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Curves of degree 1 on projective varieties

Master's thesis
in MATHEMATICS

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Abstract

We study families of singular lines on projective varieties with respect to an ample polarization. We use weighted projective spaces to construct projective varieties covered by singular lines, and we also show an example of a smooth projective surface admitting two distinct lines intersecting at two points. As an application, we show that there exists a smooth Fano variety with the Picard number equal to 1 admitting a positive dimensional family of singular lines with respect to the anti–canonical polarization.

Keywords

lines, weighted projective space, M–regular locus, singular rational curves

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Krzywe stopnia 1 na rozmaitościach rzutowych

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Introduction

Since the dawn of civilization, geometry has always been inherently related to the concept of a line. From the classical point of view, one can define the projective plane as a model of the following axiomatic theory, using points and lines as the primitive notions:

1. given two distinct points, there is exactly one line incident with both points;
2. given two distinct lines, there is exactly one point incident with both lines;
3. there exist four points such that no three of them are incident with the same line.

From a more modern but still classical point of view, one considers a variety embedded into the complex projective space $\mathbb{P}^n$. Lines on an embedded variety X are lines on the projective space $\mathbb{P}^n$, that are contained in X. One obtains very similar properties of lines to those assumed in axiomatic geometry:

1. given two distinct points, there is at most one line incident with both points;
2. given a line and a hyperplane section, either the line is contained in the hyperplane section or there is exactly one point incident with both objects;
3. every line is isomorphic to $\mathbb{P}^1$.

We consider a generalization of the concept of a line. Instead of embedding a projective variety into the projective space, we choose an ample line bundle. A polarized variety is a pair (X, L) consisting of a variety X and an ample line bundle L on X. Lines on (X, L) are rational curves of degree 1 with respect to L. If we have chosen L to be a very ample line bundle, it gives an equivalent description of lines to the classical case; see Lemma 1.11.

We study how the behaviour of lines on polarized varieties can differ from the behaviour described by properties 1,2 and 3. We are particularly interested in lines not satisfying the last property, that is, the singular lines. In Chapter 4 we see that even in the case of smooth surfaces, there is an example contradicting properties 1 and 3:

Theorem (Theorem 4.1)**.** *There exists a smooth polarized projective surface (Y, L) admitting two distinct singular lines intersecting at two points.*

Despite that, lines admit a lot of regularity. For example, a classical consequence of Mori's Bend and Break Lemma 2.8 [Kol96, Corollary II.5.5.2], states that given two distinct points, there is at most a finite number of lines incident with both points. Other examples of such regularity are the following corollaries of [Keb02, Theorem 2.4]:

Corollary. *[Keb02, Theorem 3.3] Let (X, L) be a projective polarized variety. Then for a general point $p \in X$, there is at most a finite number of singular lines containing p.*

Corollary. *[Keb02, Theorem 3.3] Let (X, L) be a projective polarized variety. Then for a general point $p \in X$, there is no line with its singular point at p.*

There is different kind of motivations, a little bit more technical.

Let X be a projective variety. Let $\mathrm{RatCurves}^n(X)$ be the normalization of the space of rational curves on X, as defined in [Kol96, Definition 2.11]. Let H be an irreducible component of $\mathrm{RatCurves}^n(X)$. From the universal property of $\mathrm{RatCurves}^n(X)$, we get the universal family U_H, together with the evaluation morphism $\xi \colon H \to X$ and the projection $\pi \colon U_H \to H$.

$$\begin{tikzcd} U_H \arrow[r, "\xi"] \arrow[d, "\pi"] & X \\ H & \end{tikzcd}$$

We say that H is a **minimal component** of $\mathrm{RatCurves}^n(X)$, if ξ is dominant and for a general point $p \in X$, the fibre $\xi^{-1}(p)$ is proper. Elements of the minimal component are called **minimal rational curves**.

Study of minimal rational curves is a very active area of research; see: [HM04], [KK04], [HK05], [FH09], [BM19], [HL21], [Hwa22], [HL22].
The connection of this research with our topic is following. The space of lines on (X, L) is a projective scheme [BKK20, Proposition 3.7]. Thus, we see that for a variety X covered by lines, lines are minimal rational curves. To some extent, lines are the simplest possible examples of minimal rational curves.

Another motivation for our study is the following Kebekus-Miyaoka-Shepherd-Barron Theorem, which characterizes the projective space. We state a version of the theorem from [BKK20, Theorem 2.8]:

Theorem. *[Keb02, Theorem 3.6] Let (X, L) be a polarized projective variety of dimension n, such that two general points are connected by a line. Then the normalization of X is the projective space $\mathbb{P}^n$*

This is a very interesting result because it shows that one can characterize a variety by studying the geometry of its lines.

Last but not least, there is an interest in studying lines on contact Fano manifolds. A contact manifold is a complex odd-dimensional manifold X with a contact structure, that is admitting a short exact sequence

$$0 \to F \to TX \xrightarrow{\theta} L \to 0,$$

where F is a sub-bundle of TX, $L \in \mathrm{Pic}(X)$ and locally $d\theta_{|F}$ induces a symplectic structure on each fibre of F. Study of lines is used to address the problem of classification of contact Fano manifolds; see [KPSW00], [Keb01], [Keb05], [PS14], [BKK20]. In this thesis, we work in a more general setting and do not assume any contact structure on X. In this context, we prove the following theorem:

Theorem (Theorem 4.7)**.** *There exists smooth Fano variety Y with the Picard number equal to 1, admitting a positive dimensional family of singular lines with respect to the anti–canonical polarization.*

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Chapter 1

Lines, parameter spaces, and smooth divisors in linear systems

1.1. Basic definitions

Throughout the whole thesis, we work over the complex numbers $\mathbb{C}$. By **scheme** we always mean a separated scheme of finite type over $\mathbb{C}$. A **variety** is a reduced and irreducible scheme. Given a variety X, we say that a property holds at a **general point** if it holds for all points in an open dense subset of points of X.

First, we state the definitions of the fundamental objects of our concern.

Definition 1.1. *Let U and W be schemes. We say that morphism $f\colon U \to W$ is an **immersion** if it restricts to an isomorphism between U and a locally closed subscheme of W.*

Definition 1.2. *Let U be a variety. A line bundle $L \in \mathrm{Pic}(U)$ is a **very ample** line bundle if and only if there exists an immersion $i\colon U \to \mathbb{P}^r$, such that $i^*\mathcal{O}_{\mathbb{P}^r}(1) \cong L$.*

Definition 1.3. *Let U be a variety. A line bundle $L \in \mathrm{Pic}(U)$ is an **ample** line bundle if and only if there exists a number $k \in \mathbb{N}$ such that $L^{\otimes k}$ is very ample.*

Definition 1.4. *A **polarized variety** is a pair (U, L) consisting of a variety U and ample line bundle L.*

Definition 1.5. *Let (U, L) be a polarized variety. Let C be a rational curve on X and denote the normalization of C by $\eta\colon \mathbb{P}^1 \to C$. A curve $C \subset X$ is a **line** on (X, L) if and only if $\eta^*(L|_C) \cong \mathcal{O}_{\mathbb{P}^1}(1)$.*

1.1.1. Intersection numbers

Intersection theory is a useful tool in our study. We recall basic definitions and properties of the intersection numbers.

Notation. *Let U be a scheme. We denote by $\mathrm{Div}(U)$ the group of Cartier divisors on U.*

Theorem 1.6. *[Laz04, Section 1.1 C] Let us fix $k \in \mathbb{N}$. Let X be a projective variety, and let V be a subvariety of X of dimension k. There exists **the intersection number** which is a function $\int_V\colon \mathrm{Div}(X)^k \to \mathbb{Z}$, satisfying the following properties:*

- *$\int_V$ is symmetric and multilinear;*

- $(D_1 \cdot \ldots \cdot D_k \cdot V) := \int_V (D_1, \ldots, D_k)$, *depends only on the linear equivalence classes of the* D_i*'s;*
- *if* $D_1|_V, \ldots, D_k|_V$ *are effective divisors with supports not containing* V*, and they meet transversally at smooth points of* V*, then*
$$(D_1 \cdot \ldots \cdot D_k \cdot V) := \#\{D_1|_V \cap \ldots \cap D_k|_V\};$$
- *let* $f \colon Y \to X$ *be a generically finite surjective projective map, then*
$$\int_Y (f^* D_1, \ldots, f^* D_k) = (\deg f) \int_X (D_1, \ldots, D_k);$$
- *let* C *be a curve on* X *and* D *a Cartier divisor, then*
$$\int_C D = \deg_C(D).$$

Remark. *Since the intersection number depends only on the linear classes of divisors, one can compute the intersection for line bundles instead of divisors.*

Definition 1.7. *[Laz04, Definition 1.1.14] Let* X *be a projective variety. Divisors* $D_1, D_2 \in \mathrm{Div}(X)$ *are* ***numerically equivalent*** *if and only if for every curve* C *on* X
$$(D_1 \cdot C) = (D_2 \cdot C).$$

Definition 1.8. *[Laz04, Definition 1.1.15]* ***The Néron–Severi group*** $\mathrm{Num}(X)$ *of a projective variety* X *is the group of numerical equivalence classes of divisors on* X*.*

Definition 1.9. *[Laz04, Definition 1.1.17] The rank of* $\mathrm{Num}(X)$ *is called the Picard number of* X*.*

In the context of intersection theory, we get an alternative description of lines.

