W)), \end{aligned}$$

which is constant by assumption.

It remains to prove that $S/\operatorname{Ann}(W)$ is a flat $A$-module. It follows from Lemma 4.12, that

$$S/\operatorname{Ann}(W) \simeq \operatorname{Hom}_A(S_A \lrcorner W, A).$$

Since $S_A \lrcorner W$ is the kernel of a surjection of flat $A$-modules, it is a flat $A$-module. Because it is finite as an $A$-module, it is locally free of finite rank. Therefore $\operatorname{Hom}_A(S_A \lrcorner W, A)$ is a locally free $A$-module of finite rank, thus flat.

(ii) Let $N = S_A \lrcorner W$. Suppose that $S_A^*/N$ is a flat $A$-module. By Corollary 4.11 the natural morphism $N \otimes_A B \to S_A^* \otimes_A B \simeq S_B^*$ sends $N \otimes_A B$ isomorphically to $S_B \lrcorner \varphi(W)$. By Lemma 4.12 $S_A/\operatorname{Ann}(W) \simeq \operatorname{Hom}_A(S_A \lrcorner W, A)$ and $S_B/\operatorname{Ann}(\varphi(W)) \simeq \operatorname{Hom}_B(S_B \lrcorner \varphi(W), B)$. Thus it is enough to show that $\operatorname{Hom}_A(N, A) \otimes_A B \simeq \operatorname{Hom}_B(N \otimes_A B, B)$. This follows from [50, Exercise 7.20(a)] and the fact that $N = S_A \lrcorner W$ is flat and finitely generated over a Noetherian ring, hence locally free of finite rank, see [12, Proposition 4.4.3].

□

**Lemma 4.14.** *Let $W \subseteq (S_A^*)_{\leq m}$ be an $A$-submodule, and let $Q = (S_A^*)_{\leq m}/W$. Let $t \in \operatorname{Spec} A$ be any closed point. If $Q$ is $A$-flat, then $\iota_t(W) = W \otimes_A k(t)$.*

*Proof.* Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r] \arrow[d, "a"] & (S_A^*)_{\leq m} \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W \otimes_A k(t) \arrow[r, "b"] & (S_{k(t)}^*)_{\leq m} \arrow[r] & Q \otimes_A k(t) \arrow[r] & 0.
\end{tikzcd}$$

The map $a$ is a surjection since tensoring is right-exact. The map $b$ is an injection, because $Q$ is $A$-flat. Hence $\iota_t(W) = W \otimes_A k(t)$. □

*Proof of Theorem 4.9.* Take any cover of $E$ by open affines $\operatorname{Spec} A_i$. We construct morphisms

$$\varphi_i : \operatorname{Spec} A_i \to \mathcal{H}ilb_r(\mathbb{A}^n),$$

and finally we show that these morphisms glue.

Let $\mathcal{U}$ be the universal subbundle on $\mathrm{Gr}(l, S^*_{\leq m})$, treated as a locally free sheaf. Let $\mathcal{U}|_{\mathrm{Spec}\, A_i} = \widetilde{W_i}$, where $W_i \subseteq S^*_{\leq m} \otimes A_i$ is a submodule. Observe that $(S_{A_i})_{\leq m}/W_i$ is $A_i$-flat from the definition of the Grassmann functor, see [50, §8.4].

Our morphism is defined by the family $\mathrm{Spec}\, S_{A_i}/\operatorname{Ann}(W_i) \to \mathrm{Spec}\, A_i$. We know that the scheme $\mathrm{Spec}\, S_{A_i}/\operatorname{Ann}(W_i)$ is a closed subscheme of $\mathbb{A}^n_{A_i}$. We use Part (i) of Lemma 4.13 for $W_i$. In order to do it, it suffices to show that for every closed point $t \in \mathrm{Spec}\, A_i$, the vector space $S_{k(t)}/\operatorname{Ann}(\iota_t(W_i))$ has dimension $r$. But this follows from the fact that $[W_i \otimes k(t)] = [\iota_t(W_i)]$ by Lemma 4.14, and the fact that $[W_i \otimes k(t)] \in E \subseteq \mathrm{Gr}(l, S^*_{\leq m})$. Hence both modules $S_{A_i}/\operatorname{Ann}(W_i)$ and $S^*_{A_i}/S_{A_i} \lrcorner W_i$ are $A_i$-flat. Then we can use Part (ii) of Lemma 4.13 to show that our family has fibers of length $r$. Hence, from the defining property of the Hilbert scheme, we have a morphism $\varphi_i : \mathrm{Spec}\, A_i \to \mathcal{H}ilb_r(\mathbb{A}^n)$. Moreover, the fiber of the family $\mathrm{Spec}\, S_{A_i}/\operatorname{Ann}(W_i)$ over the closed point $t$ is $S_{k(t)}/\operatorname{Ann}(\iota_t(W_i))$. Therefore $\varphi_i$ on closed points is defined by

$$[W] \mapsto [\mathrm{Spec}\, S/\operatorname{Ann}(W)].$$

Since the morphims $\varphi_i$ are defined on closed points by the same formula, they glue together. □

## 4.4 Cubics with Hilbert function $(1, 6, 6, 1)$

In Lemma 4.16, we give a useful characterization of cubics $f$ such that the Hilbert function of $S/\operatorname{Ann}(f)$ is $(1, 6, 6, 1)$. This is inspired by [8, Example 8]. Then we establish Lemma 4.17 about topological properties of the set of such cubics. Finally, we spread this set by the action of the product $GL(n_1 + 1) \times \cdots \times GL(n_k + 1)$, and we look at topological properties of the result. This part is taken from [47, Subsection 5.1].

In this section $S = \mathbb{C}[\alpha_1, \ldots, \alpha_n]$, and $S^* = \mathbb{C}[x_1, \ldots, x_n]$ is its graded dual. We assume that $n \geq 6$. Given $f \in S^*$, we denote by $f_j$ its homogeneous part of degree $j$. For a finite-dimensional subspace $W \subseteq S^*$, the algebra $\mathrm{Apolar}(W)$ is defined as $S/\operatorname{Ann}(W)$.

Up to this point, we only used graded Hilbert function, i.e. for a graded module $M$, and an integer $i$ we defined

$$H(M, i) = \dim_{\mathbb{C}} M_i.$$

But from now on, we also use local Hilbert function, i.e. for a local finite-dimensional algebra $A$ with maximal ideal $\mathfrak{m}$, and an integer $i$, we define

$$H(A,i) = \dim_{\mathbb{C}}\left(\frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1}}\right).$$

If the algebra is graded, the two notions coincide.

**Lemma 4.15.** *Let $W \subseteq S^*$ be a finite-dimensional linear subspace. Then*

$$H(\operatorname{Apolar}(W), k) = \operatorname{codim}_{S_k} E_k,$$

*where $E_k = \{\theta_k \in S_k |$ there exists $\theta_{\geq k+1} \in S_{\geq k+1}$ such that $(\theta_k + \theta_{\geq k+1}) \lrcorner W = 0\}$.*

*Proof.* Let $\overline{\mathfrak{m}}$ be the maximal ideal of $\operatorname{Apolar}(W)$.

$$\begin{aligned} H(\operatorname{Apolar}(W), k) &= \dim_{\mathbb{C}} \overline{\mathfrak{m}}^k / \overline{\mathfrak{m}}^{k+1} \\ &= \operatorname{codim}_{S_{\geq k}} \operatorname{Ann}(W) \cap S_{\geq k} - \operatorname{codim}_{S_{\geq k+1}} \operatorname{Ann}(W) \cap S_{\geq k+1} \\ &= \operatorname{codim}_{S_{\geq k}} S_{\geq k+1} - \operatorname{codim}_{\operatorname{Ann}(W)\cap S_{\geq k}} \operatorname{Ann}(W) \cap S_{\geq k+1} \\ &= \dim_{\mathbb{C}} S_k - \dim_{\mathbb{C}} \frac{\operatorname{Ann}(W) \cap S_{\geq k}}{\operatorname{Ann}(W) \cap S_{\geq k+1}} \\ &= \dim_{\mathbb{C}} S_k - \dim_{\mathbb{C}} E_k \end{aligned}$$

$\square$

**Lemma 4.16.** *For $[f] \in \mathbb{P}S_{\leq 3}$ the following are equivalent:*

*(a)* $\operatorname{Apolar}(f)$ *has Hilbert function* $(1, 6, 6, 1)$*,*

*(b) There exists $[U] \in \operatorname{Gr}(6, S_1)$ such that $f_3 \in \operatorname{Sym}^3 U$, $f_2 \in U \cdot S_1^*$ and $H(\operatorname{Apolar}(f_3), 1) = 6$.*

*Proof.* Notice that we can assume that $f_3 \neq 0$ since it follows from both Conditions (a) and (b).

By Iarrobino's symmetric decomposition (see [30, Theorem 2.3 and the following remarks]), the algebra $\operatorname{Apolar}(f)$ has Hilbert function $(1, c+e, c, 1)$, where $(1, c, c, 1)$ is the Hilbert function of $\operatorname{Apolar}(f_3)$. From Lemma 4.15, we know that $c + e = \operatorname{codim}_{S_1} E_1$, where $E_1 = \{\theta_1 \in S_1 |$ there exists $\theta_{\geq 2} \in S_{\geq 2}$ such that $(\theta_1 + \theta_{\geq 2}) \lrcorner f = 0\}$. We use the following computation

$$\begin{aligned}(\theta_3 + \theta_2 + \theta_1) &\lrcorner (f_3 + f_2 + f_1 + f_0) \\ &= (\theta_1 \lrcorner f_3) + (\theta_1 \lrcorner f_2 + \theta_2 \lrcorner f_3) + (\theta_1 \lrcorner f_1 + \theta_2 \lrcorner f_2 + \theta_3 \lrcorner f_3). \quad (4.8)\end{aligned}$$

Assume that $\operatorname{Apolar}(f)$ has Hilbert function $(1, 6, 6, 1)$. We show that Condition (b) is satisfied. Let $U$ be $S_2 \lrcorner f_3$, which is 6-dimensional, since the Hilbert function of $\operatorname{Apolar}(f_3)$ is $(1, 6, 6, 1)$, by the properties of Iarrobino's symmetric decomposition mentioned above. It is enough to show that $f_2 \in U \cdot S_1^*$. Assume that this does not hold. Up to a linear change of variables we can assume that $U = \langle x_1, x_2, \ldots, x_6\rangle$. Let $V = \langle x_7, x_8, \ldots, x_n\rangle$. By classification of quadratic forms over $\mathbb{C}$ we may assume that $f_2 = x_n^2 + H + K$ where $H \in \operatorname{Sym}^2(\langle x_7, x_8, \ldots, x_{n-1}\rangle)$ and $K \in S_1^* \cdot U$. Then $\alpha_n \lrcorner f_2 \notin U$ and hence $\alpha_n \notin E_1$ by Equation (4.8).

We claim that $\dim_{\mathbb{C}} E_1 \leq n - 7$. It suffices to show that the classes of $\alpha_1, \alpha_2, \ldots, \alpha_6, \alpha_n$ are linearly independent in the vector space $S_1/E_1$. Let $\omega = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_6\alpha_6$ for some $a_1, \ldots, a_6 \in \mathbb{C}$ and assume that there is $b \in \mathbb{C}$ and $\theta_{\geq 2} \in S_{\geq 2}$ such that $(\omega + b\alpha_n + \theta_{\geq 2}) \lrcorner f = 0$. Then by Equation (4.8) we get $(\omega + b\alpha_n) \lrcorner f_3 = 0$. Since $U = S_2 \lrcorner f_3 = \langle x_1, x_2, \ldots, x_6\rangle$, it follows that $f_3$ is a polynomial in $x_1, \ldots, x_6$. Therefore, $b\alpha_n \lrcorner f_3 = 0$ and as a consequence $\omega \lrcorner f_3 = 0$. We know that $\operatorname{Apolar}(f_3)$ has Hilbert function $(1, 6, 6, 1)$, thus $\omega = 0$. As a result, $b\alpha_n \in E_1$ which shows that $b = 0$.

The claim proven above contradicts Lemma 4.15 and the assumption that $H(\operatorname{Apolar}(f), 1) = 6$.

For the other direction, assume that (b) holds, we show that $\operatorname{Apolar}(f)$ has Hilbert function $(1, 6, 6, 1)$. It is enough to show that $\operatorname{codim}_{S_1} E_1 = 6$. By assumption $\dim_{\mathbb{C}} \operatorname{Ann}(f_3)_1 = n-6$, so it suffices to show that $E_1 = \operatorname{Ann}(f_3)_1$. Assume that $\theta = \theta_3 + \theta_2 + \theta_1 \in \operatorname{Ann}(f)$, then it follows from Equation (4.8) that $\theta_1 \in \operatorname{Ann}(f_3)_1$. Thus $E_1 \subseteq \operatorname{Ann}(f_3)_1$. Let us take $\theta_1 \in \operatorname{Ann}(f_3)_1$. From the assumption $f_2 = \sum_i u_i h_i$ where $u_i \in U, h_i \in S_1^*$. Therefore

$$\theta_1 \lrcorner f_2 = \sum_i u_i(\theta_1 \lrcorner h_i) \in U.$$

Since $(\cdot) \lrcorner f_3 : S_2 \to U$ is surjective, there exists $\theta_2 \in S_2$, such that $\theta_2 \lrcorner f_3 = -\theta_1 \lrcorner f_2$. By Equation (4.8) it is enough to observe that there exists $\theta_3 \in S_3$, such that $\theta_3 \lrcorner f_3 = -(\theta_1 \lrcorner f_1 + \theta_2 \lrcorner f_2)$ $\square$

**Lemma 4.17.** *The following subset of* $\mathbb{P}S^*_{\leq 3}$ *is locally closed, irreducible, and of dimension* $13n+5$

$$A = \{[f] \in \mathbb{P}S^*_{\leq 3} | \operatorname{Apolar}(f) \text{ has Hilbert function } (1,6,6,1)\}.$$

*Moreover the set*

$$B = \{[f] \in A | [\operatorname{Spec} \operatorname{Apolar}(f)] \notin \mathcal{H}ilb^{Gor,sm}_{14}(\mathbb{A}^n)\}$$

*is dense in* $A$.

*Proof.* Consider

$$\mathcal{A} = \{([U],[f]) \in \operatorname{Gr}(6, S^*_1) \times \mathbb{P}S^*_{\leq 3} \mid [f] \in \mathbb{P}(\operatorname{Sym}^3 U \oplus (S^*_1 \cdot U) \oplus S^*_{\leq 1})\}.$$

and

$$\mathcal{A}^0 = \{([U],[f]) \in \mathcal{A} \mid H(\operatorname{Apolar}(f_3), 1) = 6\}.$$

We have a pullback diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] & \operatorname{Fl}(1, 7n+42, S^*_{\leq 3}) \arrow[d] \\
\operatorname{Gr}(6, S^*_1) \arrow[r] & \operatorname{Gr}(7n+42, S^*_{\leq 3})
\end{tikzcd}$$

where $\operatorname{Fl}(1, 7n+42, S^*_{\leq 3})$ is the flag variety parametrizing flags of subspaces $M \subseteq N \subseteq S^*_{\leq 3}$ with $\dim_{\mathbb{C}} M = 1$, $\dim_{\mathbb{C}} N = 7n+42$ and the lower horizontal map sends $[U]$ to $[\operatorname{Sym}^3 U \oplus (S^*_1 \cdot U) \oplus S^*_{\leq 1}]$.

The varieties $\mathcal{A}$ and $\operatorname{Gr}(6, S^*_1)$ are projective. Moreover, the left vertical map is surjective and its fibers are irreducible varieties isomorphic to $\mathbb{P}^{7n+41}$. Since $\mathbb{P}^{7n+41}$ is irreducible, it follows from [72, Theorems 1.25-26] that $\mathcal{A}$ is irreducible and of dimension $6(n-6) + 7n + 41 = 13n + 5$.

We will show that $\mathcal{A}^0$ is open in $\mathcal{A}$. Consider the subset

$$\mathcal{B} = \{([U],[f]) \in \operatorname{Gr}(6, S^*_1) \times \mathbb{P}S^*_{\leq 3} \mid H(\operatorname{Apolar}(f_3), 1) \geq 6\}.$$

Observe that $\mathcal{A}^0 = \mathcal{A} \cap \mathcal{B}$. It is enough to show that $\mathcal{B}$ is open in $\operatorname{Gr}(6, S^*_1) \times \mathbb{P}S^*_{\leq 3}$. Let

$$\mathcal{C} = \{[f] \in \mathbb{P}S^*_{\leq 3} \mid H(\operatorname{Apolar}(f_3), 1) \geq 6\}.$$

It suffices to show that $\mathcal{C}$ is open in $\mathbb{P}S^*_{\leq 3}$, which holds since its complement is given by catalecticant minors. We have established that $\mathcal{A}^0 = \mathcal{A} \cap \mathcal{B}$ is open in $\mathcal{A}$.

By Lemma 4.16 we have $A = \pi_2(\mathcal{A}^0)$, where $\pi_2\colon \operatorname{Gr}(6, S^*_1) \times \mathbb{P}S^*_{\leq 3} \to \mathbb{P}S^*_{\leq 3}$ is the projection. Since $\pi_2|_{\mathcal{A}^0}\colon \mathcal{A}^0 \to A$ has a finite fiber over every point, it follows from [77, Theorem 11.4.1] that $A$ is irreducible and of dimension $13n + 5$.

We know that $A = \pi_2(\mathcal{A}^0) = \pi_2(\mathcal{A}) \cap \mathcal{C}$ which is locally closed since $\pi_2(\mathcal{A})$ is closed and $\mathcal{C}$ is open. Therefore we have a morphism $\mu : A \to \mathcal{H}ilb^{Gor}_{14}(\mathbb{A}^n)$ given on closed points by $[f] \mapsto [\operatorname{Spec} S/\operatorname{Ann}(f)]$, see Theorem 4.9.

By Theorem 4.2, the scheme $\mathcal{H}ilb^{Gor}_{14}(\mathbb{A}^n)$ has two irreducible components $\mathcal{H}ilb^{Gor,sm}_{14}(\mathbb{A}^n)$ and $\mathcal{H}_{1661}(\mathbb{A}^n)$. We obtain

$$B = \mu^{-1}(\mathcal{H}_{1661}(\mathbb{A}^n) \setminus \mathcal{H}ilb^{Gor,sm}_{14}(\mathbb{A}^n)),$$

so it is open in $A$. Since $B$ is non-empty (see [59, Remark 3.7], and [24, Theorem 3.16, Proposition 5.6]) and $A$ is irreducible, it follows that $B$ is dense in $A$. $\square$

We consider

$$\begin{aligned} S^* &= \mathbb{C}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}], \text{ and} \\ T^* &= \mathbb{C}[x_{1,0}, x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, x_{k,1}, \ldots, x_{k,n_k}] \\ &= \operatorname{Sym} V_1^* \otimes \cdots \otimes \operatorname{Sym} V_k^*, \end{aligned}$$

where $V_i^* = \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i} \rangle$ for $1 \leq i \leq k$. The rings $S^*$ and $T^*$ are naturally graded by $\mathbb{Z}^k$.