Lemma 1.10. *Let* (X, L) *be a polarized projective variety. A rational curve* C *on* X *is a line if and only if the intersection number* $(L \cdot C)$ *equals to* 1*.*

Proof. Let C be a rational curve on (X, L). Consider the normalization $\eta \colon \mathbb{P}^1 \to C$. Suppose that $\eta^*(L|_C) \cong \mathcal{O}_{\mathbb{P}^1}(1)$, then $(\eta^*(L|_C) \cdot \mathbb{P}^1) = 1$. The normalization morphism is finite, surjective, projective and of degree 1; thus, by the functoriality of intersection $(L|_C \cdot C) = 1$. Conversely assume that $(L|_C \cdot C) = 1$, then by the functoriality $(\eta^*(L|_C) \cdot \mathbb{P}^1) = 1$, any line bundle of degree 1 on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)$. □

1.2. Motivation from classical geometry

In this section, we describe the classical case of lines, where L is very ample as a point of reference for further study.

Lemma 1.11. *Let* V *be a linear space. A line on the projective space* $(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))$ *is a projectivization of a two-dimensional linear subspace of* V*.*

Proof. Let us take a curve C on $\mathbb{P}(V)$ and consider the normalization η_C of C

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "\eta_C"] \arrow[rr, bend left, "f"] & C \arrow[r, hook, "i"] & \mathbb{P}(V). \end{tikzcd}$$

The composition $f := i \circ \mu_C$ of the normalization μ_C with the closed embedding $i\colon C \subset \mathbb{P}^N$ is given by $n+1$ section $(h_0, \ldots, h_n)$ of $\mathcal{O}_{\mathbb{P}^1}(1)$

$$f\colon [s:t] \mapsto [h_0(s,t) : \ldots : h_n(s,t)].$$

Since the morphism f is not a constant map, one can choose two of h_i's that are not linearly dependent. Thus, the morphism f has to be a closed embedding and then the normalization μ_C is an isomorphism. □

We get these immediate corollaries.

Corollary 1.12. *Let (X, L) be polarized projective variety, and let L be a very ample line bundle. Then every line on (X, L) is smooth.*

Corollary 1.13. *Let (X, L) be polarized projective variety, and let L be a very ample line bundle. Consider two distinct closed points $x, y \in X$, there exists at most one line C on (X, L) incident to both points.*

In Chapter 4, Theorem 4.1, we show that, for L ample, but not very ample, the conclusion of Corollary 1.13 does not necessarily hold. More precisely we prove that there exists a smooth polarized projective variety (X, L) admitting two distinct lines with two common points. Even in such a situation, Theorem 2.8 tells us that there does not exist a proper positive dimensional family of lines through two points.

1.3. Bertini theorem

Another useful tool is Bertini theorem, which allows us to construct examples of smooth varieties admitting certain families of lines.

Theorem 1.14. *[Har77, Theorem 8.15] Let Y be a variety. The cotangent sheaf Ω_Y is a locally free sheaf if and only if Y is smooth.*

Theorem 1.15. *[GW20, Theorem 11.17] Let X be a Noetherian reduced scheme, and let $\mathcal{F}$ be a coherent sheaf on X and e be a non-negative integer. There exists a locally closed subscheme of X denoted $X(\mathcal{F}, e)$, which is characterized by the following universal property; for every reduced scheme T and any morphism $g\colon T \to X$, the morphism g factors through $X(\mathcal{F}, e)$ if and only if $g^*\mathcal{F}$ is locally free of rank e.*

Theorem 1.16. *[GW20, Theorem 11.17] Let X be a projective reduced scheme, and let $\mathcal{F}$ be a coherent sheaf on X. The locally closed subschemes $X(\mathcal{F}, e)$ are disjoint only finitely many of them are non-empty, and they constitute a stratification of X.*

Theorem 1.17. *Let us consider the following tuple $(Z, S, X, \mathcal{O}(1))$, where Z is a projective variety, S is a finite subset of Z containing the singular locus of Z, X is a closed reduced subscheme of Z disjoint from S and $\mathcal{O}(1)$ is a very ample line bundle on Z. Let I be the ideal sheaf of X in Z, consider $s \in \mathbb{N}$ such that $I(s) := I \otimes \mathcal{O}(1)^{\otimes s}$ is globally generated on Z and* *very ample on the open subscheme $Z \setminus X$. Let us consider the stratification of X by $X(\Omega_X, e)$ as in Theorem 1.15 and Theorem 1.16, we define E to be the set of natural numbers e such that $X(\Omega_X, e) \neq \emptyset$. Assume that the following estimate holds:*

$$\max_{e \in E}(\dim X(\Omega_X, e) + e) < \dim Z,$$

then a general section of $I(s)$ cuts out a smooth projective variety Y containing X and disjoint from S.

Proof. The strategy of the proof is to describe the incidence locus of sections that do not satisfy the required properties. We call these sections *bad ones.* Then we estimate the dimension of the set of the *bad sections* as a closed subset of the projective space of sections of $I(s)$ and conclude that the complement is open and dense.

Let $V = \Gamma(I(s), Z)$, $|V| = \dim(\mathbb{P}(V))$, $\dim(Z) = r$.

We consider the stratification of Z by locally closed subsets

$$Z = S \sqcup \bigsqcup_{e \in E} X(\Omega_X, e) \sqcup Z \setminus (S \cup X).$$

For each stratum T, we define a closed subscheme $B_T \subset \mathbb{P}(V) \times T$.

Take $T = \Omega(X, e)$, for some e. Consider the conormal sequence

$$I/I^2 \to \Omega_Z \otimes \mathcal{O}_X \to \Omega_X \to 0.$$

Since $I(s)$ is globally generated by V, we get that the following composition is a surjective one

$$V \otimes \mathcal{O}_X \to I(s) \to I/I^2(s).$$

We twist the conormal sequence by $\mathcal{O}(s)$ and get the exact sequence

$$V \otimes \mathcal{O}_X \to \Omega_Z(s) \otimes \mathcal{O}_X \to \Omega_X(s) \to 0.$$

Since we are working over T we tensor the sequence with $\mathcal{O}_T$

$$V \otimes \mathcal{O}_T \xrightarrow{\phi} \Omega_Z(s) \otimes \mathcal{O}_T \to \Omega_X(s) \otimes \mathcal{O}_T \to 0.$$

For any closed point $p \in T$ with residue field denoted by $k(p) \cong \mathbb{C}$, we get the exact sequence of complex vector spaces

$$0 \to \ker(\phi_p) \to V \xrightarrow{\phi_p} \Omega_Z(s) \otimes k(p) \to \Omega_X(s) \otimes k(p) \to 0.$$

The kernel $\ker(\phi_p)$ consists of sections from V that cut out hypersurfaces singular at p, since those sections induce 0 cotangent vector at p. As a consequence, we define

$$B_p := \mathbb{P}(\ker(\phi_p)) \subset \mathbb{P}(V) \times \{p\}.$$

The kernel $\ker(\phi_p)$ has dimension $|V| + 1 - \dim \operatorname{im}(\phi_p)$. Since, $\dim(\Omega_Z(s) \otimes k(p)) = r$ and $\dim(\Omega_X(s) \otimes k(p)) = e$, we get that $\dim(\ker(\phi_p)) = |V| + 1 - r + e$. Now we go back to the level of stratum T:

$$0 \to \ker(\phi) \to V \otimes \mathcal{O}_T \xrightarrow{\phi} \Omega_Z(s) \otimes \mathcal{O}_T.$$

The kernel $\ker(\phi)$ is a locally free coherent sheaf, since every closed fibre $\ker(\phi_p)$ has constant dimension $|V| + 1 - r + e$. Thus, we get a closed subscheme [Vak17, Definition 17.2.C]:

$$B_T := \mathrm{Proj}_T(\mathrm{Sym}(\ker(\phi)^*)) \subset \mathbb{P}(V) \times T.$$

We remark that $\dim(B_T) = |V| - r + e + \dim(T) < |V|$.

Since the case when $T = Z \setminus (S \cup X)$ is the classical Bertini theorem for a quasi–projective variety and the argument is similar but simpler than the previous one we omit the argument. Details can be found in [Vak17, 12.4.2. Bertini's Theorem]. We get a closed subscheme B_T of $\mathbb{P}(V) \times T$ of dimension $|V| - 1$, which is an incidence locus of sections from V that cut out hypersurface of Z singular at a point of T.

Now we consider the case $T = S$; *bad sections* at $p \in S$ are simply the ones that vanish at some $p \in S$. Fix $f_0 \in V$, such that f_0 does not vanish on p. Then one defines the linear map

$$\begin{aligned}\phi_p \colon V &\to \mathcal{O}_{p,Z}/m_{p,Z},\\ f &\mapsto \frac{f}{f_0}.\end{aligned}$$

The kernel of this map is the linear space of sections vanishing at the closed point p.

$$\dim(\ker(\phi_p)) = |V|,$$

so $B_p = \mathbb{P}(\ker \phi_p)$ for $p \in S$ is a projective space of dimension $|V| - 1$. The scheme B_T is a disjoint union of such B_p, since there are only finitely many of them it is a well–defined closed subscheme of $\mathbb{P}(V) \times T$.

For each stratum T the dimension of B_T is strictly lower than $|V|$. We get that the closure of each image $\pi_1(B_T)$ of projection on the first coordinate is a closed subset of $\mathbb{P}(V)$ of dimension less than $|V|$. A union of finitely many strict closed subsets of lower dimension is a proper closed subset. Thus, the complement of this union is an open dense subset of $\mathbb{P}(V)$.

Since elements of V are sections of $I(s)$, every element of $\mathbb{P}(V)$ cuts out a hypersurface containing X. Elements of $\pi_1(B_S)$ correspond to all elements of V that cut out hypersurfaces that contain any point of S. Elements of $\pi_1(B_T)$, for a chosen stratum $T \neq S$, correspond to all elements of V that cut out hypersurfaces that are singular at a point of T. Therefore, an element outside the union $\pi_1(B_T)$ is a section that cuts out hypersurface satisfying our assumptions. □

Corollary 1.18. *Let Z be a smooth projective variety of dimension at least* 1*. Then any pair of two distinct points $p \in Z$ and $q \in Z$ can be connected by a smooth curve.*

Proof. We argue by induction on dimension $\dim(Z) = m$. If $m = 1$ then the statement trivially holds. Suppose that the statement holds for $0 < k < m$. Let us choose a very ample bundle L on Z. Take $X = \{p, q\}$ with the reduced scheme structure. The tuple $(Z, \emptyset, X, L)$ satisfies the assumptions of Theorem 1.17; consequently, we get a smooth variety Y of dimension $m - 1$ containing p and q. □

Notation. *Let X and E be as in Theorem 1.17.* ***The embedding dimension of*** *X is the supremum of E.*

Remark. *Let X,Z and E be as in Theorem 1.17. Denote the embedding dimension of X by N. Note that in Theorem 1.17, instead of checking the estimate*

$$\max_{e \in E}(\dim X(\Omega_X, e) + e) < \dim Z,$$

we can check the stronger estimate

$$\dim X + N < \dim Z.$$

Chapter 2

Parametrizing space of lines

In this chapter, we study families of lines and it particular parametrizing space of lines $\mathrm{Lines}(X, L)$. The results come from [Kol96], [Keb02] and [BKK20].

Definition 2.1. *Let (X, L) be a polarized variety. A **family of lines** on (X, L) is a pair of morphisms of varieties, a projection $\pi\colon U_R \to R$, and an evaluation $\xi\colon U_R \to X$, satisfying the following properties:*

- *varieties U_R and R are normal;*
- *π is a proper map;*
- *every set-theoretic fibre of ξ endowed with the reduced scheme structure is isomorphic to $\mathbb{P}^1$;*
- *for any closed point $[l] \in R$, the morphism $\xi|_{\pi^{-1}([l])}$ is mapped onto l, line on (X, L);*
- *the morphism $\xi|_{\pi^{-1}([l])}\colon \pi^{-1}([l]) \to l$ is the normalization morphism of line l.*

We define some basic properties of the families of lines. They are intuitive, but we state them for clarity of the arguments.

Definition 2.2. *Let (X, L) be a polarized variety. The dimension of a family of lines $(\pi\colon U_R \to R,\ \xi\colon U_R \to X)$ on (X, L) is the dimension of R. We say that the family of lines is positive dimensional if $\dim(R) > 0$.*

Definition 2.3. *Let (X, L) be a polarized variety. A family of lines $(\pi\colon U_R \to R,\ \xi\colon U_R \to X)$ is a **proper family of lines** if R is a proper variety.*

Definition 2.4. *Let (X, L) be a polarized variety. A family of lines $(\pi\colon U_R \to R,\ \xi\colon U_R \to X)$ is a **family of singular lines** if each line for each closed point $[l] \in R$, the associated line $l \subset X$ is a singular curve.*

Definition 2.5. *Let (X, L) be a polarized variety. A family of lines $(\pi\colon U_R \to R,\ \xi\colon U_R \to X)$ **covers the variety** X if the evaluation ξ morphism is dominant.*

Now we state the crucial theorems, describing the parametrizing space of lines on (X, L). In our case, the parametrizing space $\mathrm{Lines}(X, L)$ is just a subscheme of Chow variety of X. For further details see [Kol96, Section 1.3 Chow Varieties] and [BKK20, Section 3.1].

Theorem 2.6. *[BKK20, Proposition 3.7.1] Let (X, L) be a polarized projective variety. Suppose that (π, ξ) is a family of lines on (X, L). Then $E = \pi_*(\xi^* L)$ is a vector bundle of rank 2 on R and $U_R \cong \mathbb{P}(E^*)$.*

Theorem 2.7. *[BKK20, Proposition 3.7] Let (X, L) be a polarized projective variety. There exists a projective reduced scheme* $\mathrm{Lines}(X, L)$*, together with function Rep from the set of closed points of* $\mathrm{Lines}(X, L)$ *to the set of lines on (X, L), satisfying the following properties:*

- *the function Rep is a bijection. We say that a closed point $r \in \mathrm{Lines}(X, L)$ represents a line $l \subset X$ if $Rep(r) = l$;*
- *consider a family of lines (π, ξ) on (X, L). There exists a morphism $f \colon R \to \mathrm{Lines}(X, L)$, such that for any closed point $[l] \in R$ the image $f([l])$ is a point representing line $l = \xi(\pi^{-1}([l]))_{red}$;*
- *suppose R is an irreducible normal variety together with a morphism $f \colon R \to \mathrm{Lines}(X, L)$. There exists a family of lines $\pi \colon U_R \to R$, such that for any point $[l] \in R$, the image of $\xi|_{\pi^{-1}([l])}$ is the line l represented by the point $f([l])$.*

2.1. Bend and break

We need a classical consequence of Mori's Bend and Break Lemma [Kol96, Corollary II.5.5.2] in a version as in [BKK20, Lemma 3.8].

Theorem 2.8. *[BKK20, Lemma 3.8] [**Bend and break**] Let (X, L) be a polarized projective variety. Let R be a positive dimensional normal variety with non-constant morphism $f \colon R \to$* $\mathrm{Lines}(X, L)$*. Consider a family of lines (π, ξ) induced by f by Theorem 2.7, then there is at most one point $p \in X$ incident to every line from the family.*

Proof. We argue by a contradiction. Suppose that the family has at least two common points. We replace R with a closed curve on R. Then, we replace R with its normalization. We have a morphism $R \to \mathrm{Lines}(X, L)$, by Theorem 2.7, $\mathrm{Lines}(X, L)$ is projective. Therefore, using curve-to-projective extension [Vak17, 16.5.1. The Curve-to-Projective Extension Theorem], we can extend the morphism f to $\tilde{f} \colon \tilde{R} \to \mathrm{Lines}(X, L)$ from $\tilde{R}$ a smooth projective compactification of R. Now we replace R with its compactification $\tilde{R}$ and f with $\tilde{f}$. Then by Theorem 2.7 we assume that $U_R \to R$ is a ruled surface.
Two common points of the family give rise to disjoint sections $\sigma_1, \sigma_2 \colon R \to U_R$, contracted by the evaluation morphism ξ.

$$\begin{tikzcd} U_R \arrow[r, "\xi"] \arrow[d, "\pi"] & X \\ R. \arrow[u, bend left, "\sigma_{1,2}"] & \end{tikzcd}$$