**Definition 4.18.** Given multi-indices $\mathbf{e}, \mathbf{d} \in \mathbb{N}^k$ with $\mathbf{e} \geq \mathbf{d}$ and a linear subspace $W \subseteq S^*_{\leq \mathbf{d}}$ we define

$$W^{\blacktriangledown \mathbf{e}} = \{\sum_{\mathbf{i} \leq \mathbf{d}} (e_1 - i_1)! \cdots (e_k - i_k)! f_{\mathbf{i}} \mid f \in W\}.$$

**Definition 4.19.** Given a polynomial $F \in T^*_{\leq \mathbf{d}}$, and for each $i$ a basis $z_i, z_{i,1}, \ldots, z_{i,n_i}$ of $V_i^*$ for $1 \leq i \leq k$, we consider the dehomogenization of $F$ with respect to bases $z_i, z_{i,1}, \ldots, z_{i,n_i}$, denoted by

$$F_{|z_1=1, z_2=1, \ldots, z_k=1} \in \mathbb{C}[z_{1,1}, \ldots, z_{1,n_1}, \ldots, z_{k,1}, \ldots, z_{k,n_k}].$$

We calculate it by writing the polynomial in the given bases, and setting $z_1 = 1, \ldots, z_k = 1$.

*Remark* 4.20. For any multi-index $\mathbf{e} \geq \mathbf{d}$, when we dehomogenize by the standard bases of $V_i^*$, we get the following equality

$$((F_{|x_{0,1}=1,\ldots,x_{0,k}=1})^{\blacktriangledown\mathbf{e}})^{\mathrm{hom},\mathbf{e}} = x_{1,0}^{e_1-d_1}\cdots x_{k,0}^{e_k-d_k}F.$$

Since in Chapter 6, we often consider forms divisible by some powers of linear forms, we need to consider the triangle operator.

Let $\mathbf{d}$ be a multi-index of length $k$ with $d_i \geq 3$ for each $i$.

Notice that in the following lemma the main actors—sets $C$ and $D$ do not depend on $\mathbf{d}$ *except* in two little places. We need a different definition of these sets to prove our main theorems for all degrees of the embedding $v_{\mathbf{d}}$ in Sections 6.2, and 6.3.

**Lemma 4.21.** *The following subset is irreducible, of dimension* $14n+6+k$*,*

$$\begin{aligned} C = \{(z_1,\ldots,z_k,P) \in \prod_{i=1}^{k} V_i^* \times T_{3,\ldots,3} \mid\ & \textit{for each } i \textit{ there exists a completion} \\ & \textit{of } z_i \textit{ to a basis } (z_i, z_{i,1},\ldots,z_{i,n_i}) \textit{ of } V_i^* \textit{ such that} \\ & \mathrm{Apolar}((P|_{z_0,z_1,\ldots,z_k=1})^{\blacktriangledown\mathbf{d}}) \textit{ has Hilbert function} \\ & (1,6,6,1)\}. \end{aligned}$$

*Moreover the set*

$$\begin{aligned} D = \{(z_1,\ldots,z_k,P) \in \prod_{i=1}^{k} V_i^* \times T_{3,\ldots,3} \mid\ & \textit{for each } i \textit{ there exists a completion} \\ & \textit{of } z_i \textit{ to a basis } (z_i, z_{i,1},\ldots,z_{i,n_i}) \textit{ of } V_i^* \textit{ such that} \\ & \mathrm{Apolar}((P|_{z_0,z_1,\ldots,z_k=1})^{\blacktriangledown\mathbf{d}}) \textit{ has Hilbert function} \\ & (1,6,6,1), \textit{ and } [\mathrm{Spec}\,\mathrm{Apolar}((P|_{z_1=1,\ldots,z_k=1})^{\blacktriangledown\mathbf{d}})] \\ & \notin \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n)\} \end{aligned}$$

*is dense in* $C$*.*

*Proof.* We consider the morphism $\varphi : GL(V_1^*)\times\cdots\times GL(V_k^*)\times T_{3,\ldots,3} \to T_{3,\ldots,3}$ given by a change of basis. Then we have a product morphism

$$\tau : GL(V_1^*)\times\cdots\times GL(V_k^*)\times T_{3,\ldots,3} \to \prod_{i=1}^{k} V_i^* \times T_{3,\ldots,3},$$

given by $(a_1,\ldots,a_k,P) \mapsto (a_1(y_1),\ldots,a_k(y_k),\varphi(a_1,\ldots,a_k,P))$.

Recall the sets $A$ and $B$ from Lemma 4.17. Let $\widehat{A}, \widehat{B}$ be the affine cones over $A$ and $B$, respectively, with the origins removed. Let $\chi : S_{\leq 3} \to T_{3,\ldots,3}$ be the $\mathbb{C}$-linear monomorphism given by $f \mapsto (f^{\blacktriangledown \mathbf{d}^{-1}})^{\underline{hom}(3,\ldots,3)}$ (here $\underline{hom}$ denotes standard homogenization to degree $(3,\ldots,3)$, equal to the composition of $((\cdot)^{\blacktriangledown(3,\ldots,3)})^{\mathrm{hom},(3,\ldots,3)})$. We have $\tau(GL(V_1^*) \times \cdots \times GL(V_k^*) \times (\chi(\widehat{A}))) = C$ and $\tau(GL(V_1^*) \times \cdots \times GL(V_k^*) \times (\chi(\widehat{B}))) = D$. These follow from the definitions of the sets $A, B, C$, and $D$, and the identity

$$\begin{aligned}(\varphi(a_1, \ldots, a_k, \chi(f))|_{a_1(z_1)=1,\ldots,a_k(z_k)=1})^{\blacktriangledown \mathbf{d}} \\ = f(a_1(x_{1,1}), \ldots, a_1(x_{1,n_1}), \ldots, a_k(x_{k,1}), \ldots, a_k(x_{k,n_k})).\end{aligned}$$

It follows from Lemma 4.17 that $C$ is irreducible, $D$ is dense in $C$, and $\dim D = \dim C = 14n + 6 + k$. □

## 4.5 Subspaces with Hilbert function $(1, 4, 3)$

In Lemma 4.22, we give a useful characterization of subspaces $W$ of a polynomial ring such that the Hilbert function of $\mathrm{Apolar}(W)$ is $(1, 4, 3)$. Then we establish Lemma 4.23 about topological properties of the set of such subspaces. Finally, in Lemma 4.24 we spread this set by the action of $GL(n+1)$, and look at topological properties of the result. This part is taken from [47, Subsection 6.1].

In this section $S = \mathbb{C}[\alpha_1, \ldots, \alpha_n]$, and $S^* = \mathbb{C}[x_1, \ldots, x_n]$ is its graded dual. We assume that $n \geq 4$. Given an integer $i$, and a linear subspace $W \subseteq S^*$, we denote by $W_i$ the image of the projection of $W$ onto the $i$-th graded part.

**Lemma 4.22.** *For $[W] \in \mathrm{Gr}(3, S^*_{\leq 2})$ the following are equivalent:*

*(a)* $\mathrm{Apolar}(W)$ *has Hilbert function* $(1, 4, 3)$*,*

*(b)* $\dim_{\mathbb{C}} W_2 = 3$*,* $[W] \in \mathrm{Gr}(3, \mathrm{Sym}^2 U \oplus S^*_{\leq 1})$ *for some* $[U] \in \mathrm{Gr}(4, S^*_1)$ *and* $H(\mathrm{Apolar}(W_2), 1) = 4$*,*

*(c)* $\mathrm{Apolar}(W_2)$ *has Hilbert function* $(1, 4, 3)$.

*Proof.* Conditions $(b)$ and $(c)$ are equivalent. We show that Conditions $(a)$ and $(c)$ are equivalent. Observe that $H(\mathrm{Apolar}(W), 2) = 3$ if and only if $\dim_{\mathbb{C}} W_2 = 3$ since

$$H(\mathrm{Apolar}(W), 2) = H(\mathrm{Apolar}(W_2), 2).$$

Therefore, we are left to show that $H(\mathrm{Apolar}(W), 1) = 4$ if and only if

$$H(\mathrm{Apolar}(W_2), 1) = 4.$$

By Lemma 4.15, we obtain $H(\mathrm{Apolar}(W), 1) = \mathrm{codim}_{S_1}(E_1)$, where

$$E_1 = \{\theta_1 \in S_1 \mid \text{ there exists } \theta_{\geq 2} \in S_{\geq 2} \text{ such that } \theta_1 + \theta_{\geq 2} \in \mathrm{Ann}(W)\}.$$

We will show that $E_1 = \mathrm{Ann}(W_2)_1$.
Let $W = \langle Q_j + L_j + C_j \mid j \in \{1, 2, 3\},\ Q_j \in S_2^*, L_j \in S_1^*$ and $C_j \in S_0^*\rangle$. Assume that $\theta_1 \in E_1$ and let $\theta_1 + \theta_{\geq 2} \in \mathrm{Ann}(W)$ for some $\theta_{\geq 2} \in S_{\geq 2}$. Then for $j \in \{1, 2, 3\}$

$$0 = (\theta_1 + \theta_{\geq 2}) \lrcorner (Q_j + L_j + C_j) = (\theta_1 \lrcorner Q_j) + (\theta_1 \lrcorner L_j + \theta_{\geq 2} \lrcorner Q_j),$$

so $\theta_1 \lrcorner Q_j = 0$ for $j \in \{1, 2, 3\}$.

Now assume that $\theta_1 \in \mathrm{Ann}(W_2)$. Since $\dim_{\mathbb{C}} W_2 = 3$, there is $\theta_2 \in S_2$ such that $\theta_2 \lrcorner Q_j = -\theta_1 \lrcorner L_j$ for $j \in \{1, 2, 3\}$. Then $\theta_1 + \theta_2 \in \mathrm{Ann}(W)$, so $\theta_1 \in E_1$.

□

**Lemma 4.23.** *The following subset of* $\mathrm{Gr}(3, S^*_{\leq 3})$ *is locally closed, irreducible, and of dimension* $7n + 8$

$$A = \{[W] \in \mathrm{Gr}(3, S^*_{\leq 2}) \mid \mathrm{Apolar}(W) \textit{ has Hilbert function } (1, 4, 3)\}.$$

*Moreover the set*

$$B = \{[W] \in A \mid [\mathrm{Spec}\,\mathrm{Apolar}(W)] \notin \mathcal{H}ilb_8^{sm}(\mathbb{A}^n)\}$$

*is dense in* $A$.

*Proof.* Consider

$$\mathcal{A} = \{([U], [W]) \in \mathrm{Gr}(4, S_1^*) \times \mathrm{Gr}(3, S^*_{\leq 2}) \mid [W] \in \mathrm{Gr}(3, \mathrm{Sym}^2 U \oplus S^*_{\leq 1})\}.$$

and

$$\mathcal{A}^0 = \{([U], [W]) \in \mathcal{A} \mid H(\mathrm{Apolar}(W_2), 1) = 4\}.$$

We have a pullback diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] & \mathrm{Fl}(3, n+11, S^*_{\leq 2}) \arrow[d] \\
\mathrm{Gr}(4, S^*_1) \arrow[r] & \mathrm{Gr}(n+11, S^*_{\leq 2})
\end{tikzcd}$$

where $\mathrm{Fl}(3, n+11, S^*_{\leq 2})$ is the flag variety parametrizing flags of subspaces $M \subseteq N \subseteq S^*_{\leq 2}$ with $\dim_{\mathbb{C}} M = 3$, $\dim_{\mathbb{C}} N = n+11$ and the lower horizontal map sends $[U]$ to $[\mathrm{Sym}^2 U \oplus S^*_{\leq 1}]$.

The varieties $\mathcal{A}$ and $\mathrm{Gr}(4, S^*_1)$ are projective. Moreover, the left vertical map is surjective and its fibers are irreducible and isomorphic to $\mathrm{Gr}(3, n+11)$. Since $\mathrm{Gr}(3, n+11)$ is irreducible, it follows from [72, Theorems 1.25-26] that $\mathcal{A}$ is irreducible and of dimension $4(n-4) + 3(n+8) = 7n+8$.

We will show that $\mathcal{A}^0$ is open in $\mathcal{A}$. Consider the subset

$$\mathcal{B} = \{([U], [W]) \in \mathrm{Gr}(4, S^*_1) \times \mathrm{Gr}(3, S^*_{\leq 2}) \mid H(\mathrm{Apolar}(W_2), 1) \geq 4\}.$$

Observe that $\mathcal{A}^0 = \mathcal{A} \cap \mathcal{B}$, therefore it is enough to show that $\mathcal{B}$ is open in $\mathrm{Gr}(4, S^*_1) \times \mathrm{Gr}(3, S^*_{\leq 2})$. Let

$$\mathcal{C} = \{[W] \in \mathrm{Gr}(3, S^*_{\leq 2}) \mid H(\mathrm{Apolar}(W_2), 1) \geq 4\}.$$

It is enough to show that $\mathcal{C}$ is open in $\mathrm{Gr}(3, S^*_{\leq 2})$. Let

$$\mathcal{D} = \{([U], [W]) \in \mathrm{Gr}(3, S^*_1) \times \mathrm{Gr}(3, S^*_{\leq 2}) \mid [W] \in \mathrm{Gr}(3, \mathrm{Sym}^2(U) \oplus S^*_{\leq 1})\},$$

and $\rho_2 : \mathrm{Gr}(3, S^*_1) \times \mathrm{Gr}(3, S^*_{\leq 2}) \to \mathrm{Gr}(3, S^*_{\leq 2})$ be the natural projection. Observe that the complement of $\mathcal{C}$ in $\mathrm{Gr}(3, S^*_{\leq 2})$ is equal to $\rho_2(\mathcal{D})$ which is closed since $\mathcal{D}$ is projective. This concludes the proof that $\mathcal{A}^0$ is open in $\mathcal{A}$.

By Lemma 4.22 we have $A = \pi_2(\mathcal{A}^0) \cap \mathcal{F}$ where

$$\mathcal{F} = \{[W] \in \mathrm{Gr}(3, S^*_{\leq 2}) \mid \dim_{\mathbb{C}} W_2 = 3\}$$

and $\pi_2 \colon \mathrm{Gr}(4, S^*_1) \times \mathrm{Gr}(3, S^*_{\leq 2}) \to \mathrm{Gr}(3, S^*_{\leq 2})$ is the projection.

Since $\pi_2|_{\mathcal{A}^0} \colon \mathcal{A}^0 \to \pi_2(\mathcal{A}^0)$ has a finite fiber over a general point, it follows from [77, Theorem 11.4.1] that $\pi_2(\mathcal{A}^0)$ is irreducible and of dimension $7n+8$. The subset $\mathcal{F} \subseteq \mathrm{Gr}(3, S^*_{\leq 2})$ is open and $\pi_2(\mathcal{A}^0) \cap \mathcal{F}$ is non-empty, so $A = \pi_2(\mathcal{A}^0) \cap \mathcal{F}$ is irreducible and of dimension $7n+8$.

We know that $A = \pi_2(\mathcal{A}) \cap \mathcal{C} \cap \mathcal{F}$, so $A$ is locally closed since $\pi_2(\mathcal{A})$ is closed and $\mathcal{C}$, $\mathcal{F}$ are open. Therefore we have a morphism $\mu : A \to \mathcal{H}ilb_8(\mathbb{A}^n)$ given on closed points by $[W] \mapsto [\operatorname{Spec} S/\operatorname{Ann}(W)]$, see Theorem 4.9. We obtain $B = \mu^{-1}(\mathcal{H}_{143}(\mathbb{A}^n) \setminus \mathcal{H}ilb_8^{sm}(\mathbb{A}^n))$, so it is open in $A$. We claim that $B$ is non-empty. Indeed, consider the subspace $W = \langle x_2x_4, x_1x_3, x_2x_3 - x_1x_4 \rangle \subseteq S^*_{\leq 2}$. By Lemma 4.22 we have that $[W] \in A$. Furthermore, we can calculate that

$$\operatorname{Ann}(W) = (\alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2, \alpha_1\alpha_2, \alpha_3\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3, \alpha_5, \alpha_6, \ldots, \alpha_n),$$

and therefore $\operatorname{Apolar} W$ is non-smoothable, see [29, the proof of Prop. 5.1]. This finishes the proof of the claim. Since $B$ is open and non-empty and $A$ is irreducible it follows that $B$ is dense in $A$. $\square$

We consider the ring $T = \mathbb{C}[\alpha_0, \ldots, \alpha_n]$, and the graded dual $T^* = \mathbb{C}[x_0, \ldots, x_n]$. Let $d \geq 3$ be an integer. Recall the notions of triangle operator from Definition 4.18 and dehomogenization with respect to a basis from Definition 4.19. In the following lemma, we use them for $k = 1$.

**Lemma 4.24.** *The following set is irreducible, and of dimension* $8n + 9$

$$C = \{(z_0, [U]) \in T_1 \times \operatorname{Gr}(3, T_2) \mid \textit{there exists a completion of } z_0 \textit{ to a basis } (z_0, z_1, \ldots, z_n) \textit{ of } T_1 \textit{ such that } \operatorname{Apolar}((U|_{z_0=1})^{\blacktriangledown d}) \textit{ has Hilbert function } (1,4,3)\}.$$

*Moreover, the set*

$$D = \{(z_0, [U]) \in T_1 \times \operatorname{Gr}(3, T_2) \mid \textit{there exists a completion of } z_0 \textit{ to a basis } (z_0, z_1, \ldots, z_n) \textit{ of } T_1 \textit{ such that } \operatorname{Apolar}((U|_{z_0=1})^{\blacktriangledown d}) \textit{ has Hilbert function } (1,4,3), \textit{ and } [\operatorname{Spec}\operatorname{Apolar}((U|_{z_0=1})^{\blacktriangledown d})] \notin \mathcal{H}ilb_8^{sm}(\mathbb{A}^n)\}.$$

*is dense in $C$.*

*Proof.* Consider the morphism $\varphi : GL(T_1^*) \times \operatorname{Gr}(3, T_2^*) \to \operatorname{Gr}(3, T_2^*)$, given by a change of basis. We have a product morphism

$$\begin{aligned} \tau : GL(T_1^*) \times \operatorname{Gr}(3, T_2^*) &\to T_1^* \times \operatorname{Gr}(3, T_2^*). \\ (a, [U]) &\mapsto (a(x_0), \varphi(a, [U])). \end{aligned}$$

Recall the sets $A, B$ from Lemma 4.23. Let $\chi : S^*_{\leq 2} \to T^*_2$ be the inverse of the $\mathbb{C}$-linear isomorphism $T^*_2 \to S^*_{\leq 2}$ given by $P \mapsto (P|_{x_0=1})^{\blacktriangledown d}$. We have $\tau(GL(T^*_1) \times A) = C$ and $\tau(GL(T^*_1) \times B) = D$. These follow from the definitions of the sets $A, B, C, D$ and the identity

$$(\varphi(a, \chi(W))|_{a(x_0)=1})^{\blacktriangledown d} = \langle f(a(x_1), \ldots, a(x_n)) \mid f \in W \rangle$$

for every $[W] \in \mathrm{Gr}(3, S^*_{\leq 2})$ and $a \in GL(T^*_1)$. It follows from Lemma 4.23 that $C$ is irreducible, $D$ is dense in $C$, and that $\dim D = \dim C = 8n+9$. $\square$

# Chapter 5

# Examples of rank calculation

Finally, we look at some examples of rank calculation. In this section, we denote the variables of the ring $T$ by Greek letters $\alpha, \beta, \dots$ and the corresponding variables in $T^*$ by $x, y, \dots$ (possibly with subscripts).