Since U_R is a ruled surface then by [Har77, Proposition V 2.3],

$$\mathrm{Num}(U_R) = \mathbb{Z} \oplus \mathbb{Z}.$$

Moreover, $\mathrm{Num}(U_R)$ is generated by a section $\sigma_1(R)$ and a fibre F (all fibres are numerically equivalent). Also, we have the following relations

$$\sigma_i \cdot F = 1, \text{ for } i = 1, 2,$$

$$F^2 = 0.$$

Since sections are contractible then, by [Kol96, Proposition II 5.3.2],

$$\sigma_i^2 < 0 \text{ for } i = 1, 2,$$

and also

$$\sigma_1 = \alpha\sigma_2 + \beta F.$$

Now we perform a simple calculation to obtain a contradiction.

$$1 = \sigma_1 \cdot F = \alpha\sigma_2 \cdot F = \alpha,$$

$$0 = \sigma_1 \cdot \sigma_2 = \sigma_2^2 + \beta \implies \beta > 0,$$

$$0 > \sigma_1^2 = \beta > 0.$$

□

Corollary 2.9. *Let (X, L) be a polarized projective variety. For any two chosen closed points $p_1, p_2 \in X$ there is at most a finite number of lines incident to both points.*

Proof. Fix two closed points $p_1 \in X$ and $p_2 \in X$. By Theorem 2.7 every such line is represented by a closed point of projective scheme $\mathrm{Lines}(X, L)$. Take an irreducible component R of $\mathrm{Lines}(X, L)$. The normalization of R, denoted by $\tilde{R}$ induces a family of lines ($\pi\colon U_{\tilde{R}} \to \tilde{R}$, $\xi\colon U_{\tilde{R}} \to X$). The subset of lines in $\tilde{R}$ incident to both points $\pi(\xi^{-1}(p_1)) \cap \pi(\xi^{-1}(p_2))$ is closed. The scheme $W := (\pi(\xi^{-1}(p_1)) \cap \pi(\xi^{-1}(p_2)))_{red}$ is projective; in particular, it has finitely many irreducible components. By Theorem 2.8, the normalization of an irreducible component of W has to be 0-dimensional. As a consequence W is finite. □

2.2. Families of singular lines

In this subsection, we cite results from [Keb02] as a motivation for Theorems 3.21 and 3.22

Definition 2.10. *Let (C, L) be a singular line. Let C_0 be the cuspidal plane cubic, and C_1 be the nodal plane cubic. We say that point $p \in C$ is a* ***special point*** *if one of the following holds:*

- *p is a singular point of C;*
- *there exists a birational morphism $\beta\colon C_i \to C$, such that $\beta^*(L) \cong \mathcal{O}_{C_i}(\beta^{-1}(p))$.*

Lemma 2.11. *[Keb02, Lemma 2.3] Let (C, L) be a singular line. Then the set of special points of C is finite.*

Theorem 2.12. *[Keb02, Theorem 2.4] Let (X, L) be a polarized projective variety. Let ($\pi\colon U_R \to R$, $\xi\colon U_R \to X$) be a proper family of lines on (X, L), such that every line in the family is singular and there exists a point p incident to every line $[l] \in R$. Then there exists a line $[l] \in R$, such that $p \in l$ is a special point.*

Natural continuation is the study of families with a common point, which is special to each line from the family. Theorems 3.21 and 3.22, show that there exist such families with a common point, singular on each line and such that it is a smooth special point on each line.

Chapter 3

Weighted projective spaces

The main tools of this thesis, used to construct various families of singular lines, are the weighted projective spaces. In the first section, we state a few important properties of such.

3.1. Basic properties of weighted projective spaces

Definition 3.1. *Let* $\mathbf{a} := (a_0, \ldots, a_n)$ *be tuple of positive integers. Let* R *be a polynomial ring* $\mathbb{C}[y_0, \ldots, y_n]$ *with not necessarily standard grading,* $\deg(y_i) = a_i$*. The* ***weighted projective space*** *with weights* $(a_0, \ldots, a_n)$ *is* $\operatorname{Proj}(R)$*.*

Notation. *Let* R *be a polynomial ring* $\mathbb{C}[y_0, \ldots, y_n]$ *with not necessarily standard grading,* $\deg(y_i) = a_i$*. The weighted projective space with weights* $(\overbrace{1, \ldots, 1,}^{m+1-\ times}\ a_0, \ldots, a_n)$ *will be denoted by*

$$\mathbb{P}((1)_{j=0}^{m}, \mathbf{a}) := \mathbb{P}(\overbrace{1, \ldots, 1,}^{m+1-\ times}\ \mathbf{a}) := \mathbb{P}(\overbrace{1, \ldots, 1,}^{m+1-\ times}\ a_0, \ldots, a_n) := \operatorname{Proj}(R[x_0, \ldots, x_m]),$$

where $\deg(x_i) = 1$*. If* $m = 0$ *then instead of* x_0 *we write* x*,*

$$\mathbb{P}(1, \mathbf{a}) := \operatorname{Proj}(R[x]).$$

Lemma 3.2. *The distinguished open set* $D(x) \subset \mathbb{P}(1, \mathbf{a})$ *is isomorphic to* $\mathbb{A}^{n+1}$*.*

Proof. By the definition of distinguished open set $D(x) = \operatorname{Spec}((R[x]_x)_0) \cong \operatorname{Spec}(\mathbb{C}\left[{}^{y_0}/_{x^{a_0}}, \ldots, {}^{y_n}/_{x^{a_n}}\right])$. Since $\mathbb{C}[{}^{y_0}/_{x^{a_0}}, \ldots, {}^{y_n}/_{x^{a_n}}] \cong \mathbb{C}\left[x_0, \ldots, x_n\right]$, the statement holds. □

Definition 3.3. *The weighted projective space* $\mathbb{P}(\mathbf{a})$ *is* ***well-formed*** *if* $\gcd(a_0, \ldots, \hat{a}_i, \ldots, a_n) = 1$ *for every* $i \in \{0, \ldots, n\}$*.*

Lemma 3.4. *[IF00, Lemma 5.7] For every positive integer* $d > 0$*, we get the isomorphism*

$$\mathbb{P}(da_0, da_1, \ldots, da_n) \cong \mathbb{P}(a_0, \ldots, a_n).$$

Assume that $\gcd(a_0, \ldots, a_n) = 1$*, then for* $q = \gcd(a_1, \ldots, a_n)$ *we get the isomorphism*

$$\mathbb{P}(a_0, a_1, \ldots, a_n) \cong \mathbb{P}(a_0, a_1/q, \ldots, a_n/q).$$

In particular, every weighted projective space $\mathbb{P}(\mathbf{a})$ *is isomorphic to a well-formed weighted projective space.*

Proof. To prove the first claim, let us recall from [Vak17, Exercise 6.4.D] that for a graded ring R and the d–th Veronese subring $R^{(d)} := \bigoplus_{m\in\mathbb{Z}} R_{dm}$,

$$\operatorname{Proj}(R) \cong \operatorname{Proj}(R^{(d)}).$$

Then consider the ring $R' = \mathbb{C}[z_0, \dots, z_n]$ with grading given by the tuple $(da_0, \dots, da_n)$. Since $R'^{(d)} \cong R$, we get the first claim,

$$\operatorname{Proj}(R) \cong \operatorname{Proj}(R'^{(d)}) \cong \operatorname{Proj}(R').$$

Second claim we prove similarly. We consider $R^{(q)}$ and look at the monomial $y_0^{d_0} \dots, y_n^{d_n} \in R^{(q)}_{kq}$ of degree kq. This leads us to the equation

$$a_0d_0 + \dots + a_nd_n = kq.$$

Since for every for $i > 0$, a_i is divisible by q, and $\gcd(a_0, q) = 1$, $q|d_0$. Thus, $R^{(q)} \cong \mathbb{C}[y_0^q, \dots, y_n]$, but then

$$\operatorname{Proj}(R) \cong \operatorname{Proj}(R^{(q)}) \cong \mathbb{P}(qa_0, a_1, \dots, a_n) \cong \mathbb{P}(a_0, a_1/q, \dots, a_n/q).$$

□

Definition 3.5. *Let M be a graded R-module. We define $M(d)$ to be M twisted by d, that is M as R-module, but with grading defined by the formula*

$$M(d)_j := M_{j+d}.$$

Definition 3.6. *We define the sheaf $\mathcal{O}_{\mathbb{P}(\mathbf{a})}(d)$ to be a coherent sheaf associated to the graded module $R(d)$.*

The definition above is analogous to the case of the standard projective space, but for a general choice of a tuple $\mathbf{a}$, those sheaves may admit a whole range of pathologies; for example, they do not have to be line bundles.