We work over $\mathbb{C}$.

We calculate the ranks, cactus ranks and border ranks (denoted by $\mathrm{r}(F)$, $\mathrm{cr}(F)$, $\mathrm{br}(F)$) of some monomials $F$ for toric surfaces embedded into projective spaces. See Definitions 3.7 and 3.10 for the definitions of these ranks.

*Remark* 5.1. It would be interesting to look at some toric varieties of higher dimension and apply the techniques of this chapter to them. In particular, to calculate the simultaneous rank of $k$ forms of different degrees $d_1, \dots, d_k$, one may consider the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^n}(d_k))$ over $\mathbb{P}^n$. This is a toric variety. The rank calculated with respect to the canonical embedding

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^n}(d_k)) \to \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_1)) \oplus \dots \oplus H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_k)))$$

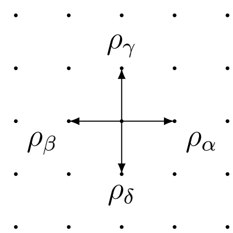
is the simultaneous rank.

## 5.1 $\mathbb{P}^1 \times \mathbb{P}^1$

Consider the set

$$\{\rho_\alpha = (1,0), \rho_\beta = (-1,0), \rho_\gamma = (0,1), \rho_\delta = (0,-1)\}.$$

Let $\Sigma$ be the only complete fan such that this set is its set of rays. Then $X_\Sigma$ is $\mathbb{P}^1 \times \mathbb{P}^1$, which is smooth.




Its class group is the free abelian group on two generators $D_{\rho_\alpha} \sim D_{\rho_\beta}$ and $D_{\rho_\gamma} \sim D_{\rho_\delta}$. Here and later in this chapter $D_\rho$ is the toric invariant divisor corresponding to $\rho$ (as in Chapter 2) and $\sim$ is the linear equivalence. Let $\alpha, \beta, \gamma, \delta$ be the variables corresponding to $\rho_\alpha$, $\rho_\beta$, $\rho_\gamma$, $\rho_\delta$. As a result, we may think of $T$ as the polynomial ring $\mathbb{C}[\alpha, \beta, \gamma, \delta]$ graded by $\mathbb{Z}^2$, where the grading is given by

| $f$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|---|---|---|---|---|
| $\deg f$ | 1 | 1 | 0 | 0 |
| | 0 | 0 | 1 | 1 |

The nef cone in $(\operatorname{Cl} X_\Sigma)_\mathbb{R}$ is generated by $D_{\rho_\alpha}$ and $D_{\rho_\gamma}$.

Let $x, y, z, w$ be the basis dual to $\alpha, \beta, \gamma, \delta$. We consider the problem of determining the cactus ranks and ranks of monomials $F = x^{(k)} y^{(l)} z^{(m)} w^{(n)}$, where $k \geq l \geq 1, m \geq n \geq 1$. The annihilator ideal is $(\alpha^{k+1}, \beta^{l+1}, \gamma^{m+1}, \delta^{n+1})$. We have

$$\dim(T/\operatorname{Ann}(F))_{(l,n)} = (l+1)(n+1).$$

It follows from Proposition 3.22 that

$$\operatorname{cr}(F) \geq (l+1)(n+1).$$

But $I = (\beta^{l+1}, \gamma^{n+1}) \subseteq \operatorname{Ann}(F)$ is a $B$-saturated ideal of a scheme of length $(l+1)(n+1)$. This is because we can look locally at the affine open set where $\alpha, \gamma \neq 0$. There our scheme becomes

$$\operatorname{Spec} \mathbb{C}\left[\frac{\beta}{\alpha}, \frac{\delta}{\gamma}\right] / \left(\frac{\beta^{l+1}}{\alpha^{l+1}}, \frac{\delta^{n+1}}{\gamma^{n+1}}\right) \cong \operatorname{Spec} \mathbb{C}[u, v]/(u^{l+1}, v^{n+1})$$

for some variables $u, v$. The scheme constructed in this way has the desired length. Hence, by Theorem 3.9

$$\operatorname{cr}(F) = (l+1)(n+1).$$

Now we address the problem of finding the ranks of such monomials. We prove Theorem 1.9. The formulation and the proof come from the author's article [46].

Let $S$ be the polynomial ring $\mathbb{C}[u,v]$. In the following, we consider the dehomogenization given by the ring homomorphism $T \to S$, $\alpha \mapsto u$, $\beta \mapsto 1$, $\gamma \mapsto v$, $\delta \mapsto 1$, and the corresponding homogenization.

**Lemma 5.2.** *Consider the ideal $I = (u^o v^p - 1, u^q - v^r) \subseteq S$, where $o, p \geq 1$ and at least one of the integers $q, r$ is greater than or equal to 1. Then $V(I) \subset \mathbb{A}^2$ consists of $or+pq$ reduced points, and the ideal $I$ is $(uv)$-saturated.*

*Proof.* First we show that $u^o v^p - 1, u^q - v^r$ intersect transversally. Let $s \in \mathbb{N}$ be the smallest number such that $u^s v^t - 1 \in I$ for some $t \in \mathbb{Z}_{>0}$. Let $i \in \mathbb{N}$ be the smallest number such that $u^i - v^j \in I$ for some $j \in \mathbb{Z}_{>0}$. We claim that $s = 0$ or $i = 0$. Assume to the contrary, that $\min(s,i) > 0$. If $s \geq i$, then we have

$$u^s v^t - 1 - u^{s-i} v^t (u^i - v^j) = u^{s-i} v^{j+t} - 1 \in I,$$

which contradicts the minimality of $s$. If $s < i$, then

$$v^t (u^i - v^j) - u^{i-s} (u^s v^t - 1) = u^{i-s} - v^{t+j} \in I,$$

which contradicts the minimality of $i$.

We get that $v^i - 1 \in I$ for some $i \in \mathbb{Z}_{>0}$. Similarly, by interchanging the roles of $i, s$ with $j, t$, we get that $u^j - 1 \in I$ for some $j \in \mathbb{Z}_{>0}$. The polynomials $v^i - 1$ and $u^j - 1$ intersect transversally in $ij$ points, so $u^o v^p - 1, u^q - v^r$ also intersect transversally.

We want to use Bézout's theorem for $\mathbb{P}^1 \times \mathbb{P}^1$. In order to do so, we homogenize generators of $I$ and check that they have no roots at infinity. Then the generators of $I$ become

$$\alpha^o \gamma^p - \beta^o \delta^p, \alpha^q \delta^r - \beta^q \gamma^r. \tag{5.1}$$

Now we can see that if $\beta = 0$, then $\alpha \neq 0$, so if we put this into the first generator in Equation 5.1, we get that $\gamma = 0$, and if we put it into the second generator, we get $\delta = 0$. But $\gamma$ and $\delta$ cannot simultaneously be 0. Similarly, if $\delta = 0$, then $\gamma \neq 0$. From the first generator, we get that $\alpha = 0$, and from the second we have $\beta = 0$, but the two equalities cannot hold at the same time.

This means that the polynomials $u^o v^p - 1, u^q - v^r$ have no common roots at infinity, so we can use multihomogeneous Bézout's theorem (see [72, Example 4.9]) to get that $u^o v^p - 1, u^q - v^r$ have $or + pq$ common roots.

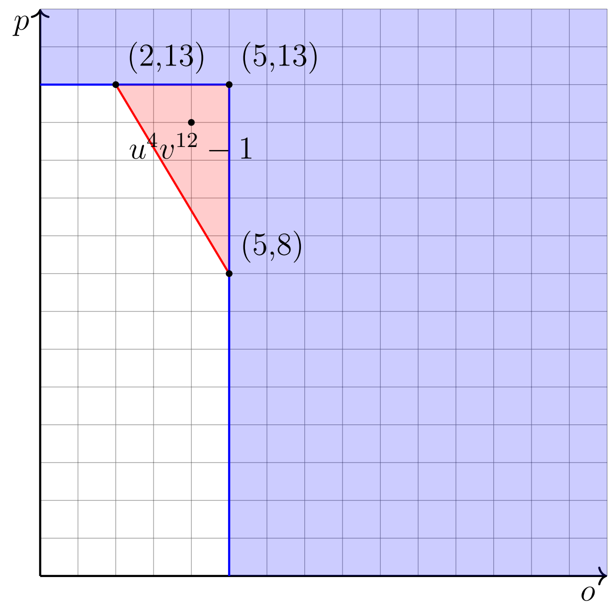
The second claim follows from the fact that $u^j - 1, v^i - 1 \in I$, hence all the points of $V(I)$ lie outside of the lines $uv = 0$. $\square$

The proof of Theorem 1.9(i) can be better understood when we put binomials on the $\mathbb{Z}^2$ lattice. We put the binomial $u^o v^p - 1$ in degree $(o, p)$ (that is the degree of the homogenization). We also put the binomial $u^q - v^r$ in degree $(q, r)$. In order to prove Point (i), we define the ideal of $(k+1)(n+1) + (l+1)(m-n)$ points

$$I = (u^{l+1} - v^{n+1}, u^{k+1} v^{m-n} - 1).$$

The part in the proof that is hardest to picture is showing that $I^{\text{hom}} \subseteq \operatorname{Ann}(F)$. For this, we need to argue that for each binomial $u^o v^p - 1 \in I$, we have $(u^o v^p - 1)^{\text{hom}} \in \operatorname{Ann}(F)$, and that for each binomial $u^q - v^r \in I$, we have $(u^q - v^r)^{\text{hom}} \in \operatorname{Ann}(F)$. To get ready for the proof, we study the case of $F = x^{(4)} y^{(2)} z^{(12)} w^{(4)}$.

**Example 5.3.** Let $F = x^{(4)} y^{(2)} z^{(12)} w^{(4)}$. Then $I = (u^3 - v^5, u^5 v^8 - 1)$. By Lemma 5.2, $I$ is a radical ideal of 49 points. First we tackle the binomials of the form $u^o v^p - 1$. We need to show that their homogenizations are in the ideal $(\alpha^5, \beta^3, \gamma^{13}, \delta^5)$. We calculate that $(u^o v^p - 1)^{\text{hom}} \in \operatorname{Ann}(F)$ if and only if $o \geq 5$ or $p \geq 13$.




So suppose $u^o v^p - 1 \in I$ for some nonnegative integers $o, p$. Since $(u^o v^p - 1, u^3 - v^5) \subseteq I$, by Lemma 5.2 we get that $5o + 3p \geq 49$. Geometrically, this means that the monomial $u^o v^p - 1$ lies on or above the line $5o + 3p = 49$. Hence, it suffices to show that neither the points in the interior of the red triangle nor the ones in the interior of the segment $(5, 8) - (2, 13)$ can be in $I$. Take for instance the point $(4, 12)$, corresponding to the monomial $u^4v^{12} - 1$. Assume that $u^4v^{12} - 1 \in I$. But then

$$u^4v^{12} - 1 - (u^5v^8 - 1) = u^4v^8(u - v^4) \in I.$$

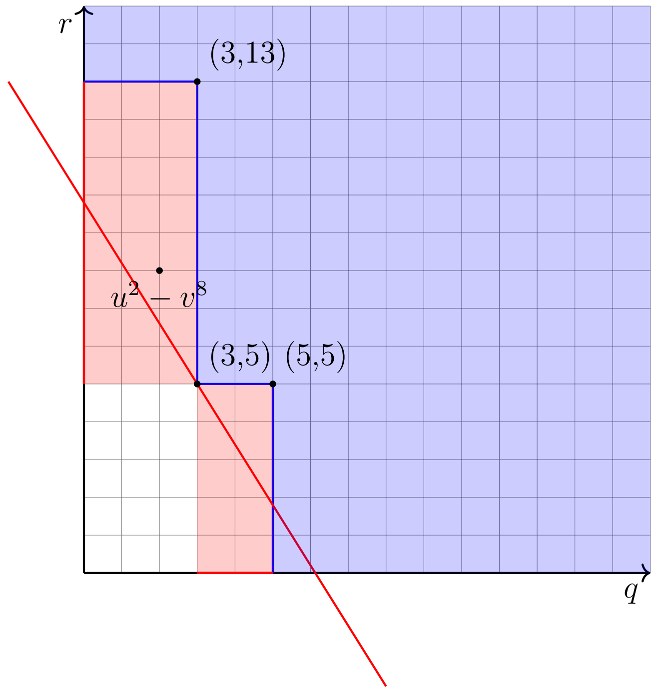
As $I$ is $uv$-saturated by Lemma 5.2, we get that $u - v^4 \in I$. It follows that

$$(u - v^4, u^5v^8 - 1) \subseteq I,$$

however, by Lemma 5.2 the vanishing locus of the ideal $(u - v^4, u^5v^8 - 1)$ has 28 points, a contradiction.

Now we want to show that binomials of the form $u^q - v^r$ belonging to $I$ satisfy $(u^q - v^r)^{\mathrm{hom}} \in \mathrm{Ann}(F)$. We calculate that this condition is equivalent to

$$(q \geq 5 \text{ or } r \geq 5) \text{ and } (q \geq 3 \text{ or } r \geq 13).$$




Suppose $u^q - v^r \in I$ for some nonnegative integers $q, r$. Since $(u^q - v^r, u^5v^8 - 1) \subseteq I$, by Lemma 5.2, we get that $8q + 5r \geq 49$. Geometrically, this means that the monomial $u^q - v^r$ lies on or above the line $8q + 5r = 49$. Hence, it suffices to show that the points in the interior of the red rectangles as well as the ones in the interior of the segments $(3,0)-(5,0)$ and $(0,5)-(0,13)$ cannot be in $I$. Take for instance the point $(2,8)$, corresponding to the monomial $u^2 - v^8$. Assume that $u^2 - v^8 \in I$. But then we have

$$u^2 - v^8 - v^3(u^3 - v^5) = u^2(1 - uv^3).$$

As $I$ is $uv$-saturated by Lemma 5.2, we get $uv^3 - 1 \in I$. It follows that we have the following inclusion

$$(uv^3 - 1, u^3 - v^5) \subseteq I.$$

But the ideal $(uv^3 - 1, u^3 - v^5)$ has 14 points, a contradiction.

*Proof of Point (i) of Theorem 1.9.* If $k = l$ or $m = n$, Point (i) becomes Equation (1.1), so it is true. Now assume $k > l$ and $m > n$ and consider $I = (u^{l+1} - v^{n+1}, u^{k+1}v^{m-n} - 1)$. Let

$$M = (k+1)(n+1) + (l+1)(m+1) - (l+1)(n+1).$$

From Proposition 2.18 we know that $I^{\text{hom}}$ is $\beta\delta$-saturated, which implies that it is $B$-saturated. By Lemma 5.2, we get that $I^{\text{hom}}$ is a radical ideal of $M$ points. We need to show that $I^{\text{hom}} \subseteq \operatorname{Ann}(F)$. From Proposition 2.21 it suffices to show that for $u^o v^p - 1 \in I$ we have $(u^o v^p - 1)^{\text{hom}} \in \operatorname{Ann}(F)$ and that for $u^q - v^r \in I$ we have $(u^q - v^r)^{\text{hom}} \in \operatorname{Ann}(F)$.

Since $u^{l+1} - v^{n+1} \in I$, from Lemma 5.2 we get that any element of the form $u^o v^p - 1 \in I$ must satisfy

$$(l+1)p + (n+1)o \geq M. \tag{5.2}$$

Then the binomials $u^o v^p - 1 \in I$ lie above or on the line connecting two points: $(k+1, n-m)$ and $(k-l, m+1)$. Similarly, the binomials $u^q - v^r \in I$ satisfy

$$(k+1)r + (n-m)q \geq M. \tag{5.3}$$

We have:

*Claim 1:* for each binomial of the form $u^o v^p - 1 \in I$ we have either $o \geq k+1$ or $p \geq m+1$. It suffices to argue that there are no elements $u^o v^p - 1 \in I$ in the interior of the segment connecting points $(k+1, m-n)$ and $(k-l, m+1)$ nor in the interior of the triangle with vertices $(k+1, m-n), (k-l, m+1), (k+1, m+1)$. Suppose that $u^o v^p - 1$ is such, then

$$u^o v^p - 1 - (u^{k+1} v^{m-n} - 1) = u^o v^{m-n}(v^{p-m+n} - u^{k+1-o}) \in I.$$

Since $I$ is $uv$-saturated by Lemma 5.2, we get that $u^{k+1-o} - v^{p-m+n} \in I$. But $k+1-o < l+1$ and $p-m+n < n+1$ (because $o > k-l$ and $p < m+1$, respectively), so the point $(q, r) = (k+1-o, p-m+n)$ lies below the line $(k+1)r + (n-m)q = M$, contradicting Equation (5.3).

*Claim 2:* there are no binomials $u^q - v^r \in I$ lying in the interior of the rectangle with vertices $(k+1, 0), (l+1, 0), (l+1, n+1), (k+1, n+1)$ nor in the interior of the segment $(l+1, 0) - (k+1, 0)$. Indeed, for any such binomial $u^q - v^r$ we have

$$u^q - v^r - u^{q-l-1}(u^{l+1} - v^{n+1}) = v^r(u^{q-l-1} v^{n+1-r} - 1) \in I,$$

so also $u^{q-l-1} v^{n+1-r} - 1 \in I$. But $q-l-1 < k-l$ and $n+1-r < m+1$ (since $q < k+1$ and $r > 0$, respectively), so the point $(o, p) = (q-l-1, n+1-r)$ lies below the line $(l+1)p + (n+1)o = M$, contradicting Equation (5.2).

*Claim 3:* there are no binomials $u^q - v^r \in I$ lying in the interior of the rectangle with vertices $(0, n+1), (0, m+1), (l+1, n+1), (l+1, m+1)$ nor in the interior of the segment $(0,n+1)-(0,m+1)$. This is just *Claim 2* with the roles of the axes reversed.

From *Claim 1, 2* and *3* it follows that for each $u^o v^p - 1 \in I$ we have $(u^o v^p - 1)^{\mathrm{hom}} \in \mathrm{Ann}(F)$ and for $u^q - v^r \in I$ we have $(u^q - v^r)^{\mathrm{hom}} \in \mathrm{Ann}(F)$. From it we conclude that for any binomial $b \in I$ we have $b^{\mathrm{hom}} \in \mathrm{Ann}(F)$ (since homogenization is well-behaved with respect to multiplication by monomials), so from Proposition 2.21 we have $I^{\mathrm{hom}} \subseteq \mathrm{Ann}(F)$, and we are done with Point (i). $\square$

*Proof of Point (ii).* Suppose that $\mathrm{r}(F) < (k+1)(n+1)$. Then by Theorem 3.9 there is a radical $B$-saturated ideal $I$ of at most $(k+1)(n+1)-1$ points such that $I \subseteq \mathrm{Ann}(F) = (\alpha^{k+1}, \beta^{l+1}, \gamma^{m+1}, \delta^{n+1})$. By Proposition 3.18 we have that $\dim(T/I)_{k,n} \leq (k+1)(n+1)-1$. We know that $\dim T_{k,n} = (k+1)(n+1)$. But this means that $\dim I_{k,n} \geq 1$. We have

$$\begin{aligned}&\mathrm{Ann}(F)_{k,n} = \\ &\beta^{l+1} \cdot \langle \alpha^{k-l-1}, \alpha^{k-l-2}\beta, \ldots, \beta^{k-l-1} \rangle \cdot \langle \gamma^n, \gamma^{n-1}\delta, \ldots, \delta^n \rangle\end{aligned}$$

Hence there is a non-zero polynomial

$$\beta^{l+1}(\kappa_{k-l-1}\alpha^{k-l-1} + \kappa_{k-l-2}\alpha^{k-l-2}\beta + \ldots + \kappa_0 \beta^{k-l-1}) \in I, \tag{5.4}$$

where $\kappa_i \in \langle \gamma^n, \gamma^{n-1}\delta, \ldots, \delta^n \rangle$.

We prove that $\kappa_j = 0$ for $j = 0, \ldots, k-l-1$. We do this by descending induction on $j$. Assume the following equation holds for a given $j$

$$\kappa_{k-l-1} = \kappa_{k-l-2} = \cdots = \kappa_{j+1} = 0.$$

We prove that $\kappa_j = 0$. By Equation (5.4) for some $i \geq 1$

$$\beta^i(\kappa_j \alpha^j + \kappa_{j-1}\alpha^{j-1}\beta + \ldots + \kappa_0 \beta^j) \in I.$$

The ideal $I$ is radical, so we know that

$$\beta(\kappa_j \alpha^j + \kappa_{j-1}\alpha^{j-1}\beta + \ldots + \kappa_0 \beta^j) \in I.$$

But $I \subseteq \mathrm{Ann}(F)$, so

$$\beta(\kappa_j \alpha^j + \kappa_{j-1}\alpha^{j-1}\beta + \ldots + \kappa_0 \beta^j) \lrcorner F = 0.$$

Since $F$ is a monomial, we get that

$$(\kappa_j \alpha^j \beta) \lrcorner F = 0.$$

However, we know that, as $l \geq 1$ and $j < k$, we get

$$(\kappa_j \alpha^j \beta) \lrcorner F = \bar{\kappa}_j x^{(k-j)} y^{(l-1)},$$

where $\bar{\kappa}_j = \kappa_j \lrcorner z^{(m)} w^{(n)} \in \mathbb{C}_{dp}[z, w]_m$. Hence,

$$\bar{\kappa}_j = 0. \tag{5.5}$$

Observe that since we have

$$\kappa_j \in \langle \gamma^n, \gamma^{n-1}\delta, \ldots, \delta^n \rangle,$$

and $m \geq n$, we know that Equation (5.5) implies that $\kappa_j = 0$, as desired.

The fact that $\kappa_i$ are all zero gives a contradition with the fact that the polynomial was non-zero. This proves Point (ii). □

*Proof of Point (iii).* Let $I \subseteq \mathrm{Ann}(F)$ be a $B$-saturated radical ideal of at most $(l+2)(n+2)-2$ points. Then $\dim(T/I)_{l+1,n+1} \leq (l+2)(n+2)-2$, so $\dim I_{l+1,n+1} \geq 2$. Since

$$\begin{aligned} \mathrm{Ann}(F)_{l+1,n+1} = {} & \beta^{l+1} \langle \gamma^{n+1}, \gamma^n \delta, \ldots, \gamma \delta^n \rangle \\ & + \langle \alpha^{l+1}, \alpha^l \beta, \ldots, \alpha \beta^l \rangle \cdot \delta^{n+1} + \langle \beta^{l+1} \delta^{n+1} \rangle, \end{aligned}$$

we get that $I_{l+1,n+1}$ has a basis consisting of

$$\begin{aligned} t_1 &= \beta^{l+1}(\kappa_1 \gamma^n \delta + \cdots + \kappa_n \gamma \delta^n) \\ &\quad + (\lambda_0 \alpha^{l+1} + \cdots + \lambda_l \alpha \beta^l)\delta^{n+1} + \eta \beta^{l+1} \delta^{n+1}, \\ t_2 &= \beta^{l+1}(\mu_0 \gamma^{n+1} + \mu_1 \gamma^n \delta + \cdots + \mu_n \gamma \delta^n) \\ &\quad + (\nu_0 \alpha^{l+1} + \cdots + \nu_l \alpha \beta^l)\delta^{n+1} + \zeta \beta^{l+1} \delta^{n+1}, \end{aligned}$$

where $\kappa_i, \lambda_i, \mu_i, \nu_i, \eta, \zeta \in \mathbb{C}$. If $\kappa_1 = 0$, then (since $I$ is radical and $t_1$ is divisible by $\delta^2$) $t_1/\delta \in I$. Out of the monomials of $t_1/\delta$, only the ones divisible by $\beta^{l+1}$ are in $\mathrm{Ann}(F)$. Hence, $\lambda_i = 0$ for all $i$. But then from the fact that $I$ is radical and that $t_1/\delta$ is divisible by $\beta^2$ we get that $t_1/(\beta\delta) \in I$. Since none of the monomials of $t_1/(\beta\delta)$ are in $\mathrm{Ann}(F)$, we get $\eta = \kappa_2 = \kappa_3 = \cdots = \kappa_n = 0$, a contradiction. It follows that we may assume that $\kappa_1 = 1$.

If $\mu_0 = 0$, then we consider the element $\mu_1 t_1 - t_2$. It is divisible by $\delta^2$, so in the same way as before, we get that all the coefficients of $\mu_1 t_1 - t_2$ are 0. It follows that we may assume that $\mu_0 = 1$.

In this case

$$\begin{aligned}
&\gamma t_1 - \delta t_2 \\
&= \beta^{l+1}((\kappa_2 - \mu_1)\gamma^n \delta^2 + \cdots + (\kappa_n - \mu_{n-1})\gamma^2 \delta^n - \mu_n \gamma \delta^{n+1}) \\
&+ (\lambda_0 \alpha^{l+1} + \lambda_1 \alpha^l \beta + \ldots \lambda_l \alpha \beta^l)\gamma \delta^{n+1} + \eta \beta^{l+1} \gamma \delta^{n+1} \\
&- (\nu_0 \alpha^{l+1} + \nu_1 \alpha^l \beta + \ldots \nu_l \alpha \beta^l)\delta^{n+2} + \zeta \beta^{l+1} \delta^{n+2}.
\end{aligned}$$

This is divisible by $\delta^2$, so also

$$\frac{\gamma t_1 - \delta t_2}{\delta} \in I.$$

The monomials $\alpha^{l+1}\gamma\delta^n, \ldots, \alpha\beta^l\gamma\delta^n$ are not in $\mathrm{Ann}(F)$ (which is a monomial ideal), so $\lambda_i = 0$ for all $i$. Hence $t_1$ is divisible by $\beta^{l+1}$, and therefore $t_1/\beta^l \in I \subseteq \mathrm{Ann}(F)$. The monomial $\beta\gamma^n\delta$ is not in $\mathrm{Ann}(F)$, but its coefficient in $t_1/\beta^l$ is $\kappa_1 = 1$, a contradiction. $\square$

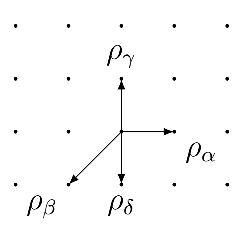
## 5.2 Hirzebruch surface $\mathbb{F}_1$

This section is taken from the author's master thesis. Still, we put it also here, since it contains an important remark (Remark 5.5) about the wildness or tameness of monomials.

Consider the set

$$\{\rho_\alpha = (1, 0), \rho_\beta = (-1, -1), \rho_\gamma = (0, 1), \rho_\delta = (0, -1)\}.$$

Let $\Sigma$ be the only complete fan such that this set is the set of rays of $\Sigma$. Then $X_\Sigma$ is called the Hirzebruch surface $\mathbb{F}_1$. It is smooth. We present a picture of the lattice of $\mathbb{F}_1$.



Its class group is the free abelian group on two generators $D_{\rho_\alpha} \sim D_{\rho_\beta}$ and $D_{\rho_\delta}$. Moreover, $D_{\rho_\gamma} \sim D_{\rho_\delta} + D_{\rho_\alpha}$. Let $\alpha, \beta, \gamma, \delta$ be the variables corresponding to $\rho_\alpha$, $\rho_\beta$, $\rho_\gamma$, $\rho_\delta$. As a result, we may think of $T$ as the polynomial ring $\mathbb{C}[\alpha, \beta, \gamma, \delta]$ graded by $\mathbb{Z}^2$, where the grading is given by

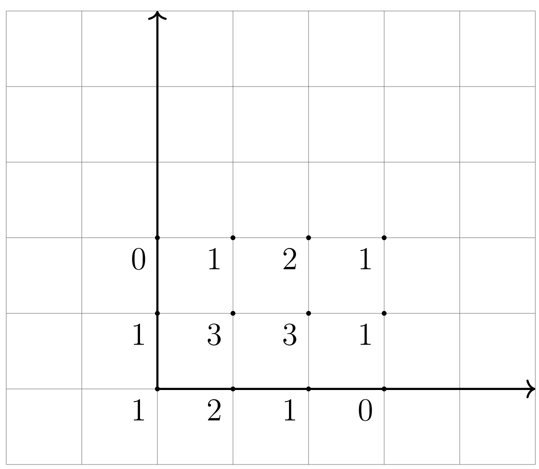
| $f$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|---|---|---|---|---|
| $\deg f$ | 1 | 1 | 1 | 0 |
| | 0 | 0 | 1 | 1 |

The nef cone in $(\mathrm{Cl}\, X_\Sigma)_\mathbb{R}$ is generated by $D_{\rho_\alpha}$ and $D_{\rho_\gamma} \sim dD_{\rho_\alpha} + D_{\rho_\delta}$.

**Example 5.4.** Consider the monomial $F \coloneqq xyzw$, where $x, y, z, w$ is the basis dual to $\alpha, \beta, \gamma, \delta$. It has degree $(3, 2)$, so it is in the interior of the nef cone, hence the corresponding line bundle is very ample. We claim that the rank and the cactus rank of $F$ are four, and that the border rank is three:

| $\mathrm{r}(F)$ | $\mathrm{cr}(F)$ | $\mathrm{br}(F)$ |
|---|---|---|
| 4 | 4 | 3 |

Let us compute the Hilbert function of the apolar algebra of $F$.




Notice that it can only be non-zero in the first quadrant. Hence, the symmetry of the Hilbert function ([46, Proposition 4.5]) implies it can only be non-zero in the rectangle with vertices $(0, 0)$, $(3, 0)$, $(3, 2)$, $(0, 2)$.

The apolar ideal $\mathrm{Ann}(F)$ is $(\alpha^2, \beta^2, \gamma^2, \delta^2)$. (It is independent of the grading, so we can just copy the result from the Waring rank case, see [71].)

Firstly, we show that the rank is at most four. By Theorem 3.9, it is enough to find a reduced zero-dimensional subscheme of length four $R$ of $X_\Sigma$

(i.e. a set of four points in $X_\Sigma$) such that $I_{\mathrm{Cl}}(R) \subseteq \operatorname{Ann}(F)$. The subscheme defined by $I = (\alpha^2 - \beta^2, \gamma^2 - \beta^2\delta^2) \subseteq \operatorname{Ann}(F)$ satisfies these requirements. This scheme is a reduced union of four points:

$$[1,1;1,1], [1,1;1,-1], [1,-1;1,1], [1,-1;1,-1].$$

As a consequence, we may write

$$xyzw = \frac{1}{4}\left(\varphi(1,1;1,1) - \varphi(1,1;1,-1) - \varphi(1,-1;1,1) + \varphi(1,-1;1,-1)\right).$$

We show that the cactus rank is at least four. Suppose it is at most three. Then there is a $B$-saturated homogeneous ideal $I \subseteq \operatorname{Ann}(F)$ defining a zero-dimensional subscheme $R$ of length at most three. From the calculation of the Hilbert function, we know that $\dim_{\mathbb{C}} \operatorname{Ann}(F)_{2,1} = 2$. Let us calculate $\dim_{\mathbb{C}} I_{2,1}$. Since $I$ is $B$-saturated, the vector subspace $I_{2,1} \subseteq T_{2,1}$ are the sections which are zero on $R$. But from Proposition 3.18

$$3 \geq \text{length of } R \geq \dim_{\mathbb{C}} T_{2,1} - \dim_{\mathbb{C}} I_{2,1} = 5 - \dim_{\mathbb{C}} I_{2,1},$$

so

$$\dim_{\mathbb{C}} I_{2,1} \geq 2.$$

By Theorem 3.9, we have $I_{2,1} \subseteq (\operatorname{Ann}(F))_{2,1}$. As the dimensions are equal, it follows that $I_{2,1} = (\operatorname{Ann}(F))_{2,1}$. This means $\alpha^2\delta, \beta^2\delta \in I$. But $I$ is $B$-saturated, so $\alpha\beta\delta \in I \subseteq \operatorname{Ann}(F)$, which implies that $\alpha\beta\delta \lrcorner\, xyzw = 0$, a contradiction.

In order to argue that $\operatorname{br}(F) = 3$, we show that the third secant variety $\sigma_3(X) = \mathbb{P}^8$. It suffices to show that $\dim \sigma_3(X_\Sigma)$ is eight. We will use Terracini's Lemma (Proposition 3.2).

Since $X_\Sigma \to \mathbb{P}(H^0(X_\Sigma, \mathcal{O}(3,2))^*)$ is given by a parametrization, we can calculate the projectivized tangent space. Take points of the form $[1,\lambda;\mu,1]$, where $\lambda, \mu \in \mathbb{C}$. Then

$$\varphi([1,\lambda;\mu,1]) = [1, \lambda, \lambda^2, \lambda^3, \mu, \mu\lambda, \mu\lambda^2, \mu^2, \mu^2\lambda].$$

The coordinates are in the standard monomial basis of $H^0(X_\Sigma, \mathcal{O}(3,2))^*$. The affine tangent space at $\varphi([1,\lambda;\mu,1])$ is spanned by the vector

$$v = [1, \lambda, \lambda^2, \lambda^3, \mu, \mu\lambda, \mu\lambda^2, \mu^2, \mu^2\lambda]$$

and its two derivatives with respect to $\lambda$ and $\mu$:

$$\frac{\partial v}{\partial \lambda} = [0, 1, 2\lambda, 3\lambda^2, 0, \mu, 2\mu\lambda, 0, \mu^2],$$
$$\frac{\partial v}{\partial \mu} = [0, 0, 0, 0, 1, \lambda, \lambda^2, 2\mu, 2\mu\lambda],$$

If we take three general points, say $[1, p, q, 1], [1, s, t, 1], [1, u, v, 1]$, we can look at the space spanned by the three tangent spaces. This will be the space spanned by the rows of the following matrix:

$$M = \begin{pmatrix} 1 & p & p^2 & p^3 & q & qp & qp^2 & q^2 & q^2p \\ 0 & 1 & 2p & 3p^2 & 0 & q & 2qp & 0 & q^2 \\ 0 & 0 & 0 & 0 & 1 & p & p^2 & 2q & 2qp \\ 1 & s & s^2 & s^3 & t & ts & ts^2 & t^2 & t^2s \\ 0 & 1 & 2s & 3s^2 & 0 & t & 2ts & 0 & t^2 \\ 0 & 0 & 0 & 0 & 1 & s & s^2 & 2t & 2ts \\ 1 & u & u^2 & u^3 & v & vu & vu^2 & v^2 & v^2u \\ 0 & 1 & 2u & 3u^2 & 0 & v & 2vu & 0 & v^2 \\ 0 & 0 & 0 & 0 & 1 & u & u^2 & 2v & 2vu \end{pmatrix}$$

We can calculate the determinant using for instance Macaulay2

$$\det M = (s-u)(u-p)(s-p)(qs - pt - qu + tu + pv - sv)^4.$$

This is non-zero for general points on the variety. This means that the tangent space of the cone of the third secant variety at a general point has dimension nine, so $\dim \sigma_3(X_\Sigma) = 8$, hence $\sigma_3(X)$ fills the whole space.

Finally, the border rank is at least three by Proposition 3.22. We use it for the class $(2, 1)$.

*Remark* 5.5. We could also define the smoothable $X$-rank:

$$\mathrm{sr}_X(F) = \min\{\text{length } R | R \hookrightarrow X, \dim R = 0, F \in \langle R \rangle, R \text{ smoothable}\}.$$

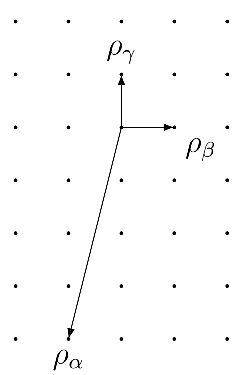
For the definition of a smoothable scheme, see [57, Definition 5.16]. For more on the smoothable rank, see [20]. We always have $\mathrm{cr}(F) \leq \mathrm{sr}(F) \leq \mathrm{r}(F)$, so in the case of $\mathbb{F}_1$ we get $\mathrm{sr}(xyzw) = 4$. In particular, we obtain what the authors in [20] call a "wild" case, i.e. the border rank is strictly less than the smoothable rank.