Example 3.7. *Let us consider $\mathbb{P}(1,1,2) = \operatorname{Proj}(\mathbb{C}[x_0, x_1, y_0])$. The sheaf $\mathcal{O}_{\mathbb{P}(1,1,2)}(1)$ is not a line bundle. On the open set $D(y_0) \cong \operatorname{Spec}(\mathbb{C}\left[{x_0^2}/{y_0}, {x_0x_1}/{y_0}, {x_1^2}/{y_0}\right])$ it is given by the module*

$$\frac{x_0}{y_0}\mathbb{C}\left[{x_0^2}/{y_0}, {x_0x_1}/{y_0}, {x_1^2}/{y_0}\right] + \frac{x_1}{y_0}\mathbb{C}\left[{x_0^2}/{y_0}, {x_0x_1}/{y_0}, {x_1^2}/{y_0}\right],$$

and it is not a free $\mathbb{C}\left[{x_0^2}/{y_0}, {x_1x_0}/{y_0}, {x_1^2}/{y_0}\right]$–module of rank 1.

Lemma 3.8. *[Rob84, Theorem 1.13] Let us assume that $\gcd(\mathbf{a}) = 1$. Let us denote the least common multiplier $\operatorname{lcm}(\mathbf{a})$ by l. Then the following claims hold:*

- *$\mathcal{O}_{\mathbb{P}(\mathbf{a})}(l)$ is an ample line bundle;*
- *$\mathcal{O}_{\mathbb{P}(\mathbf{a})}(dl) \otimes \mathcal{O}_{\mathbb{P}(\mathbf{a})}(c) \cong \mathcal{O}_{\mathbb{P}(\mathbf{a})}(dl + c)$, for every $c, d \in \mathbb{Z}$;*
- *$\mathcal{O}_{\mathbb{P}(\mathbf{a})}(c) \cong \mathcal{H}om(\mathcal{O}_{\mathbb{P}(\mathbf{a})}(dl), \mathcal{O}_{\mathbb{P}(\mathbf{a})}(dl + c))$ for every $c, d \in \mathbb{Z}$;*
- *$R(d) \cong H^0(\mathbb{P}(\mathbf{a}), \mathcal{O}_{\mathbb{P}(\mathbf{a})}(d))$ for every $d \in \mathbb{Z}$.*

The following example shows why one should be careful when approaching the $\mathcal{O}_{\mathbb{P}(\mathbf{a})}(c)$ sheaves.

Example 3.9. *Let us consider $\mathbb{P}(2,3) = \mathrm{Proj}(\mathbb{C}[y_0, y_1])$, by Lemma 3.4 it is isomorphic to $\mathbb{P}^1$. The sheaf $\mathcal{O}_{\mathbb{P}(2,3)}(1)$ is not an ample line bundle. Since $\mathbb{P}(2,3) \cong \mathbb{P}^1$ as varieties, a line bundle is ample if and only if it has at least two-dimensional space of sections. The sheaf $\mathcal{O}_{\mathbb{P}}(1)$ is a line bundle, we see that on the affine cover $\mathbb{P}(2,3) = D(y_0) \cup D(y_1)$. On the affine set $D(y_0)$ it is given by a free module of rank 1,*

$$\frac{y_1}{y_0}\mathbb{C}\left[y_1^2/y_0^3\right],$$

On the affine set $D(y_1)$ it is given by a free module of rank 1 as well,

$$\frac{y_0^2}{y_1}\mathbb{C}\left[y_0^3/y_1^2\right].$$

But $\mathcal{O}_{\mathbb{P}(2,3)}(1)$ has no global sections by Lemma 3.8. We also see that $\mathcal{O}_{\mathbb{P}}(6)$ is very ample bundle, by considering the 6–th Veronesse ring of $\mathbb{C}[y_0, y_1]$. Thus, $\mathcal{O}_{\mathbb{P}(2,3)}(1)^{\otimes 6} \not\cong \mathcal{O}_{\mathbb{P}(2,3)}(6)$, otherwise $\mathcal{O}_{\mathbb{P}(2,3)}(1)$ would be ample.

3.2. M–regular locus

We would like to study the lines with respect to the sheaf $\mathcal{O}_{\mathbb{P}(\mathbf{a})}(1)$, but this sheaf is in general not an ample line bundle; see Example 3.7 and Example 3.9. That is the reason why we consider the M–regular locus, which is only quasi-projective, but $\mathcal{O}_{\mathbb{P}(\mathbf{a})}(1)$ restricted to it is an ample line bundle.

Definition 3.10. *[BR86, Definition 5.4]* ***The regular locus of Mori*** *(M–regular locus) of a weighted projective space $\mathbb{P}(\mathbf{a})$ is an open subscheme $U(\mathbf{a}) \subset \mathbb{P}(\mathbf{a})$, given as the union of the distinguished open subschemes $\bigcup D(Y_I)$, where $Y_I := \Pi_{i\in I} y_i$, for $I \subset \{0, \ldots, n\}$ satisfying*

$$\gcd((a_i)_{i\in I}) = 1.$$

Theorem 3.11. *[Mor75, Theorem 1.7] The M–regular locus $U(\mathbf{a})$ is the largest open subscheme of $\mathbb{P}(\mathbf{a})$ with the following two properties:*

- *for every $d \in \mathbb{Z}$, the sheaf $\mathcal{O}_{U(\mathbf{a})}(d)$ is a line bundle;*
- *for every $d \in \mathbb{Z}$, the natural morphism*

  $$\mathcal{O}_{U(\mathbf{a})}(1)^{\otimes d} \cong \mathcal{O}_{U(\mathbf{a})}(d),$$

  induced by the homomorphism

  $$R(1)^{\otimes d} \to R(d),$$

  is an isomorphism.

Example 3.12. *Let us consider $\mathbb{P}(2,3) = \mathrm{Proj}(\mathbb{C}[y_0, y_1]) = \mathrm{Proj}(R)$. The M–regular locus of $\mathbb{P}(2,3)$ is, by Definition 3.10, equal to $D(y_0y_1)$. The coherent sheaf $\mathcal{O}_{\mathbb{P}(2,3)}(1)$ on the affine set $D(y_0)$ it is given by a free module of rank 1*

$$\frac{y_1}{y_0}\mathbb{C}\left[y_1^2/y_0^3\right].$$

Note that $D(y_0) = \mathrm{Spec}(\mathbb{C}\left[y_1^2/y_0^3\right]) \cong \mathbb{A}^1$, thus every line bundle on $D(y_0)$ is trivial. In particular the following isomorphism holds:

$$\mathcal{O}_{D(y_0)}(1)^{\otimes 6} \cong \mathcal{O}_{D(y_0)}(6),$$

but this isomorphism is not induced by the homomorphism $R(1)^{\otimes 6} \to R(6)$*. The line bundle* $\mathcal{O}_{D(y_0)}(6)$ *is given by the free module*

$$y_0{}^3\mathbb{C}\left[y_1^2/y_0^3\right] \neq \left(\frac{y_1}{y_0}\right)^6 \mathbb{C}\left[y_1^2/y_0^3\right],$$

where the right hand side is the module associated to $\mathcal{O}_{D(y_0)}(1)^{\otimes 6}$*.*

Corollary 3.13. *The coherent sheaf* $\mathcal{O}_{U(\mathbf{a})}(1)$ *is an ample line bundle.*

Proof. By Theorem 3.11, $\mathcal{O}_{U(\mathbf{a})}(1)^{\otimes l} \cong \mathcal{O}_{U(\mathbf{a})}(l)$. By Lemma 3.8 we know that there is a number $l \in \mathbb{N}$ such that $\mathcal{O}_{\mathbb{P}(\mathbf{a})}(l)$ is ample; consequently, $\mathcal{O}_{U(\mathbf{a})}(l)$ is ample. Thus, $\mathcal{O}_{U(\mathbf{a})}(1)$ is an ample bundle as well. $\square$

The following lemma is crucial because taking generic hypersurface of considered weighted projective space, we can always omit fixed finite subset; see Bertini theorem 1.17.

Lemma 3.14. *Let us consider* $\mathbb{P}(\mathbf{a})$ *and suppose that* $\gcd(a_i, a_j) = 1$*, for* $i \neq j$*. Then the* M*-regular locus is a co-finite subset of* $\mathbb{P}(\mathbf{a})$ *and it is contained in the smooth locus of* $\mathbb{P}(\mathbf{a})$*.*

Proof. By Definition 3.10, for $i \neq j$, the distinguished open subsets $D(x_i x_j)$ are contained in the M–regular locus of $\mathbb{P}(\mathbf{a})$. Thus, the complement of the M–regular locus consists of points $[b_0 : \ldots : b_n] \in \mathbb{P}(\mathbf{a})$ with $b_i \neq 0$ for only one index i. There are only finitely many such points. From [IF00, The singular locus 5.15] we see that M–regular locus is contained in the smooth locus. $\square$

One needs to take caution, the M–regular locus is in general not equal to the smooth locus.