This is important, since there is a conjecture that the monomials on $\mathbb{P}^n$ are always "tame" (i.e. not wild) ([70, Conjecture 1.1]). This example shows that if we look at more general smooth toric varieties, this conjecture does not hold.

## 5.3 Weighted projective plane $\mathbb{P}(1,1,4)$

This section is taken from the author's article [46].

Consider a set of rays $\{\rho_\alpha = (-1,-4), \rho_\beta = (1,0), \rho_\gamma = (0,1)\}$. Let $\Sigma$ be the complete fan determined by these rays. This is a fan of $\mathbb{P}(1,1,4)$, the weighted projective space with weights $1,1,4$, see [39, Section 2.0, Subsection Weighted Projective Space; and Example 3.1.17].




The class group is $\mathbb{Z}$, generated by $D_{\rho_\alpha} \sim D_{\rho_\beta}$, and we know that $D_{\rho_\gamma} \sim 4D_{\rho_\alpha}$. The Cox ring is $\mathbb{C}[\alpha,\beta,\gamma]$, where $\alpha,\beta,\gamma$ correspond to $\rho_\alpha,\rho_\beta,\rho_\gamma$, and the degrees are given by the vector $(1,1,4)$. Let $x,y,z$ denote the dual coordinates. The Picard group is generated by $\mathcal{O}(4)$. The only singular point is $[0,0,1]$.

Consider the embedding given by the complete linear system of the line bundle $\mathcal{O}_{X_\Sigma}(4)$. It maps $X$ into $\mathbb{P}^5$ (since there are six monomials of degree 4: $x^{(4)}, x^{(3)}y, x^{(2)}y^{(2)}, xy^{(3)}, y^{(4)}, z$). We calculate various ranks of $F = x^{(2)}y^{(2)}$. The results are shown it the following table

| $\mathrm{r}(F)$ | $\mathrm{cr}(F)$ | $\mathrm{br}(F)$ |
|---|---|---|
| 3 | 2 | 3 |

The Hilbert function of $\mathrm{Apolar}(F)$ is $(1,2,3,2,1)$ (here the first element of the sequence corresponds to $\mathcal{O}_{X_\Sigma}$, the next to $\mathcal{O}_{X_\Sigma}(1)$, and so on). This means (by Proposition 3.22) that $\mathrm{br}(F) \geq 3$.

We know that $\mathrm{Ann}(F) = (\alpha^3, \beta^3, \gamma)$, since the annihilator remains the same if we change the grading. Let $I = (\alpha^3, \beta^3) \subseteq \mathrm{Ann}(F)$. We show that the length of the scheme $R \coloneqq V(I)$ is two. This will mean that $\mathrm{cr}(F) \leq 2$. Since $R$ is supported at the point $[0, 0, 1]$, we can look at it on the affine open $U_\sigma$, where $\sigma = \mathrm{Cone}(\rho_\alpha, \rho_\beta)$. After localizing $T = \mathbb{C}[\alpha, \beta, \gamma]$ at $\gamma$ and taking degree 0, we get the ring

$$\mathbb{C}\left[\frac{\alpha^4}{\gamma}, \frac{\alpha^3\beta}{\gamma}, \frac{\alpha^2\beta^2}{\gamma}, \frac{\alpha\beta^3}{\gamma}, \frac{\beta^4}{\gamma}\right].$$

Ideal $I$ becomes the ideal generated by $\frac{\alpha^4}{\gamma}, \frac{\alpha^3\beta}{\gamma}, \frac{\alpha\beta^3}{\gamma}, \frac{\beta^4}{\gamma}$ in this ring, so the quotient is a two-dimensional vector space with basis $1, \frac{\alpha^2\beta^2}{\gamma}$. Hence the length of $R$ is two.

But the cactus rank cannot be 1, since $x^{(2)}y^{(2)}$ is not in the image of $\varphi_{|\mathcal{O}(4)|}$ (see Proposition 2.26). It follows that $\mathrm{cr}(F) = 2$.

Now consider the ideal $I = (\alpha^3 - \beta^3, \gamma) \subseteq \mathrm{Ann}(F)$. We show that the length of the scheme defined by $I$ is three. Since $I$ is radical, the scheme given by $I$ is reduced, hence this will show that $\mathrm{r}(F) \leq 3$, as desired. But $I = (\alpha - \beta, \gamma) \cap (\alpha - \varepsilon\beta, \gamma) \cap (\alpha - \varepsilon^2\beta, \gamma)$, where $\varepsilon$ is a cubic root of unity, so the scheme given by $I$ is the reduced union of $[1, 1, 0], [\varepsilon, 1, 0], [\varepsilon^2, 1, 0]$.

*Remark* 5.6. Since in this example

$$\dim_{\mathbb{C}}(T/\mathrm{Ann}(F))_2 = 3,$$

and $\mathrm{cr}(F) = 2$, we see that the bound stated in point (1) of Proposition 1.12 does not hold for the cactus rank (and reflexive sheaves of rank one that are not line bundles).

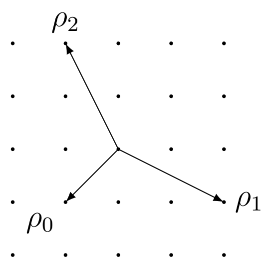
*Remark* 5.7. One can also calculate that the projective tangent space in this embedding at the singular point $[0, 0, 1]$ is the whole $\mathbb{P}^5$ (this is a straightforward application of Proposition 2.14). It follows that the cactus rank of every point in $\mathbb{P}^5$ is at most two, since any point of the tangent space at $[0, 0, 1]$ can be reached by a linear span of a scheme of length two supported at $[0, 0, 1]$.

## 5.4 Fake weighted projective plane

This section also appeared in the author's master thesis. However, we give it also here, since this version contains an additional picture of the polytope $P$

of the embedding, and an explanation how the tangent spaces at the singular points fit into this picture.

Consider the set of rays $\{\rho_0 = (-1,-1), \rho_1 = (2,-1), \rho_2 = (-1,2)\}$. Let $\Sigma$ be the complete fan determined by these rays. Then $X_\Sigma$ is an example of a fake weighted projective space, see [18, Example 6.2].




Let $\alpha_0, \alpha_1, \alpha_2$ be the corresponding coordinates in $T$. The class group is generated by $D_{\rho_0}, D_{\rho_1}, D_{\rho_2}$ with relations $D_{\rho_0} \sim 2D_{\rho_1} - D_{\rho_2} \sim 2D_{\rho_2} - D_{\rho_1}$. This is the same as a group with two generators $D_{\rho_0}$ and $D_{\rho_2} - D_{\rho_1}$ with the relation $3(D_{\rho_2} - D_{\rho_1}) \sim 0$. This choice gives an isomorphism with $\mathbb{Z} \times \mathbb{Z}/3$ sending $D_{\rho_0}$ to $(1,0)$ and $D_{\rho_2} - D_{\rho_1}$ to $(0,1)$. The Picard group is the subgroup generated by $3D_{\rho_0}$. It is free.

As a result, $T = \mathbb{C}[\alpha_0, \alpha_1, \alpha_2]$ is graded by $\operatorname{Cl} X_\Sigma = \mathbb{Z} \times \mathbb{Z}/3$, where

$$\begin{aligned}
\deg \alpha_0 &= (1,0),\\
\deg \alpha_1 &= (1,1),\\
\deg \alpha_2 &= (1,-1) = (1,2),
\end{aligned}$$

and $\operatorname{Pic} X_\Sigma$ is generated by $(3,0)$. The singular points of $X_\Sigma$ are $[1,0,0]$, $[0,1,0]$, $[0,0,1]$.

Consider the line bundle $\mathcal{O}(6,0)$. It is ample, because by [39, Proposition 6.3.25] every complete toric surface is projective, and the line bundles $\mathcal{O}(-3m,0)$ for $m < 0$ have no non-zero sections. By [39, Proposition 6.1.10, (b)] it is very ample. It gives an embedding $\varphi : X_\Sigma \hookrightarrow \mathbb{P}^9$. We denote the dual coordinates by $x_0, x_1, x_2$.

**Example 5.8.** Let $F = x_0^{(4)} x_1 x_2 \in H^0(X_\Sigma, \mathcal{O}(6,0))^*$. The apolar ideal is $(\alpha_0^5, \alpha_1^2, \alpha_2^2)$. We claim that the cactus rank is two, the rank is at most five, and the border rank is two.

| r($F$) | cr($F$) | br($F$) |
|---|---|---|
| $\leq 5$ | 2 | 2 |

Note that $F$ is not in the image of $\varphi_{|\mathcal{O}(6,0)|}$, so the cactus rank and the border rank are at least two.

We show that the cactus rank is two. Consider the ideal $I = (\alpha_1^2, \alpha_2^2) \subseteq \mathrm{Ann}(F)$. It is saturated, since $B$ in this case is $(\alpha_0, \alpha_1, \alpha_2)$, so it is the same as in the case of $\mathbb{P}^2$. We show that the length of the subscheme given by $I$ is two. Since the support of the scheme is the point $[1, 0, 0]$, we check it on the set $U_\sigma$, where $\sigma = \mathrm{Cone}(\rho_1, \rho_2)$. We localize with respect to $\alpha_0$, take degree zero, and get the ring

$$\mathbb{C}\left[\frac{\alpha_1^3}{\alpha_0^3}, \frac{\alpha_2^3}{\alpha_0^3}, \frac{\alpha_1\alpha_2}{\alpha_0^2}\right] \cong \mathbb{C}[u, v, w]/(w^3 - uv). \tag{5.6}$$

If we factor out by the ideal generated by $\alpha_1^2$ and $\alpha_2^2$, we get

$$\mathbb{C}[u, v, w]/(w^3 - uv, u, v, w^2) \cong \mathbb{C}[w]/(w^2),$$

so the length of the scheme defined by $I$ is two.

Now we show that the rank is at most five. Take a homogeneous ideal $I = (\alpha_0^5 - \alpha_1^4\alpha_2, \alpha_1^3 - \alpha_2^3) \subseteq \mathrm{Ann}(F)$. We show that the length of the subscheme defined by $I$ is five. From these equations we know that no coordinate can be zero, so we can check the length on the open subset $U_\sigma$, where $\sigma = \mathrm{Cone}(\rho_1, \rho_2)$. We get the same ring as in Equation 5.6, and we want to factor it out by the ideal generated by $\alpha_0^5 - \alpha_1^4\alpha_2$ and $\alpha_1^3 - \alpha_2^3$. The second generator gives the relation $u - v$, and the first one the relation $1 - vw$. So we get the ring

$$\mathbb{C}[v, w]/(w^3 - v^2, 1 - vw).$$

But notice that $1 = vw$ implies that $w$ is non-zero. Hence

$$\begin{aligned}\mathbb{C}[v, w]/(w^3 - v^2, 1 - vw) &\cong \mathbb{C}[v, w, w^{-1}]/(w^3 - v^2, 1 - vw)\\ &\cong \mathbb{C}[v, w, w^{-1}]/(w^5 - 1, w^{-1} - v) \cong \mathbb{C}[w, w^{-1}]/(w^5 - 1).\end{aligned}$$

We get a reduced scheme of length five, so the rank is at most five.

Now we show that $\mathrm{br}(F) = 2$. Consider the equations given by rank one reflexive sheaves $\mathcal{O}(3, 0)$ and $\mathcal{O}(3, 1)$ (given by minors of matrices). In order to find these equations, we give coordinates to every point $p \in H^0(X_\Sigma, \mathcal{O}(6, 0))^*$:

$$\begin{aligned}p = {}& t_{6,0,0}x_0^{(6)} + t_{0,6,0}x_1^{(6)} + t_{0,0,6}x_2^{(6)} + t_{4,1,1}x_0^{(4)}x_1x_2 + t_{1,4,1}x_0x_1^{(4)}x_2\\ &+ t_{1,1,4}x_0x_1x_2^{(4)} + t_{3,3,0}x_0^{(3)}x_1^{(3)} + t_{0,3,3}x_1^{(3)}x_2^{(3)} + t_{3,0,3}x_0^{(3)}x_2^{(3)} + t_{2,2,2}x_0^{(2)}x_1^{(2)}x_2^{(2)}.\end{aligned}$$

Now we write down the matrix of the map $(\cdot \lrcorner p) : T_{(3,0)} \to T^*_{(3,0)}$ in the standard monomial bases $\alpha_0^3, \alpha_1^3, \alpha_2^3, \alpha_0\alpha_1\alpha_2$ and $x_0^{(3)}, x_1^{(3)}, x_2^{(3)}, x_0x_1x_2$:

$$M = \begin{pmatrix} t_{6,0,0} & t_{3,3,0} & t_{3,0,3} & t_{4,1,1} \\ t_{3,3,0} & t_{0,6,0} & t_{0,3,3} & t_{1,4,1} \\ t_{3,0,3} & t_{0,3,3} & t_{0,0,6} & t_{1,1,4} \\ t_{4,1,1} & t_{1,4,1} & t_{1,1,4} & t_{2,2,2} \end{pmatrix}$$

We also write down the matrix of the map $(\cdot \lrcorner p) : T_{(3,1)} \to T^*_{(3,-1)}$ in the bases $\alpha_0^2\alpha_1, \alpha_1^2\alpha_2, \alpha_2^2\alpha_0$ and $x_0^{(2)}x_2, x_1^{(2)}x_0, x_2^{(2)}x_1$:

$$N = \begin{pmatrix} t_{4,1,1} & t_{2,2,2} & t_{3,0,3} \\ t_{3,3,0} & t_{1,4,1} & t_{2,2,2} \\ t_{2,2,2} & t_{0,3,3} & t_{1,1,4} \end{pmatrix}$$

We compute that the 3 by 3 minors of $M$ and $N$ define a variety of dimension 5 over $\mathbb{Q}$. But it can be found by the same method as in Section 5.2 that the dimension of the second secant variety of the embedding $X_\Sigma \hookrightarrow \mathbb{P}(H^0(X_\Sigma, \mathcal{O}(6,0))^*)$ is 5. Hence, the $\sigma_2(X_\Sigma)$ is given set-theoretically by the 3 by 3 minors of $M$ and $N$ over $\mathbb{Q}$. But this means that it is also defined by these equations over $\mathbb{C}$. Finally, since $F$ satisfies these equations, the claim follows.

**Example 5.9.** Take

$$F = x_0^{(2)}x_1^{(2)}x_2^{(2)} \in H^0(X_\Sigma, \mathcal{O}(6,0))^*.$$

Here the apolar ideal is $\operatorname{Ann}(F) = (\alpha_0^3, \alpha_1^3, \alpha_2^3)$. We calculate the following

| $\mathrm{r}(F)$ | $\mathrm{cr}(F)$ | $\mathrm{br}(F)$ |
|---|---|---|
| 3 | 3 | 3 |

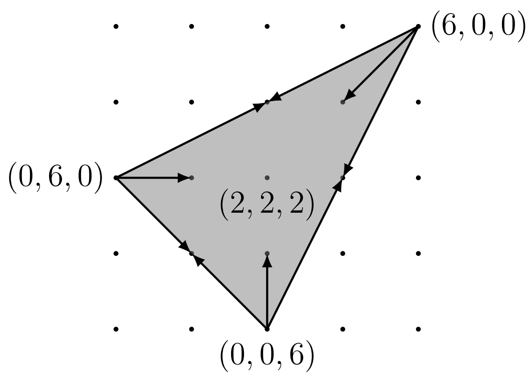
Let $I = (\alpha_0^3 - \alpha_1^3, \alpha_1^3 - \alpha_2^3)$. In this case also no coordinate can be zero, so we may calculate the length on $U_\sigma$ (where $\sigma$ is as before). We get the ring as in Equation 5.6 and the two generators become $1-u$ and $u-v$. So here the quotient ring is

$$\mathbb{C}[w]/(w^3 - 1).$$

This means that the rank is at most three (notice that we get a reduced scheme). We can calculate the Hilbert function of $\operatorname{Apolar}(F) = T/\operatorname{Ann}(F)$.

We have $\dim_{\mathbb{C}} \operatorname{Apolar}(F)_{(3,1)} = 3$, so from Proposition 3.22 we get that $\operatorname{br}(F) \geq 3$.

Now we show that $\operatorname{cr}(F) = 3$. We look at the polytope $P$ of the embedding by $\mathcal{O}(6,0)$.




The projective tangent space at the vertex $v$ is given by the Hilbert basis of the semigroup $\mathbb{N}(P \cap M - v)$ (see Proposition 2.14). The vector $(2,2,2)$ is in none of the three Hilbert bases, which means that $x_0^{(2)} x_1^{(2)} x_2^{(2)}$ is in none the of three tangent spaces at the singular points. But the fact that $\operatorname{br}(F) \geq 3$ means that $F$ is neither in any projective tangent space at a smooth point nor at any secant line passing through two points. It follows that $\operatorname{cr}(F) > 2$.

# Chapter 6

# Distinguishing between secant varieties and cactus varieties

The goal of this chapter is to prove three theorems describing cactus varieties. In this chapter, we work over the field of complex numbers.

We present the three main theorems. Each one has two points. The first point always claims that there are two components of the cactus variety. The fact that there are at most two components in the cases considered is nothing new, it follows from [29] and [30]. But in general, it is possible that there is just one component. Consider for instance the case of $v_3 : \mathbb{P}(\mathbb{C}^7) \to \mathbb{P}(\operatorname{Sym}^3 \mathbb{C}^7)$, where the secant variety $\sigma_{14}(v_3(\mathbb{P}(\mathbb{C}^7)))$ fills the ambient space (which follows from the Alexander-Hirschowitz theorem).

However, in each of the theorems, the main result is Point (ii), describing the irreducible component of the respective cactus variety which is not the secant variety.

We define the coordinate ring and the graded dual ring to be

$$\begin{aligned} T &= \mathbb{C}[\alpha_{1,0}, \dots, \alpha_{1,n_1}, \dots, \alpha_{k,0}, \dots, \alpha_{k,n_k}], \\ T^* &= \mathbb{C}_{dp}[x_{1,0}, \dots, x_{1,n_1}, \dots, x_{k,0}, \dots, x_{k,n_k}] \\ &= \operatorname{Sym} V_1^* \otimes \cdots \operatorname{Sym} V_k^*, \end{aligned}$$

where $V_i^* = \langle x_{i,0}, \dots, x_{i,n_i} \rangle$ for $1 \le i \le k$. The rings $T$, and $T^*$ are naturally graded by $\mathbb{Z}^k$. As before, for a vector of non-negative integers $\mathbf{d} = (d_1, \dots, d_k)$, we define $T_{\mathbf{d}}$ ($T^*_{\mathbf{d}}$) to be the graded piece of $T(T^*)$ of degree $\mathbf{d}$.

In the following theorems, we consider secant varieties with respect to the Segre-Veronese embedding $v_{\mathbf{d}}$ of degree $\mathbf{d}$ for a multi-index $\mathbf{d}$. This is the map attached to the linear system $|\mathcal{O}(\mathbf{d})|$ or, equivalently, it is given on points by

$$\begin{aligned} v_{\mathbf{d}} : \mathbb{P}T^*_{1,0,\dots,0} \times \cdots \times \mathbb{P}T^*_{0,\dots,0,1} &\to \mathbb{P}T^*_{\mathbf{d}} \\ ([l_1], \dots, [l_k]) &\mapsto [l_1^{d_1} \cdots l_k^{d_k}]. \end{aligned}$$

In Theorems 6.1 and 6.3 we consider the special case of the Veronese embedding ($k = 1$).

Recall the triangle operator from Definition 4.18, the notion of dehomogenizing with respect to bases (Definition 4.19) and that $\operatorname{Apolar}(W) = S/\operatorname{Ann}(W)$.

**Theorem 6.1.** *Let $n \geq 6$ and $d \geq 5$ be integers.*

*(i) The cactus variety $\kappa_{14}(v_d(\mathbb{P}T_1^*))$ has two irreducible components, one of which is $\sigma_{14}(v_d(\mathbb{P}T_1^*))$, and we denote the other one by $\eta_{14}(v_d(\mathbb{P}T_1^*))$.*

*(ii) The irreducible component $\eta_{14}(v_d(\mathbb{P}T_1^*))$ is the closure of the following set*

$$\begin{aligned} \{[z_0^{d-3}F] \in \mathbb{P}T_d^* \mid z_0 \in T_1^* \setminus \{0\}, [F] \in \mathbb{P}T_3^*, &\text{ and there exists} \\ &\text{a completion of } z_0 \text{ to a basis } (z_0, z_1, \dots, z_n) \text{ of } T_1^* \text{ such that} \\ &\operatorname{Apolar}((F|_{z_0=1})^{\blacktriangledown d}) \text{ has Hilbert function } (1,6,6,1)\}. \end{aligned}$$

**Theorem 6.2.** *Let $n = n_1 + \cdots + n_k$ and suppose $n \geq 6$. Let $\mathbf{d} = (d_1, \dots, d_k)$ be a multi-index with $d_i \geq 7$ for $1 \leq i \leq k$.*

*(i) The cactus variety $\kappa_{14}(v_{\mathbf{d}}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}))$ has two irreducible components, one of which is $\sigma_{14}(v_{\mathbf{d}}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}))$, and we denote the other one by $\eta_{14}(v_{\mathbf{d}}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}))$.*

*(ii) The irreducible component $\eta_{14}(v_{\mathbf{d}}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}))$ is the closure of the following set*

$$\begin{aligned} \{[z_1^{d_1-3} \cdots z_k^{d_k-3}F] \in \mathbb{P}T_{\mathbf{d}}^* \mid z_i \in V_i^* \setminus \{0\}, [F] \in \mathbb{P}T_{3,\dots,3}, &\text{ for each } i \text{ there} \\ &\text{exists a completion of } z_i \text{ to a basis } (z_i, z_{i,1}, \dots, z_{i,n_i}) \\ &\text{of } V_i^* \text{ such that } \operatorname{Apolar}((F|_{z_0,z_1,\dots,z_k=1})^{\blacktriangledown \mathbf{d}}) \text{ has} \\ &\text{Hilbert function } (1,6,6,1)\} \end{aligned}$$

**Theorem 6.3.** *Let $n \geq 4$ and $d \geq 5$ be integers and consider the polynomial ring $T^* = \mathbb{C}[x_0, \ldots, x_n]$ graded in a standard way by $\mathbb{N}$.*

*(i) The Grassmann cactus variety $\kappa_{8,3}(\nu_d(\mathbb{P}T_1^*))$ has two irreducible components, one of which is the Grassmann secant variety $\sigma_{8,3}(\nu_d(\mathbb{P}T_1^*))$, and we denote the other one by $\eta_{8,3}(\nu_d(\mathbb{P}T_1^*))$.*

*(ii) The irreducible component $\eta_{8,3}(\nu_d(\mathbb{P}T_1^*))$ is the closure of the following set*

$$\{[z_0^{d-2}U] \in \mathrm{Gr}(3, T_d^*) \mid z_0 \in T_1^* \setminus \{0\}, U \in \mathrm{Gr}(3, T_2^*), \textit{ and there exists a completion of } z_0 \textit{ to a basis } (z_0, z_1, \ldots, z_n) \textit{ of } T_1^* \textit{ such that } \mathrm{Apolar}((U|_{z_0=1})^{\blacktriangledown d}) \textit{ has Hilbert function } (1,4,3)\}.$$

Theorems 6.1 and 6.3 have the following corollaries (Corollary 6.4 has been already stated in the Introduction as Theorem 1.21):

**Corollary 6.4.** *For $d \geq 5$, the cactus variety $\kappa_{14}(v_d(\mathbb{P}^6))$ has two irreducible components: the secant variety $\sigma_{14}(v_d(\mathbb{P}^6))$, and the variety $\eta_{14}(v_d(\mathbb{P}^6))$ consisting of degree $d$ forms divisible by the $(d-3)$-rd power of a linear form.*

**Corollary 6.5.** *Let $d \geq 5$, the Grassmann cactus variety $\kappa_{8,3}(v_d(\mathbb{P}^4))$ has two irreducible components: the Grassmann secant variety $\sigma_{8,3}(v_d(\mathbb{P}^4))$ and the variety $\eta_{8,3}(v_d(\mathbb{P}^4))$ consisting of 3-dimensional vector spaces divisible by a $(d-2)$-nd power of a linear form.*

Furthermore for $n \geq 6$ and $d \geq 6$ we present an algorithm (Theorem 6.6) for deciding whether $[G] \in \kappa_{14}(v_d(\mathbb{P}^n))$ is in $\sigma_{14}(v_d(\mathbb{P}^n))$.

**Theorem 6.6.** *Let $T^* = \mathbb{C}[x_0, \ldots, x_n]$ be a polynomial ring with $n \geq 6$. Given an integer $d \geq 6$ and $[G] \in \kappa_{14}(v_d(\mathbb{P}T_1^*)) \subseteq \mathbb{P}T_d^*$ the following algorithm checks if $[G] \in \sigma_{14}(v_d(\mathbb{P}T_1^*))$.*

***Step 1*** *Compute the ideal $\mathfrak{a} = \sqrt{((\mathrm{Ann}\, G)_{\leq d-3})}$.*

***Step 2*** *If $\mathfrak{a}_1$ is not $n$-dimensional, then $[G] \in \sigma_{14}(v_d(\mathbb{P}T_1^*))$ and the algorithm terminates. Otherwise compute $\{K \in T_1^* \mid \mathfrak{a}_1 \lrcorner K = 0\}$. Let $z_0$ be a generator of this one dimensional $\mathbb{C}$-vector space.*

***Step 3*** *Let $e$ be the maximal integer such that $z_0^e$ divides $G$. If $e \neq d-3$, then $[G] \in \sigma_{14}(v_d(\mathbb{P}T_1^*))$ and the algorithm terminates. Otherwise let $G = z_0^{d-3}F$, pick a basis $(z_0, z_1, \ldots, z_n)$ of $T_1^*$ and compute $f = F|_{z_0=1} \in R^* := \mathbb{C}[z_1, \ldots, z_n]$.*

***Step 4*** *Let $I = \operatorname{Ann}(f^{\blacktriangledown d}) \subseteq R$. If the Hilbert function of $R/I$ is not equal to $(1,6,6,1)$, then $[G] \in \sigma_{14}(v_d(\mathbb{P}T_1^*))$, and the algorithm terminates.*

***Step 5*** *Compute $r = \dim_{\mathbb{C}} \operatorname{Hom}_R(I, R/I)$. Then $[G] \in \sigma_{14}(v_d(\mathbb{P}T_1^*))$ if and only if $r > 14n - 8$.*

Using the description of the irreducible component $\eta$ given in Theorem 6.3, we are able to determine algorithmically if a given point from the Grassmmann cactus variety is in the Grassmann secant variety.

**Theorem 6.7.** *Let $n$ be at least 4 and $T^* = \mathbb{C}[x_0, \dots, x_n]$ be a polynomial ring. Given an integer $d \geq 5$ and $[V] \in \kappa_{8,3}(v_d(\mathbb{P}T_1^*)) \subseteq \operatorname{Gr}(3, T_d^*)$ the following algorithm checks if $[V] \in \sigma_{8,3}(v_d(\mathbb{P}T_1^*))$.*

***Step 1*** *Compute the ideal $\mathfrak{a} = \sqrt{((\operatorname{Ann} V)_{\leq d-2})}$.*

***Step 2*** *If $\mathfrak{a}_1$ is not $n$-dimensional, then $[V] \in \sigma_{8,3}(v_d(\mathbb{P}T_1^*))$ and the algorithm terminates. Otherwise compute $\{K \in T_1^* \mid \mathfrak{a}_1 \lrcorner K = 0\}$. Let $z_0$ be a generator of this one dimensional $\mathbb{C}$-vector space.*

***Step 3*** *Let $e$ be the maximal integer such that $z_0^e$ divides $V$. If $e \neq d-2$, then $[V] \in \sigma_{8,3}(v_d(\mathbb{P}T_1^*))$ and the algorithm terminates. Otherwise let $V = z_0^{d-2}U$, pick a basis $(z_0, z_1, \dots, z_n)$ of $T_1^*$ and compute $W = U|_{z_0=1} \subseteq R^* := \mathbb{C}[z_1, \dots, z_n]$.*

***Step 4*** *Let $I = \operatorname{Ann}(W^{\blacktriangledown d}) \subseteq R$. If the Hilbert function of $R/I$ is not $(1,4,3)$, then $[V] \in \sigma_{8,3}(v_d(\mathbb{P}T_1^*))$, and the algorithm terminates.*

***Step 5*** *Compute $r = \dim_{\mathbb{C}} \operatorname{Hom}_R(I, R/I)$. Then $[V] \in \sigma_{8,3}(v_d(\mathbb{P}T_1^*))$ if and only if $r > 8n - 7$.*

Section 6.1 is intented as a warm-up. We cover the case of the third Veronese there and argue that this case is more difficult than the cases with higher degree. The goal of the three remaining sections is two prove Theorems 6.1–6.3, and to prove that Algorithms 6.6 and 6.7 work.

While Sections 6.2 and 6.4 come from the article [47], Sections 6.1 and 6.3 are the author's own work that has not appeared anywhere before.

## 6.1 Initial cases

The first example that we would like to study is the case of cubics. However, it is extremely hard to say anything about this case. If $\dim V \leq 13$, we do not know anything. But for at least 14 variables we know that the cactus variety has two components, and we know their dimensions.

But first let us recall a fact mentioned in [57, Theorem 1.69].

**Proposition 6.8.** *Let $I \subseteq T$ be the saturated ideal of a zero-dimensional scheme in $\mathbb{P}^n$. Let $\tau$ be the degree where the Hilbert function of $T/I$ stabilizes. Then $I$ is generated in degrees $\leq \tau + 1$.*

Recall the function $h_r$, as in Definition 3.11.

**Proposition 6.9.** *Let $n \geq 13$. Let $G$ be a general homogeneous cubic in $n+1$ variables $x_0, \ldots, x_n$ such that* $\mathrm{Apolar}(G)$ *has Hilbert function* $1, 14, 14, 1$ *and* $\mathrm{cr}(G) = 14$. *There exists exactly one saturated ideal $I \subseteq \mathrm{Ann}(G)$ of degree 14. Moreover, any ideal $J \subseteq \mathrm{Ann}(G)$ with Hilbert function $h_{14}$ is equal to $I$.*

*Proof.* As $I$ is saturated of degree 14, and contained in $\mathrm{Ann}(G)$, it has Hilbert function $1, 14, 14, 14, 14, \ldots$. This means that we can apply Proposition 6.8 to get that $I$ is generated in degrees at most 2. But because the Hilbert functions of $\mathrm{Ann}(G)$ and $I$ at degree 2 are equal, and $I \subseteq \mathrm{Ann}(G)$, we have

$$I_2 = \mathrm{Ann}(G)_2.$$

Since $I$ has no generators in higher degree, we get $I = (\mathrm{Ann}(G)_{\leq 2})$.

Such an ideal $I$ exists because of Theorem 3.9. Take any $J \subseteq \mathrm{Ann}(G)$ such that $T/J$ has Hilbert function $h_{14}$. Then $J_2 = \mathrm{Ann}(G)_2 = I_2$, but $I$ is generated in degrees at most 2, so $I \subseteq J$. The ideals have the same Hilbert function, so they are the same. □

**Theorem 6.10.** *Suppose* $\dim V \geq 14$. *Then*

$$\kappa_{14}(v_3(\mathbb{P}V)) = \sigma_{14}(v_3(\mathbb{P}V)) \cup \eta_{14}(v_3(\mathbb{P}V))$$

*is a decomposition into irreducible components. Moreover,*

$$\dim \sigma_{14}(v_3(\mathbb{P}V)) = 14 \dim V - 1,$$
$$\dim \eta_{14}(v_3(\mathbb{P}V)) = 14 \dim V - 9.$$

*The component $\eta_{14}(v_3(\mathbb{P}V))$ is the closure of the following set*

$$\{[F] \in \mathbb{P}\,\mathrm{Sym}^3 V \mid \text{there exists a basis } l, l_1, \ldots, l_n \text{ of } V$$
$$\text{such that } \mathrm{Apolar}(F|_{l=1}) \text{ has Hilbert function } 1, 6, 6, 1\}.$$

*Proof.* Proposition 6.9 means that the projection map

$$\chi_{|Y_2} : Y_2 \to \eta_{14}(v_3(\mathbb{P}V))$$

is generically one-to one ($Y_2$ is defined in Equation (4.1)). The claim on the dimension of $\eta_{14}(v_3(\mathbb{P}V))$ follows.

To prove that $\sigma_{14}(v_3(\mathbb{P}V)))$ and $\eta_{14}(v_3(\mathbb{P}V))$ are irreducible components, notice that the general point of $\eta_{14}(v_3(\mathbb{P}V))$ cannot belong to $\sigma_{14}(v_3(\mathbb{P}V))$, because then by Theorem 3.12 there would exist an ideal $J \subseteq \mathrm{Ann}(G)$ in $\mathrm{Slip}_{r,\mathbb{P}^n}$ such the Hilbert function of $T/J$ is $h_{14}$, which would mean that $I = J$. But since Slip surjects onto the smoothable component of the standard Hilbert scheme, we get that $V(I)$ would be smoothable, a contradiction.

The final claim follows from [8, Page 61]. □

## 6.2 14th secant variety of higher Veronese embeddings

We will consider the polynomial ring $T = \mathbb{C}[\alpha_0, \alpha_1, \dots, \alpha_n]$, and its graded dual $T^* = \mathbb{C}[x_0, x_1, \dots, x_n]$, where $n \geq 6$. Given $f \in T^*$, we denote by $f_j$ its homogeneous part of degree $j$.

Our goal is to characterize for $d \geq 5$ and $n \geq 6$ the closure of the set-theoretic difference between the cactus variety $\kappa_{14}(v_d(\mathbb{P}T_1^*))$ and the secant variety $\sigma_{14}(v_d(\mathbb{P}T_1^*))$. For $n = 6$ and $d \geq 5$ this closure consists of points $[G] \in \mathbb{P}T_d^*$ with $G$ divisible by $(d-3)$-th power of a linear form. However for $n > 6$ the situation is more complicated.

*Remark* 6.11. Notice that we omitted the case $d = 4$. This is because neither the methods for $d = 3$, nor the ones for $d \geq 5$ seem to work here.

**Proposition 6.12.** *Let $(z_0, z_1, \dots, z_n)$ be a $\mathbb{C}$-basis of $T_1^*$. Assume that $G = z_0^{d-3}F$ for some natural number $d \geq 5$ and $F \in T_3^*$. Define $f := F|_{z_0=1} \in R^* := \mathbb{C}[z_1, \dots, z_n]$. If $f$ satisfies the following conditions:*

*(a)* $\mathrm{Apolar}(f^{\blacktriangledown d})$ *has Hilbert function* $(1, 6, 6, 1)$*,*

*(b)* $[\mathrm{Spec}\,\mathrm{Apolar}(f^{\blacktriangledown d})] \notin \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n)$*,*

*then* $[G] \in \eta_{14}(v_d(\mathbb{P}^n)) \setminus \sigma_{14}(v_d(\mathbb{P}^n))$*.*

*Proof.* By Condition (a) we have $\dim_{\mathbb{C}}(R/\operatorname{Ann}(f^{\blacktriangledown d})) = 14$. Therefore from Theorem 3.31 (i)

$$\operatorname{cr}(G) = \operatorname{cr}\,(\sum_{i=0}^{3} z_0^{(d-i)} f_i^{\blacktriangledown d}) \leq 14.$$

From Theorem 3.12, if $[G] \in \sigma_{14}(v_d(\mathbb{P}^n))$ then there exists $J \subseteq \operatorname{Ann}(G)$ with $[J] \in \operatorname{Slip}_{14,\mathbb{P}T_1} \subseteq \operatorname{Hilb}_T^{h_{14}}$. Thus $[\operatorname{Proj}(T/\operatorname{sat}(J,B))] \in \mathcal{H}ilb_{14}^{sm}(\mathbb{P}^n)$. The Hilbert function of $R/\operatorname{Ann}(f_3^{\blacktriangledown d})$ is $(1,6,6,1)$ by [30, Theorem 2.3 and the following remarks]. In particular, $f_3^{\blacktriangledown d}$ is not a power of a linear form. From Theorem 3.31(iii) it follows that $\operatorname{sat}(J,B) = \operatorname{Ann}(f^{\blacktriangledown d})^{\mathrm{hom}}$, so

$$[\operatorname{Spec}(R/\operatorname{Ann}(f^{\blacktriangledown d}))] \in \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n).$$

This contradicts Condition (b). $\square$

Finally we present the proof of the characterization of points of the second irreducible component of the cactus variety.