Example 3.15. *Let us take* $\mathbb{P}(2,3) \cong \mathbb{P}^1$*. Every point of* $\mathbb{P}(2,3)$ *is smooth, but points* $[0,1]$ *and* $[1,0]$ *are not* M*–regular. On the other hand, the* M*–regular locus of* $\mathbb{P}(1,1)$ *is the whole* $\mathbb{P}(1,1)$*. So taking a* M*–regular locus takes into account more than the class of isomorphism of weighted projective space, but also the graded structure of the underlying ring.*

3.3. Cones on weighted projective spaces

Definition 3.16. *Let us consider the surjective rational map*

$$\pi\colon \mathbb{P}(1,\mathbf{a}) \dashrightarrow \mathbb{P}(\mathbf{a}),$$
$$\pi\colon [c : b_0 : \ldots : b_n] \mapsto [b_0 : \ldots : b_n].$$

The domain of π *is* $\mathrm{Dom}(\pi) = \mathbb{P}(1,\mathbf{a}) \setminus \{[1:0:\ldots:0]\}$*. Let* X *be a closed subvariety of* $\mathbb{P}(\mathbf{a})$*, then* ***the cone over*** X*, denoted by* $\mathrm{Cone}(X)$*, is the closure of* $\pi|_{\mathrm{Dom}(\pi)}{}^{-1}(X)$ *in* $\mathbb{P}(1,\mathbf{a})$*.*

Example 3.17. *Let us take* $\mathbb{P}(\mathbf{a}) = \mathbb{P}(2,3)$*. By Lemma 3.4, we know that* $\mathbb{P}(2,3) \cong \mathbb{P}^1 = \mathbb{P}(1,1)$*. But* $\mathrm{Cone}(\mathbb{P}(2,3)) = \mathbb{P}(1,2,3) \ncong \mathbb{P}^2 = \mathbb{P}(1,1,1) = \mathrm{Cone}(\mathbb{P}(1,1))$*, the former is singular and latter is a smooth variety. So taking a weighted cone takes into account more than the class of isomorphism of our variety, but also the graded structure of the underlying ring.*

We want to show that for a sufficiently general tuple of weights $\mathbf{a}$, the cone over a closed point of the weighted projective space $\mathbb{P}(\mathbf{a})$ is a line. To do so, we first show the following two lemmas.

Lemma 3.18. *Let us consider the action of $\mathbb{G}_m^{n+1}$ on $\mathbb{P}(\mathbf{a})$ given by the formula*

$$\mathbb{G}_m^{n+1} \times \mathbb{P}(\mathbf{a}) \to \mathbb{P}(\mathbf{a}),$$
$$((\lambda_0, \dots, \lambda_n), [b_0 : \dots : b_n]) \mapsto [\lambda_0^{a_0} b_0 : \dots : \lambda_n^{a_n} b_n].$$

Consider X a closed subvariety of $\mathbb{P}(\mathbf{a})$, and fix an element of the torus $\lambda = (\lambda_0, \dots, \lambda_n) \in \mathbb{G}_m^{n+1}$, then we have an isomorphism of cones $\mathrm{Cone}(X) \cong \mathrm{Cone}(\lambda \cdot X)$.

Proof. On $\mathbb{P}(1, \mathbf{a})$ we define an action of $\mathbb{G}_m^{n+1}$ given by the formula

$$\mathbb{G}_m^{n+1} \times \mathbb{P}(1, \mathbf{a}) \to \mathbb{P}(1, \mathbf{a}),$$
$$((\lambda_0, \dots, \lambda_n), [c : b_0 : \dots : b_n]) \mapsto [c : \lambda_0^{a_0} b_0 : \dots : \lambda_n^{a_n} b_n].$$

The action above is a lift of action on $\mathbb{P}(\mathbf{a})$ by π; that is, we get the following commutative diagram:

$$\begin{tikzcd} \mathbb{P}(1, \mathbf{a}) \arrow[r, "\cdot\lambda"] \arrow[d, dashed, "\pi"] & \mathbb{P}(1, \mathbf{a}) \arrow[d, dashed, "\pi"] \\ \mathbb{P}(\mathbf{a}) \arrow[r, "\cdot\lambda"] & \mathbb{P}(\mathbf{a}). \end{tikzcd}$$

As a consequence, we get that

$$\mathrm{Cone}(\lambda \cdot X) = \lambda \cdot \mathrm{Cone}(X) \cong \mathrm{Cone}(X).$$

□

Lemma 3.19. *Fix a tuple of positive integers $(c_0, \dots, c_m)$. Consider the embedding*

$$\phi \colon \mathbb{P}(\mathbf{a}) \to \mathbb{P}(c_0, \dots, c_m, \mathbf{a}),$$
$$\phi \colon [b_0 : \dots : b_n] \mapsto [\overbrace{0 : \dots : 0}^{m+1-\ times} : b_0 : \dots : b_n].$$

For a fixed closed subvariety $X \subset \mathbb{P}(\mathbf{a})$ the cone $\mathrm{Cone}(X) \subset \mathbb{P}(1, \mathbf{a})$ is isomorphic to the cone $\mathrm{Cone}(\phi(X)) \subset \mathbb{P}(1, c_0, \dots, c_m, \mathbf{a})$.

Proof. Let us take the closed embedding ϕ_1 defined analogously to ϕ,

$$\phi_1 \colon \mathbb{P}(1, \mathbf{a}) \to \mathbb{P}(1, c_0, \dots, c_m, \mathbf{a}),$$
$$\phi_1 \colon [c : b_0 : \dots : b_n] \mapsto [c : \overbrace{0 : \dots : 0}^{m+1-\ \text{times}} : b_0 : \dots : b_n].$$

We get the following commutative diagram:

$$\begin{tikzcd} \mathrm{Cone}(X) \arrow[r, hook, "cl"] \arrow[d, dashed, "\pi|_{\mathrm{Cone}(X)}"] & \mathbb{P}(1, \mathbf{a}) \arrow[r, hook, "\phi_1"] \arrow[d, dashed, "\pi"] & \mathbb{P}(1, c_0, \dots, c_m, \mathbf{a}) \arrow[d, dashed, "\pi_{c_0}"] \\ X \arrow[r, hook, "cl"] & \mathbb{P}(\mathbf{a}) \arrow[r, hook, "\phi"] & \mathbb{P}(c_0, \dots, c_m, \mathbf{a}), \end{tikzcd}$$

where π_{c_0} is defined as in Definition 3.16, thus π is a restriction of π_{c_0} to $\mathrm{Cone}(\mathbb{P}(\mathbf{a})) = \mathbb{P}(1, \mathbf{a})$ and the rest follows from the definition of cone. □

Theorem 3.20. *Let us consider $\mathbb{P}(\mathbf{a})$, suppose that for $i \neq j$, $\gcd(a_i, a_j) = 1$, and for all $i \in \{0, \dots, n\}$, $a_i > 1$. Then the cone over any M–regular point is a singular line on $(U(1, \mathbf{a}), \mathcal{O}_{U(1,\mathbf{a})}(1))$. The point $[1 : 0 : \dots : 0] \in \mathbb{P}(1, \mathbf{a})$ is a singular point of the considered cone.*

Proof. Let us choose a M–regular point $p \in U(\mathbf{a})$ and denote the homogeneous prime ideal of p in $\mathbb{P}(\mathbf{a})$ by $J \subset R$. We consider the closed embedding of $\mathbb{P}(\mathbf{a})$ into $\mathbb{P}(1, \mathbf{a})$ and the rational projection π in the other direction, and translate it to algebra,

$$\begin{tikzcd}
\mathbb{P}(\mathbf{a}) \arrow[r, hook, "cl"] & \mathbb{P}(1, \mathbf{a}) \arrow[r, dashed, "\pi"] & \mathbb{P}(\mathbf{a}),
\end{tikzcd}$$

$$\begin{tikzcd}
R[x]/(x) & R[x] \arrow[l, two heads] & R, \arrow[l, hook']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{P}(\mathbf{a}) \cap \operatorname{Cone}(\{p\}) \arrow[r, hook, "cl"] & \operatorname{Cone}(\{p\}) \arrow[r, dashed, "\pi"] & \{p\},
\end{tikzcd}$$

$$\begin{tikzcd}
J \otimes_R R[x]/(x) & J \otimes_R R[x] \arrow[l, two heads] & J. \arrow[l, hook']
\end{tikzcd}$$

With the identification $R \cong R[x]/(x)$, the ideal $J \otimes_R R[x]/(x)$ is equal to J as an ideal of R; that is, we have the following commutative diagram:

$$\begin{tikzcd}
& R[x] \arrow[dr, two heads] & \\
R \arrow[ur, hook] \arrow[rr, "\cong"] & & R[x]/(x) \\
J \arrow[u, hook] \arrow[rr, dashed, "\cong"] & & J \otimes_R R[x]/(x) \arrow[u, hook]
\end{tikzcd}$$

As a consequence $\operatorname{Cone}(\{p\})$ intersects $\mathbb{P}(\mathbf{a})$ transversally in exactly one point, in p. Since the M–regular locus of $\mathbb{P}(1, \mathbf{a})$ is equal to $U(\mathbf{a}) \cup D(x)$, $\operatorname{Cone}(\{p\})$ is contained in $U(1, \mathbf{a})$. By Theorem 1.6 the intersection number $(\operatorname{Cone}(p) \cdot \mathcal{O}_{\operatorname{Cone}(p)}(1))$ is equal to 1.