*Proof of Theorem 6.1 and Corollary 6.4.* We set $k = 1$. Recall the sets $C$ and $D$ from Lemma 4.21. Let $\psi\colon (T_1^* \setminus \{0\}) \times (T_3^* \setminus \{0\}) \to \mathbb{P}T_d^*$ be given by $(z_0, F) \mapsto [z_0^{d-3}F]$. By Proposition 6.12 if $(z_0,F) \in D$, then $[z_0^{d-3}F] \in \eta_{14}(v_d(\mathbb{P}T_1^*))$. Hence $\overline{\psi(C)} \subseteq \eta_{14}(v_d(\mathbb{P}T_1^*))$. The fibers of $\psi_{|C} : C \to \psi(C)$ are two-dimensional, so $\dim \overline{\psi(C)} = 14n+5$. But $\dim \eta_{14}(v_d(\mathbb{P}T_1^*)) \leq 14n+5$ by Proposition 4.7. Therefore $\overline{\psi(C)} = \eta_{14}(v_d(\mathbb{P}T_1^*))$. This concludes the proof of Theorem 6.1(ii).

We proceed to the proof of Theorem 6.1(i). Let $[f] \in B$, where $B$ is as in Lemma 4.17. Then by Proposition 6.12 we get $[f^{\mathrm{hom},d}] \in \kappa_{14}(v_d(\mathbb{P}T_1^*)) \setminus \sigma_{14}(v_d(\mathbb{P}T_1^*))$. This is enough, since

$$\dim \sigma_{14}(v_d(\mathbb{P}T_1^*)) > \dim \eta_{14}(v_d(\mathbb{P}T_1^*)).$$

Now we prove Corollary 6.4. Assume that $n = 6$. Then in the above notation the closure of $C$ in $(T_1^* \setminus \{0\}) \times (T_3^* \setminus \{0\})$ has the maximal dimension $14 \cdot 6 + 7 = 91$. Thus

$$\overline{C} = (T_1^* \setminus \{0\}) \times (T_3^* \setminus \{0\}). \tag{6.1}$$

Notice that $\psi$ can be factored by

$$(T_1^* \setminus \{0\}) \times (T_3^* \setminus \{0\}) \xrightarrow{q} \mathbb{P}T_1^* \times \mathbb{P}T_3^* \xrightarrow{\psi'} \mathbb{P}T_d^*,$$

where $q$ is the product of projections, and $\psi'([z_0],[P]) = [z_0^{d-3}P]$. Since $\psi'$ is a morphism from a projective variety, the set $\psi'(\mathbb{P}T_1^* \times \mathbb{P}T_3^*)$ is closed. We have

$$\psi(\overline{C}) \subseteq \overline{\psi(C)}, \tag{6.2}$$

but the set

$$\psi(\overline{C}) = \psi'(q(\overline{C})) = \psi'(\mathbb{P}T_1^* \times \mathbb{P}T_3^*)$$

is closed, hence Inclusion 6.2 is an equality, which means that

$$\eta_{14}(v_d(\mathbb{P}T_1^*)) = \psi(\overline{C}).$$

But then the statement of the Corollary follows from Equation 6.1. $\square$

**Proposition 6.13.** *Let $d \geq 4$ be an integer, $z_0 \in T_1^*$, $Q \in T_2^*$. Assume $z_0 \neq 0$ and $Q \neq 0$. Define $G = z_0^{d-2}Q \in T_d^*$. If $[G] \in \kappa_r(\nu_d(\mathbb{P}^n))$ for a positive integer $r$, then $[G] \in \sigma_r(\nu_d(\mathbb{P}^n))$.*

*Proof.* Complete $z_0$ to a basis $\{z_0, z_1, \ldots, z_n\}$ of $T_1^*$. Let $S^* = \mathbb{C}[z_1, ..., z_n]$ and $q = q_2 + q_1 + q_0 \in S$ be such that $G = q_2 z_0^{(d-2)} + q_1 z_0^{(d-1)} + q_0 z_0^{(d)}$. By Theorem 3.31(ii) we have $\dim_{\mathbb{C}} S/\operatorname{Ann}(q) = s$ for some $s \leq r$. Therefore,

$$[\operatorname{Proj} T/\operatorname{Ann}(q)^{\mathrm{hom}}] \in \mathcal{H}ilb_s(\mathbb{P}^n).$$

By [29, Prop. 4.9] this subscheme is smoothable. Hence, it follows from Lemma 3.35 that $[G] \in \sigma_r(\nu_d(\mathbb{P}^n))$. $\square$

The following lemma gives a description of the set-theoretic difference of the cactus variety and the secant variety. We need it to give a clear proof of Theorem 6.6.

**Lemma 6.14.** *Let $d \geq 6, n \geq 6$. The point $[G] \in \kappa_{14}(_d(\mathbb{P}^n))$ does not belong to $\sigma_{14}(v_d(\mathbb{P}^n))$ if and only if there exists a linear form $z_0 \in T_1^*$, and $F \in T_3^*$ such that $G = z_0^{d-3}F$ and for any completion of $z_0$ to a basis $(z_0, \ldots, z_n)$ of $T_1^*$ (with dual basis equal to $(\gamma_0, \ldots, \gamma_n)$) we have:*

*(a)* $\operatorname{Apolar}((F|_{z_0=1})^{\blacktriangledown d})$ *has Hilbert function* $(1, 6, 6, 1)$,

*(b)* $[\operatorname{Spec}\operatorname{Apolar}((F|_{z_0=1})^{\blacktriangledown d})] \notin \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n)$.

*Proof.* If $z_0 \in T_1^*$ and $F \in T_3^*$ are such that $G = z_0^{d-3}F$, and there exists a completion of $z_0$ to a basis $(z_0, \ldots, z_n)$ of $T_1^*$, for which Conditions (a),(b) hold, we get

$$[G] \notin \sigma_{14}(v_d(\mathbb{P}^n))$$

by Proposition 6.12.

Assume that $[G] \notin \sigma_{14}(v_d(\mathbb{P}^n))$. Then by Theorem 6.1 there exists a linear form $z_0 \in T_1^*$ such that $z_0^{d-3}|G$. Hence we showed that $G = z_0^{d-3}P$ for some $F \in T_3^*$. Extend $z_0$ to a basis $z_0, z_1, \ldots, z_n$. Let $f = F|_{z_0=1}$. Suppose $f = f_3 + f_2 + f_1 + f_0$.

Now we prove Conditions (a), (b) hold. It follows from the definition of $()^{\blacktriangledown d}$ that $f^{\blacktriangledown d} = f_3^{\blacktriangledown d} + f_2^{\blacktriangledown d} + f_1^{\blacktriangledown d} + f_0^{\blacktriangledown d} \in \mathbb{C}[z_1, \ldots, z_n]$. We have

$$G = \sum_{i=0}^{3} z_0^{(d-i)} f_i^{\blacktriangledown d}.$$

By Lemma 2.33 (i)

$$\operatorname{Ann}(f^{\blacktriangledown d})^{\mathrm{hom}} \subseteq \operatorname{Ann}(G).$$

If $\dim_{\mathbb{C}}(\operatorname{Apolar}(f^{\blacktriangledown d})) \leq 13$, then $\operatorname{cr}(G) \leq 13$ by Theorem 3.9. Therefore, $[G] \in \kappa_{13}(v_d(\mathbb{P}^n)) = \sigma_{13}(v_d(\mathbb{P}^n)) \subseteq \sigma_{14}(v_d(\mathbb{P}^n))$, a contradiction.

From Theorem 3.31(ii) we obtain $\dim_{\mathbb{C}}(\operatorname{Apolar}(f^{\blacktriangledown d})) \leq 14$. We proved that $\dim_{\mathbb{C}}(\operatorname{Apolar}(f^{\blacktriangledown d})) = 14$.

Since we assumed that $[G] \notin \sigma_{14}(v_d(\mathbb{P}^n))$, it follows from Lemma 3.35 that $\operatorname{Spec}\operatorname{Apolar}(f^{\blacktriangledown d})$ is not smoothable. This implies Condition (b) holds. By [30, Theorem 2.3] and [30, Proposition 6.11] the algebra $\operatorname{Apolar}(f^{\blacktriangledown d})$ has Hilbert function $(1, 6, 6, 1)$. Thus we proved Condition (a) holds. □

Steps 2–5 of the algorithm from Theorem 6.6 verify if $G$ satisfies necessary conditions to be of the form given by Lemma 6.14.

*Proof of Theorem 6.6.* Assume that $[G] \notin \sigma_{14}(v_d(\mathbb{P}^n))$. Then there exist a basis $(z_0, \ldots, z_n)$ of $T_1^*$ and $F \in T_3^*$ as in Lemma 6.14. Let $f = F|_{z_0=1}$. It follows from the definition of $()^{\blacktriangledown d}$ that $f^{\blacktriangledown d} = f_3^{\blacktriangledown d} + f_2^{\blacktriangledown d} + f_1^{\blacktriangledown d} + f_0^{\blacktriangledown d} \in \mathbb{C}[z_1, \ldots, z_n]$. Therefore $G = z_0^{(d-3)} f_3^{\blacktriangledown d} + z_0^{(d-2)} f_2^{\blacktriangledown d} + z_0^{(d-1)} f_1^{\blacktriangledown d} + z_0^{(d)} f_0^{\blacktriangledown d}$. By Lemma 2.33(ii), we have $\operatorname{Ann}(G)_{\leq d-3} = (\operatorname{Ann}(f^{\blacktriangledown d})^{\mathrm{hom}})_{\leq d-3}$. Moreover, by Lemma 2.34,

$$((\operatorname{Ann}(f^{\blacktriangledown d})^{\mathrm{hom}})_{\leq d-3}) = \operatorname{Ann}(f^{\blacktriangledown d})^{\mathrm{hom}} \tag{6.3}$$

(The assumptions of the Lemma are satisfied since the algebras $\mathrm{Apolar}(f_3^{\blacktriangledown d})$ and $\mathrm{Apolar}(f^{\blacktriangledown d})$ have the same Hilbert function by [30, Theorem 2.3 and the following remarks].) Therefore we have

$$\mathfrak{a} = \sqrt{(\mathrm{Ann}(G)_{\leq d-3})} = \sqrt{\mathrm{Ann}(f^{\blacktriangledown d})^{\mathrm{hom}}} = (\gamma_1, \ldots, \gamma_n),$$

where $\gamma_1, \ldots, \gamma_n \in T_1$ are dual to $z_1, \ldots, z_n \in T_1^*$. This shows that if the $\mathbb{C}$-linear space $\left(\sqrt{(\mathrm{Ann}(G)_{\leq d-3})}\right)_1$ is not $n$-dimensional then $[G] \in \sigma_{14}(v_d(\mathbb{P}^n))$. Therefore, in that case, algorithm stops correctly at Step 2.

Assume that the algorithm did not stop at Step 2. Then if $G$ is of the form as in Lemma 6.14, then $z_0$ divides $G$ exactly $d-3$ times. Otherwise $[G] \in \sigma_{14}(v_d(\mathbb{P}^n))$ and the algorithm stops correctly at Step 3.

Assume that the algorithm did not stop at Step 3. Then the algorithm does not stop at Step 4 if and only if Condition (a) of Lemma 6.14 is fulfilled. Therefore, if the Hilbert function of $R/\mathrm{Ann}(f^{\blacktriangledown d})$ is not equal to $(1, 6, 6, 1)$, the algorithm stops correctly at Step 4.

Assume that the algorithm did not stop at Step 4. Then $F$ satisfies Condition (a) from Lemma 6.14. Hence $[G]$ is in $\sigma_{14}(v_d(\mathbb{P}^n))$ if and only if $F$ does not satisfy Condition (b). Using Lemma 4.5, this is equivalent to

$$\dim_{\mathbb{C}} \mathrm{Hom}_R(I, R/I) > 14n - 8.$$

The left term is the dimension of the tangent space to the Hilbert scheme $\mathcal{H}ilb_{14}^{Gor}(\mathbb{A}^n)$ at the point $[\mathrm{Spec}\, R/I]$ (see [53, Proposition 2.3] or [68, Theorem 18.29]). $\square$

*Remark* 6.15. The algorithm is stated for $d \geq 6$ even though it is based on Theorem 6.1 which works for $d \geq 5$. The reason for this is that we needed $d \geq 6$ to obtain Equation (6.3) and for Lemma 6.14 to work. We do not know a counterexample for the algorithm in case $d = 5$.

Equations defining the cactus variety $\kappa_{14}(v_6(\mathbb{P}^n))$ for $n \geq 6$ are unknown and there is no example of an explicit equation of the secant variety $\sigma_{14}(v_6(\mathbb{P}^n))$ which does not vanish on the cactus variety. We present some known results about 14-th secant and cactus varieties of Veronese embeddings of $\mathbb{P}^6$.

*Remark* 6.16. Let $V$ be a 7-dimensional complex vector space. The catalecticant minors define a subscheme of $\mathbb{P}(\mathrm{Sym}^6 V)$ one of whose irreducible components is the secant variety $\sigma_{14}(v_6(\mathbb{P}V))$ (see [57, Theorem 4.10A]). Moreover, these equations are known to vanish on the cactus variety $\kappa_{14}(v_6(\mathbb{P}V))$

(see [19, Proposition 3.6], or Theorem 1.17). Example 6.17 shows that the catalecticant minors do not define $\kappa_{14}(v_6(\mathbb{P}V))$ set-theoretically. However, if we consider the $d$-th Veronese for $d \geq 28$, then the catalecticant minors are enough to define $\kappa_{14}(v_d(\mathbb{P}V))$ set-theoretically, see [19, Theorem 1.5]. The article [64] gives an extensive list of results on equations of secant varieties but in the case of $\sigma_{14}(v_6(\mathbb{P}V))$ it does not improve the result in [57].

**Example 6.17.** Let $F = x_0^6 + x_1^2x_2^2x_3^2 + x_4^3x_5^2x_6 \in T^* = \mathbb{C}[x_0, \dots, x_6]$. Then Hilbert function of $T/\operatorname{Ann}(F)$ is $(1, 7, 12, 14, 12, 7, 1)$ but there is only one minimal homogeneous generator of $\operatorname{Ann}(F)$ in degree 4. Therefore there is no homogeneous ideal $J$ in $T$ such that $T/J$ has an 14-standard Hilbert function and $J$ is contained in $\operatorname{Ann}(F)$. Thus $\operatorname{bcr}(F) > 14$ by Theorem 3.15, even though the Hilbert function of $T/\operatorname{Ann}(F)$ is bounded by 14.

## 6.3 14th secant variety of Segre-Veronese embeddings

The aim of this section is to prove Theorem 6.2. We let $T = \operatorname{Sym} V_1 \otimes \cdots \otimes \operatorname{Sym} V_k$, and the graded dual $T^* = \operatorname{Sym} V_1^* \otimes \cdots \otimes \operatorname{Sym} V_k^*$. Here $\dim_{\mathbb{C}} V_i = n_i + 1$, and $n = n_1 + \cdots + n_k$.

**Proposition 6.18.** *For each $i = 1, \dots, k$, let $z_i, z_{i,1}, \dots, z_{i,n_i}$ be a $\mathbb{C}$-basis of $V_i$. Assume that $G = z_1^{d_1-3} \cdots z_k^{d_k-3}F$ for some natural numbers $d_1, \dots, d_k \geq 7$ and $F \in T^*_{3,\dots,3}$. Define*

$$f := F|_{z_1=1,\dots,z_k=1} \in R^* := \mathbb{C}[z_{1,1}, \dots, z_{1,n_1}, \dots, z_{k,1}, \dots, z_{k,n_k}].$$

*If $f$ satisfies the following conditions:*

*(a)* $\operatorname{Apolar}(f^{\blacktriangledown,\mathbf{d}})$ *has Hilbert function* $(1, 6, 6, 1)$*,*

*(b)* $[\operatorname{Spec}\operatorname{Apolar}(f^{\blacktriangledown,\mathbf{d}})] \notin \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n)$*,*

*then* $[G] \in \eta_{14}(v_{\mathbf{d}}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})) \setminus \sigma_{14}(v_{\mathbf{d}}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}))$.

*Proof.* By Condition (a) we have $\dim_{\mathbb{C}}(R/\operatorname{Ann}(f^{\blacktriangledown\mathbf{d}})) = 14$. Therefore from Theorem 3.30 (i)

$$\operatorname{cr}(G) = \operatorname{cr}((f^{\blacktriangledown\mathbf{d}})^{\mathrm{hom},\mathbf{d}})) \leq 14.$$

From Theorem 3.12, if $[G] \in \sigma_{14}(v_d(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}))$ then there exists $J \subseteq \operatorname{Ann}(G)$ with $[J] \in \operatorname{Slip}_{14,\mathbb{P}V_1\times\cdots\times\mathbb{P}V_k} \subseteq \operatorname{Hilb}^{h_{14}}_{T^*}$. Thus $[\operatorname{Proj}(T/\operatorname{sat}(J,B))] \in \mathcal{H}ilb^{sm}_{14}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$. From Theorem 3.30 (iii) it follows that $\operatorname{sat}(J,B) = \operatorname{Ann}(f^{\blacktriangledown d})^{\mathrm{hom}}$, so

$$[\operatorname{Spec}(R/\operatorname{Ann}(f^{\blacktriangledown \mathbf{d}}))] \in \mathcal{H}ilb^{Gor,sm}_{14}(\mathbb{A}^n).$$

This contradicts Condition (b). $\square$

Finally we present the proof of the characterization of points of the second irreducible component of the cactus variety.

*Proof of Theorem 6.2.* Recall the sets $C$ and $D$ from Lemma 4.21. Let

$$\psi : \prod_{i=1}^{k}(V_i^* \setminus \{0\}) \times (T^*_{3,\ldots,3} \setminus \{0\}) \to \mathbb{P}T^*_{\mathbf{d}}$$

be given by $(z_1, \ldots, z_k, F) \mapsto [z_1^{d_1-3} \cdots z_k^{d_k-3} F]$.

By Proposition 6.18 if $(z_1, \ldots, z_k, F) \in D$, then

$$[z_1^{d_1-3} \cdots z_k^{d_k-3} F] \in \eta_{14}(v_{\mathbf{d}}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_k^*)).$$

Hence $\overline{\psi(C)} \subseteq \eta_{14}(v_{\mathbf{d}}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_k^*))$. The fibers of $\psi_{|C} : C \to \psi(C)$ are $(k+1)$-dimensional, so $\dim \overline{\psi(C)} = 14n + 5$. But

$$\dim \eta_{14}(v_{\mathbf{d}}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_k^*)) \leq 14n + 5$$

by Proposition 4.7. Therefore $\overline{\psi(C)} = \eta_{14}(v_{\mathbf{d}}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_k^*))$. This concludes the proof of Theorem 6.2(ii).