From now on we check properties dependent only on isomorphism classes, before going further we simplify the statement a little bit. Suppose that $p = [b_0 : \dots : b_n] \in \mathbb{P}(\mathbf{a})$, by Lemma 3.19 we can assume that none of b_i's is 0, then by Lemma 3.18, we can assume that each b_i is 1.

Now we prove that $\operatorname{Cone}(\{p\})$ is a rational curve, because by Lemma 1.10 this implies that $\operatorname{Cone}(\{p\})$ is a line on $(\operatorname{Cone}(\{p\}), \mathcal{O}_{\operatorname{Cone}(\{p\})}(1))$, consequently it is a line on $(U(1, \mathbf{a}), \mathcal{O}_{U(1,\mathbf{a})}(1))$ as well. We consider the distinguished open subset $D(x) \subset \mathbb{P}(1, \mathbf{a})$, the coordinate ring is $\mathbb{C}\left[{}^{y_0}/_{x^{a_0}}, \dots {}^{y_n}/_{x^{a_n}}\right]$ as a non-graded ring is isomorphic to $R = \mathbb{C}[y_0, \dots, y_n]$. With this identification, the prime ideal of $\operatorname{Cone}(\{p\})$ restricted to $D(x)$ is equal to J. Let us consider the following homomorphism:

$$f \colon R/J \to \mathbb{C}[t],$$
$$f \colon y_i \mapsto t^{a_i}.$$

Since J is the homogeneous prime ideal of the point $p = [1 : \dots : 1] \in \mathbb{P}(\mathbf{a})$, the homomorphism f is well-defined. It is a finite, injective morphism into integrally closed domain, thus $\mathbb{C}[t]$ is the integral closure of R/J. Consequently $\mathrm{Cone}(\{p\}) \cap D(x)$ is rational, thus $\mathrm{Cone}(\{p\})$ is as well.

Last thing to prove is that $\mathrm{Cone}(\{p\})$ is a singular curve. Suppose that the weights are ordered, $a_0 < a_1 < \ldots < a_n$. We prove that $q = [1 : 0 : \ldots : 0] \in \mathbb{P}(1, \mathbf{a})$ is a singular point of $\mathrm{Cone}(\{p\})$. We again consider open dense subset $\mathrm{Cone}(\{p\}) \cap D(x)$ and examine it algebraically. With the identification $D(x) \cong \mathrm{Spec}(\mathbb{C}[y_0, \ldots, y_n])$, q is $0 \in \mathrm{Spec}(\mathbb{C}[y_0, \ldots, y_n])$. We show that for every homogeneous polynomial $W \in J$:

$$\frac{\partial W}{\partial y_0}(0) = 0,$$

$$\frac{\partial W}{\partial y_1}(0) = 0.$$

Assume otherwise, that there exists a homogeneous polynomial $W \in J$ with one of those derivatives non–zero at 0. Then it would be of the form (up to a scalar):

$$W = y_i + F_i, \text{ for } i = 0 \text{ or i=1},$$

where F_i is zero or linearly independent of y_i and $\deg(F_i) = a_i$. $F_i \neq 0$ since J is also the ideal of $p = [1 : \ldots : 1] \in \mathbb{P}(\mathbf{a})$. Since $1 < a_0 < a_1 < a_2 < \ldots < a_n$ and $\gcd(a_0, a_1) = 1$, it is impossible for F to be linearly independent of y_i. Thus, at the $\mathrm{Cone}(\{p\})$ has at least two dimensional tangent space at the point q. □

Theorem 3.21. *For any chosen positive integer n, there exists a polarized projective variety (X, L) of dimension n and such that X is covered by a family R of singular lines, together with a closed point $p \in X$ incident to every line $[l]$ from R and p is singular on each $[l] \in R$.*

Proof. Let us consider a weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ with weights a_i pairwise coprime and greater than 1. By Lemma 3.14 the M–regular locus of $\mathbb{P}(a_0, \ldots, a_n)$ is cofinite. By Bertini Theorem 1.17 we can find closed subvariety X' of dimension $n-1$ contained in the M–regular locus. Let us consider $X := \mathrm{Cone}(X')$. It has dimension equal to n. Since X is contained in the M–regular locus, the pair $(X, \mathcal{O}_X(1))$ is a polarized variety. By Theorem 3.20, X as a cone is covered by lines intersecting in point $[1 : 0 : \ldots : 0]$, which is a singular point of each line obtained as a cone over point. □

Theorem 3.22. *For any chosen positive integer M, there exists a polarized projective variety (X, L) of dimension M and such that X is covered by a family R of singular lines, together with a closed point $p \in X$ incident to every line $[l]$ from R and p is a smooth special point of each $[l] \in R$.*

Proof. Let us consider a weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ with weights a_i pairwise coprime and greater than 1. The point $p = [1 : \ldots : 1]$ is M–regular. By Theorem 3.20 the cone $\mathrm{Cone}(\{p\})$ is a line on $(U(\mathbf{a}), \mathcal{O}_{U(\mathbf{a})}(1))$. Let us now consider $X := \overbrace{\mathrm{Cone} \circ \ldots \circ \mathrm{Cone}}^{m+1-\text{ times}}(\{p\})$. Note that X is contained in the M–regular locus of $\mathbb{P}((1)_{j=0}^m, \mathbf{a})$. We define a family of lines on X. First, we consider the following morphism, surjective onto X:

$$\eta\colon \mathbb{P}^{m+1} \to \mathbb{P}((1)_{j=0}^m, \mathbf{a}),$$
$$\eta\colon [s_0 : \ldots : s_m : t] \mapsto [s_0 : \ldots : s_m : t^{a_0} : \ldots : t^{a_n}].$$

It is a well-defined morphism, see [Vak17, Exercise 6.4A].

Let us fix a tuple $\mathbf{c} := (c_0, \dots, c_m)$ of complex numbers, not all equal to 0. Take the embedding

$$\mu_{\mathbf{c}} \colon \mathbb{P}^1 \to \mathbb{P}^m,$$
$$\mu_{\mathbf{c}} \colon [s' : t'] \to [c_0 s' : \dots : c_m s' : t'].$$

The composition $\eta \circ \mu_{\mathbf{c}}$ is the normalization morphism of its image. And its image is up to a linear change of degree 1 coordinates the cone over the point p. Moreover, the images of $\mu_{\mathbf{c}}$'s are not disjoint; they intersect at the point $q' = [0 : \dots : 0 : 1] \in \mathbb{P}^{m+1}$ and $\mathbb{P}^{m+1}$ is not a $\mathbb{P}^1$-bundle over $\mathbb{P}^m$. Thus, we consider the blowup of $\mathbb{P}^{m+1}$ in q'

$$\begin{tikzcd}
\mathbb{P}^{m+1} \times \mathbb{P}^m \arrow[drr] & \mathrm{Bl}_{q'} \mathbb{P}^{m+1} \arrow[l, hook'] \arrow[r, "f"'] \arrow[rr, bend left, "\xi"] \arrow[d, "\pi"] & \mathbb{P}^{m+1} \arrow[r, "\eta"] & \mathbb{P}((1)_{j=0}^m, \mathbf{a}) \\
 & \mathbb{P}^m. & &
\end{tikzcd}$$

Fix an element $[l] = [c_0 : \dots : c_m] \in \mathbb{P}^m$, the morphism $f|_{\pi^{-1}([l])}$ is exactly $\mu_{\mathbf{c}}$. Consequently, the pair (ξ, π) forms a family of lines on $(X, \mathcal{O}_X(1))$. The point which is common for all lines described above is the point $q := \eta(q') = [\overbrace{0 : \dots : 0}^{m+1\text{–times}} : 1 : \dots : 1] \in \mathbb{P}((1)_{j=0}^m, \mathbf{a})$. If we fix a line $l \subset X$, represented by $[l] \in \mathbb{P}^m$, and look at the normalization $\xi|_{\pi^{-1}([l])}$, we see that q is a special smooth point of l. We have obtained $M = (m+1)$-dimensional polarized projective variety $(X, \mathcal{O}_X(1))$ as in the statement. $\square$

Example 3.23. *An example of surface constructed in Theorem 3.22 is the subvariety X of $\mathbb{P}(1,1,2,3)$ cut out by the equation*

$$y_0^3 - y_1^2 = 0.$$

The variety X is a double cone $\mathrm{Cone}(\mathrm{Cone}(p))$ *over the point $p = [1 : 1] \in \mathbb{P}(2,3)$.*

Chapter 4

Families of lines on smooth varieties

Now we show an interesting result to be compared with Corollary 1.13.