We proceed to the proof of Theorem 6.2(i). Let $[f] \in B$, where $B$ is as in Lemma 4.17. Then by Proposition 6.18 we get

$$[f^{\mathrm{hom},\mathbf{d}}] \in \kappa_{14}(v_d(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_k^*)) \setminus \sigma_{14}(v_d(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_k^*)).$$

Moreover, the class $[z_1^{d_1} \cdots z_k^{d_k} + w_1^{d_1} \cdots w_k^{d_k}]$ is in $\sigma_{14}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_k^*))$, but is not divisible by any $(d_1-3)$-rd power of a linear form in $V_1^*$, hence it is not in $\eta(v_{\mathbf{d}}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k^*))$. $\square$

## 6.4 Grassmann secant variety of Veronese embeddings

In this section we show that the Grassmann cactus variety $\kappa_{8,3}(v_d(\mathbb{P}^n))$ has 2 components for $d \geq 5$ and $n \geq 4$, one of which is the Grassmann secant variety $\sigma_{8,3}(v_d(\mathbb{P}^n))$ and the other one is described in Theorem 6.3. Furthermore, we present an algorithm (Theorem 6.7) for deciding whether $[V] \in \kappa_{8,3}(v_d(\mathbb{P}^n))$ is in $\sigma_{8,3}(v_d(\mathbb{P}^n))$.

*Remark* 6.19. By Theorem 4.3, we know that $\sigma_{r,k}(v_d(\mathbb{P}^n)) = \kappa_{r,k}(v_d(\mathbb{P}^n))$ for $r \leq 7$, and any $k, n, d$, and that $\sigma_{8,k}(v_d(\mathbb{P}^n)) = \kappa_{8,k}(v_d(\mathbb{P}^n))$ for $n \leq 3$, and any $k, d$. In addition, we claim that $\sigma_{8,2}(v_d(\mathbb{P}^n)) = \kappa_{8,2}(v_d(\mathbb{P}^n))$ for any $n$. We sketch the proof. All local algebras of length at most 8 and socle dimension at most 2 are smoothable by Theorem 4.3. Hence the claim follows from the fact that $\kappa_{r,k}(v_d(\mathbb{P}^n))$ is the closure of the following set

$$\{[V] \in \mathrm{Gr}(k, \mathbb{C}^{n+1}) \mid \exists R \hookrightarrow \mathbb{P}^n \text{ such that } V \subseteq \langle v_d(R)\rangle, \text{length}\, R \leq r,$$
$$H^0(R, \mathcal{O}_R) \text{ is a product of local algebras of socle dimension at most } k\}$$

(a generalization of [19, Prop. 2.2]). Detailed proof of this fact is outside the main interests of this thesis, hence we skip it.

It follows from the above discussion that the number $k = 3$ is the smallest one such that the variety $\kappa_{8,k}(v_d(\mathbb{P}^n))$ can be reducible for some $d, n$. That is why we focus on studying $\kappa_{8,3}(v_d(\mathbb{P}^n))$ for $n \geq 4$.

We consider the polynomial ring $T = \mathbb{C}[\alpha_0, \alpha_1, \ldots, \alpha_n]$, and its graded dual $T^* = \mathbb{C}[x_0, x_1, \ldots, x_n]$, where $n \geq 4$. Recall Definition 4.18.

Our goal is to characterize for $d \geq 5$ and $n \geq 4$ the closure of the set-theoretic difference between the cactus variety $\kappa_{8,3}(v_d(\mathbb{P}T_1^*))$ and the secant variety $\sigma_{8,3}(v_d(\mathbb{P}T_1^*))$. For $n = 4$ and $d \geq 5$ this closure consists of points $[V] \in \mathrm{Gr}(3, T_d^*)$ such that each form in $V$ is divisible by $(d-2)$-th power of the same linear form. However for $n > 4$ the situation is more complicated. We start with showing that points of $\mathrm{Gr}(3, T_d^*)$ corresponding to subspaces divisible by $(d-1)$-st power of a linear form are in the Grassmann secant variety $\sigma_{8,3}(v_d(\mathbb{P}T_1))$.

**Proposition 6.20.** *Let $d \geq 2$ and $n \geq 4$ be integers, $z_0 \in T_1^*$ and $[U] \in \mathrm{Gr}(3, T_1^*)$. Define $V = z_0^{d-1}U \in \mathrm{Gr}(3, T_d^*)$. Then* $\mathrm{cr}(V) \leq 4$*, so $[V] \in \kappa_{4,3}(v_d(\mathbb{P}T_1^*)) = \sigma_{4,3}(v_d(\mathbb{P}T_1^*)) \subseteq \sigma_{8,3}(v_d(\mathbb{P}T_1^*))$.*

*Proof.* Up to a linear change of variables, $V$ is of one of the following forms

(a) $V = \langle x_0^{d-1}x_1, x_0^{d-1}x_2, x_0^{d-1}x_3\rangle$ or

(b) $V = \langle x_0^d, x_0^{d-1}x_1, x_0^{d-1}x_2\rangle$.

Then $V = W^{\mathrm{hom},d}$, where $W$ is respectively

(a) $W = \langle x_1, x_2, x_3\rangle$ or

(b) $W = \langle 1, x_1, x_2\rangle$.

In either case, $\dim_{\mathbb{C}} S/\operatorname{Ann}(W) \leq 4$, so $\operatorname{cr}(V) = \operatorname{cr}(W^{\mathrm{hom},d}) \leq 4$ by Theorem 3.30(i). $\square$

In the rest of the section we use the notation $W^{\blacktriangledown d}$ from Definition 4.18.

In the following proposition we identify many points from the Grassmann cactus variety which are outside of the Grassmann secant variety. In fact the closure of the set of these points is the second irreducible component of the Grassmann cactus variety. This will be established in Theorem 6.3.

**Proposition 6.21.** *Let $T^*$ be defined as at the beginning of this subsection and let $(z_0, z_1, \ldots, z_n)$ be a $\mathbb{C}$-basis of $T_1^*$. Assume that $V = z_0^{d-2}U$ for some natural number $d \geq 5$ and $[U] \in \operatorname{Gr}(3, T_2^*)$. Define $[W] := [U|_{z_0=1}] \in \operatorname{Gr}(3, R_{\leq 2}^*)$ where $R^* := \mathbb{C}[z_1, \ldots, z_n]$. If $W$ satisfies the following conditions:*

*(a)* $\operatorname{Apolar}(W^{\blacktriangledown d})$ *has Hilbert function* $(1, 4, 3)$*,*

*(b)* $[\operatorname{Spec}\operatorname{Apolar}(W^{\blacktriangledown d})] \notin \mathcal{H}ilb_8^{sm}(\mathbb{A}^n)$*,*

*then* $[V] \in \eta_{8,3}(v_d(\mathbb{P}^n)) \setminus \sigma_{8,3}(v_d(\mathbb{P}^n))$*.*

*Proof.* By Condition (a) we have $\dim_{\mathbb{C}}(R/\operatorname{Ann}(W^{\blacktriangledown d})) = 8$. Therefore from Theorem 3.30 (i)

$$\operatorname{cr}(V) = \operatorname{cr}((W^{\blacktriangledown d})^{\mathrm{hom},d}) \leq 8.$$

From Theorem 3.12, if $[V] \in \sigma_{8,3}(v_d(\mathbb{P}^n))$ then there exists $J \subseteq \operatorname{Ann}(V)$ with $[J] \in \operatorname{Slip}_{8,\mathbb{P}T_1} \subseteq \operatorname{Hilb}_T^{h_8}$. Thus $[\operatorname{Proj}(T/\operatorname{sat}(J, B))] \in \mathcal{H}ilb_8^{sm}(\mathbb{P}^n)$. From Theorem 3.30 (iii) it follows that $\operatorname{sat}(J, B) = \operatorname{Ann}(W^{\blacktriangledown d})^{\mathrm{hom}}$, so

$$[\operatorname{Spec}(R/\operatorname{Ann}(W^{\blacktriangledown d}))] \in \mathcal{H}ilb_8^{sm}(\mathbb{A}^n).$$

This contradicts Condition (b). $\square$

Finally we present the proof of the characterization of points of the second irreducible component of the Grassmann cactus variety.

*Proof of Theorem 6.3 and Corollary 6.5.* Recall the sets $C$ and $D$ from the statement of Lemma 4.24. Let

$$\psi : (T_1^* \setminus \{0\}) \times \mathrm{Gr}(3, T_2^*) \to \mathrm{Gr}(3, T_d^*)$$
$$\text{be given by } (z_0, [U]) \mapsto [z_0^{d-2}U].$$

By Proposition 6.21 if $(z_0, [U]) \in D$, then $[z_0^{d-2}U] \in \eta_{8,3}(v_d(\mathbb{P}T_1^*))$. Hence $\overline{\psi(C)} \subseteq \eta_{8,3}(v_d(\mathbb{P}T_1^*))$. The fibers of $\psi_{|C} : C \to \psi(C)$ are one-dimensional, so $\dim \overline{\psi(C)} = 8n + 8$. But $\dim \eta_{8,3}(v_d(\mathbb{P}T_1^*)) \leq 8n + 8$ by Proposition 4.8. Therefore $\overline{\psi(C)} = \eta_{8,3}(v_d(\mathbb{P}T_1^*))$. This concludes the proof of Theorem 6.3(ii).

We proceed to the proof of Theorem 6.3(i). Let $[W] \in B$, where $B$ is as in Lemma 4.23. Then by Proposition 6.21 we get $[W^{\mathrm{hom},d}] \in \kappa_{8,3}(v_d(\mathbb{P}^n)) \setminus \sigma_{8,3}(v_d(\mathbb{P}^n))$. It is enough to show that $\eta_{8,3}(v_d(\mathbb{P}^n)) \neq \kappa_{8,3}(v_d(\mathbb{P}^n))$. By Point (ii) every $[V] \in \eta_{8,3}(v_d(\mathbb{P}^n))$ is divisible by the $(d-2)$-nd power of a linear form. Therefore, $[\langle x_0^d, x_1^d, x_2^d \rangle] \in \kappa_{8,3}(v_d(\mathbb{P}^n)) \setminus \eta_{8,3}(v_d(\mathbb{P}^n))$.

Now we prove Corollary 6.5. Assume that $n = 4$. Then in the above notation the closure of $C$ in $(T_1^* \setminus \{0\}) \times \mathrm{Gr}(3, T_2^*)$ has the maximal dimension 41. Thus

$$\overline{C} = (T_1^* \setminus \{0\}) \times \mathrm{Gr}(3, T_2^*). \tag{6.4}$$

Notice that $\psi$ can be factored by

$$(T_1^* \setminus \{0\}) \times \mathrm{Gr}(3, T_2^*) \xrightarrow{q} \mathbb{P}T_1^* \times \mathrm{Gr}(3, T_2^*) \xrightarrow{\psi'} Gr(3, T_d^*),$$

where $q$ is the product of projection and the identity, and $\psi'([z_0], [P]) = [z_0^{d-2}P]$. Since $\psi'$ is a morphism from a projective variety, the set $\psi'(\mathbb{P}T_1^* \times \mathrm{Gr}(3, T_2^*))$ is closed. We have

$$\psi(\overline{C}) \subseteq \overline{\psi(C)}, \tag{6.5}$$

but the set

$$\psi(\overline{C}) = \psi'(q(\overline{C})) = \psi'(\mathbb{P}T_1^* \times \mathrm{Gr}(3, T_2^*))$$

is closed, hence Inclusion 6.5 is an equality, which means that

$$\eta_{8,3}(v_d(\mathbb{P}T_1^*)) = \psi(\overline{C}).$$

But then the statement of the corollary follows from Equation 6.4. $\square$

Using the description of the irreducible component $\eta$ given in Theorem 6.3, we are able to determine algorithmically if a given point from the Grassmann cactus variety is in the Grassmann secant variety.

The following lemma gives a description of the set-theoretic difference of the Grassmann cactus variety and the Grassmann secant variety. We need it to give a clear proof of Theorem 6.7.

**Lemma 6.22.** *Let $d \geq 5, n \geq 4$. The point $[V] \in \kappa_{8,3}(v_d(\mathbb{P}^n))$ does not belong to $\sigma_{8,3}(v_d(\mathbb{P}^n))$ if and only if there exists a linear form $z_0 \in T_1^*$, and $[U] \in \mathrm{Gr}(3, T_2^*)$ such that $V = z_0^{d-2}U$ and for any completion of $z_0$ to a basis $(z_0, \ldots, z_n)$ of $T_1^*$ (with dual basis equal to $(\gamma_0, \ldots, \gamma_n)$) we have:*

*(a)* $\mathrm{Apolar}((U|_{z_0=1})^{\blacktriangledown d})$ *has Hilbert function* $(1, 4, 3)$,

*(b)* $[\mathrm{Spec}\,\mathrm{Apolar}((U|_{z_0=1})^{\blacktriangledown d})] \notin \mathcal{H}ilb_8^{sm}(\mathbb{A}^n)$.

*Proof.* If $z_0 \in T_1$ and $[U] \in \mathrm{Gr}(3, T_2^*)$ are such that $V = z_0^{d-2}U$, and there exists a completion of $z_0$ to a basis $(z_0, \ldots, z_n)$ of $T_1^*$, for which Conditions (a),(b) hold, we get

$$[V] \notin \sigma_{8,3}(v_d(\mathbb{P}^n))$$

by Proposition 6.21.

Assume that $[V] \notin \sigma_{8,3}(v_d(\mathbb{P}^n))$. Then by Theorem 6.3 there exists a linear form $z_0 \in T_1^*$ such that $z_0^{d-2}|V$. Using Proposition 6.20 we conclude that $V$ is not divisible by $z_0^{d-1}$. Hence we showed that $V = z_0^{d-2}U$ for some $U \in \mathrm{Gr}(3, T_2^*)$. Extend $z_0$ to a basis $z_0, \ldots, z_n$. Let $W = U|_{z_0=1}$.

Now we prove Conditions (a), (b) hold. We have

$$V = (W^{\blacktriangledown d})^{\mathrm{hom},d}.$$

By Lemma 2.33 (i)

$$\mathrm{Ann}(W^{\blacktriangledown d})^{\mathrm{hom}} \subseteq \mathrm{Ann}(V).$$

If $\dim_{\mathbb{C}}(\mathrm{Apolar}(W^{\blacktriangledown d})) \leq 7$, then $\mathrm{cr}(V) \leq 7$ by Theorem 3.9, since the ideal $\mathrm{Ann}(W^{\blacktriangledown d})^{\mathrm{hom}}$ is saturated. Therefore,

$$[V] \in \kappa_{7,3}(v_d(\mathbb{P}^n)) = \sigma_{7,3}(v_d(\mathbb{P}^n)) \subseteq \sigma_{8,3}(v_d(\mathbb{P}^n)),$$

a contradiction.

From Theorem 3.30(ii') we obtain $\dim_{\mathbb{C}}(\mathrm{Apolar}(W^{\blacktriangledown d})) \leq 8$. We proved that $\dim_{\mathbb{C}}(\mathrm{Apolar}(W^{\blacktriangledown d})) = 8$. Since we assumed that $[V] \notin \sigma_{8,3}(v_d(\mathbb{P}^n))$, it follows by Lemma 3.35 that $\mathrm{Spec}(\mathrm{Apolar}(W^{\blacktriangledown d}))$ is not smoothable. This implies Condition (b) holds. From [29, Theorem 4.20], the algebra $\mathrm{Apolar}(W^{\blacktriangledown d})$ has Hilbert function $(1, 4, 3)$. We proved Condition (a) holds. $\square$

Steps 2–5 of the algorithm verify if $V$ satisfies necessary conditions to be of the form given by Lemma 6.22.

*Proof of Theorem 6.7.* Assume that $[V] \notin \sigma_{8,3}(v_d(\mathbb{P}^n))$. Then there exist a basis $(z_0, \ldots, z_n)$ of $T_1^*$ and $U \subseteq \mathbb{C}[z_0, \ldots, z_n]$ as in Lemma 6.22. Let $W = U|_{z_0=1} \subseteq \mathbb{C}[z_1, ..., z_n]$.

We get

$$V = (W^{\blacktriangledown d})^{\mathrm{hom},d}.$$

By Lemma 2.33(ii), we have $\mathrm{Ann}(V)_{\leq d-2} = (\mathrm{Ann}(W^{\blacktriangledown d})^{\mathrm{hom}})_{\leq d-2}$. Moreover, since $W^{\blacktriangledown d} \subseteq \mathbb{C}[z_1, \ldots, z_n]_{\leq 2}$, and $d \geq 5$, we obtain $((\mathrm{Ann}(W^{\blacktriangledown d})^{\mathrm{hom}})_{\leq d-2}) = \mathrm{Ann}(W^{\blacktriangledown d})^{\mathrm{hom}}$. Therefore we have

$$\mathfrak{a} = \sqrt{(\mathrm{Ann}(V)_{\leq d-2})} = \sqrt{\mathrm{Ann}(W^{\blacktriangledown d})^{\mathrm{hom}}} = (\gamma_1, \ldots, \gamma_n),$$

where $\gamma_1, \ldots, \gamma_n \in T_1$ are dual to $z_1, \ldots, z_n \in T_1^*$. This shows that if the $\mathbb{C}$-linear space $\left(\sqrt{(\mathrm{Ann}(V)_{\leq d-2})}\right)_1$ is not $n$-dimensional then $[V] \in \sigma_{8,3}(v_d(\mathbb{P}^n))$. Therefore, in that case, the algorithm stops correctly at Step 2.

Assume that the algorithm did not stop at Step 2. Then if $V$ is of the form as in Lemma 6.22, then $z_0$ divides $V$ exactly $(d-2)$-times. Otherwise $[V] \in \sigma_{8,3}(v_d(\mathbb{P}^n))$ and the algorithm stops correctly at Step 3.

Assume that the algorithm did not stop at Step 3. Then the Hilbert function of $R/I$ computed in Step 4 is $(1, 4, 3)$ if and only if Condition (a) of Lemma 6.22 is fulfilled. Therefore, if it is not $(1, 4, 3)$, the algorithm stops correctly at Step 4.

Assume that the algorithm did not stop at Step 4. Then $V$ satisfies Condition (a) from Lemma 6.22. Hence $[V]$ is in $\sigma_{8,3}(v_d(\mathbb{P}^n))$ if and only if $V$ does not satisfy Condition (b). Using Lemma 4.6, this is equivalent to

$$\dim_{\mathbb{C}} \mathrm{Hom}_R(I, R/I) > 8n - 7.$$

The left term is the dimension of the tangent space to the Hilbert scheme $\mathcal{H}ilb_8(\mathbb{A}^n)$ at the point $[\mathrm{Spec}\, R/I]$ (see [53, Prop. 2.3.] or [68, Theorem 18.29]). $\square$

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