Theorem 4.1. *There exists a smooth polarized projective surface (Y, L) admitting a pair of lines intersecting at two points.*

Proof. Let us take the weighted projective space $\mathbb{P}(1,2,3) = \operatorname{Proj}(\mathbb{C}[x, y_0, y_1])$. We consider two distinct lines l_0 and l_1 given by the following equations:

$$l_0 : y_0^3 - y_1^2 = 0,$$
$$l_1 : y_0^3 - y_1^2 + xy_0^2 = 0.$$

The curve l_0 is a line since it is a cone over the point $[1,1] \in \mathbb{P}(2,3)$, see Theorem 3.20. The curve l_1 intersects $\mathbb{P}(2,3)$ at the point $[0,1,1]$ transversally. On the affine open set $D(x)$, l_1 it is a standard nodal cubic curve, thus it is rational. Consequently, by Lemma 1.10, l_1 is a line as well.

Note that the points $[0,1,1]$ and $[1,0,0]$ are contained in both lines. Denote the scheme theoretic union of l_0 and l_1 by X. Let us observe that X is contained in the M–regular locus of $\mathbb{P}(1,2,3)$, thus the embedding dimension of X is at most 2. The dimension of X is 1. There is a finite number of singular points of X. We consider the stratification of X for the cotangent sheaf Ω_X as in Theorem 1.16. Since the embedding dimension of X is limited by 2, it has only two strata, one of dimension 1, the smooth locus, and the singular locus, consisting of points with the embedding dimension equal to 2. We take the closed embedding of $\mathbb{P}(1,2,3)$ into $Z := \mathbb{P}(1,1,2,3)$, as in Lemma 3.19. The dimension of Z is 3, which is larger than $\max_{e \in E}(\dim X(\Omega_X, e) + e)$, where in this case $E = \{1,2\}$. Since Z is projective it admits a very ample bundle L. Let us denote the complement of the M–regular locus of Z by S, by Lemma 3.14 this is a finite set. The tuple (Z, S, X, L) satisfies the assumptions for Bertini theorem 1.17. As a consequence, there exists a smooth surface Y containing X and omitting S. In particular, $\mathcal{O}_Y(1)$ is ample and l_0 and l_1 are lines on $(Y, \mathcal{O}_Y(1))$. □

4.1. Fano variety

The final result is a construction of a smooth Fano variety with the Picard number equal to 1, admitting a positive dimensional family of singular lines. We do this in a few steps. First, we need to investigate the properties of a certain simple example of weighted projective space.

Lemma 4.2. *Consider $Z_m := \mathbb{P}((1)_{j=0}^m, 2, 3)$. The line bundle $\mathcal{O}_{Z_m}(6)$ is very ample and the sheaf $\mathcal{O}_{Z_m}(6+r)$, for $r > 0$ is globally generated.*

Proof. Let us take a monomial $W := x_0^{r_0} \dots x_m^{r_m} y_0^b y_1^c \in \mathbb{C}[x_0, \dots x_m, y_0, y_1] = R[x_0, \dots x_m]$ of degree $\sum r_i + 2b + 3c \geq 12$. We claim that W is divisible by a monomial of degree 6. Assume to the contrary, then $\sum r_i \leq 5$, $b \leq 2$ and $c \leq 1$. Since $\sum r_i + 2b + 3c \geq 12$, we get that $\sum r_i = 5$, $b = 2$ and $c = 1$, but a monomial with such exponents is divisible by a monomial of degree 6 of form $x_i y_0 y_1$.

We consider the Veronese subring $R^{(6)} = \bigoplus_{n \in \mathbb{Z}} R_{6n}$, from the calculation above $R^{(6)}$ is generated in degree 6. Therefore $R^{(6)}$ induces a closed embedding into the projective space; thus, $\mathcal{O}_{\mathbb{P}(\mathbf{a})}(6)$ is very ample.

Let us choose $r > 0$. To prove that $\mathcal{O}_{\mathbb{P}(\mathbf{a})}(6 + r)$ is globally generated it is enough to prove that for the distinguished open sets $D(T)$, $T \in \{x_0, \dots, x_m, y_0, y_1\}$, the following map is surjective:

$$H^0(\mathbb{P}(\mathbf{a}), \mathcal{O}_{\mathbb{P}(\mathbf{a})}(6 + r)) \otimes \mathcal{O}_{D(T)} \to \mathcal{O}_{D(T)}(6 + r).$$

Since $D(T)$ is an affine scheme it is enough to prove this claim on the level of global sections. Take an element $u \in H^0(D(T), \mathcal{O}_{\mathbb{P}(\mathbf{a})}(6 + r))$ which is a monomial in $R[1/T]$, and express it as a fraction

$$u = U/T^s,$$

where $U \in R$, $s > 0$, and $U/T^s = UT^{5s}/T^{6s}$. The monomial UT^{5s} has degree $d = \deg(u) + 6s \deg(T) \geq 12 + r$. So from the calculation above, we can decompose it into a product of two monomials, one of degree 6 and the second of degree $d - 6$. We take the second one and repeat this procedure until we get the following decomposition:

$$UT^{5s} = AB,$$

where $\deg(A) = 6+r$ and $6 | \deg(B)$. So by Lemma 3.8, A defines a global section of $\mathcal{O}_{\mathbb{P}(\mathbf{a})}(6+r)$ and $B/T^{6s} \in H^0(D(T), \mathcal{O}_{\mathbb{P}(\mathbf{a})})$, which proves that $\mathcal{O}_{\mathbb{P}(\mathbf{a})}(6 + r)$ is globally generated. ☐

Lemma 4.3. *Let us fix $m \in \mathbb{N}$. Let X be a closed subvariety of Z_m cut out by the following system of equations:*

$$x_i = 0 \quad \text{for } i > 1,$$
$$y_0^3 - y_1^2 = 0.$$

Let I be the ideal sheaf of X in Z_m. Then the following claims hold:

- *the sheaf $I(12) := I \otimes \mathcal{O}_{Z_m}(6)^{\otimes 2}$ is globally generated;*
- *the line bundle $I(18)|_{Z_m \setminus X} := (I \otimes \mathcal{O}_{Z_m}(6)^{\otimes 3})|_{Z_m \setminus X}$ is very ample.*

Proof. Fix $m \in \mathbb{N}$ and denote $Z := Z_m$. The ideal sheaf I is a coherent sheaf associated to the ideal $J = (x_2, \dots, x_m, y_0^3 - y_1^2) \subset R[x_0, \dots, x_m]$. We have the following exact sequence of graded $R[x_0, \dots, x_m]$-modules:

$$\bigoplus_i R[x_0, \dots, x_m](-1) \oplus R[x_0, \dots, x_m](-6) \to J \to 0.$$

Since the functor of taking the coherent sheaf associated to a graded module is exact [Sta22, Tag 01M7], we get the following exact sequence:

$$\bigoplus_i \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z(-6) \to I \to 0.$$

If we twist it by $\mathcal{O}_Z(6)^{\otimes 2}$, by Lemma 3.8 we get the following exact sequence:

$$\bigoplus_i \mathcal{O}_Z(-1+12) \oplus \mathcal{O}_Z(-6+12) \to I(12) \to 0.$$

By Lemma 4.2 if $r \geq 6$ then $\mathcal{O}_Z(r)$ is globally generated. Thus, the first claim holds.

Let us observe that if $I(12)$ is globally generated then in particular, $I(12)_{|Z\setminus X}$ is a base point free line bundle. The tensor product of a base point free line bundle with a very ample one is very ample [Vak17, Exercise 16.6.C]. Thus, the second claim of the statement is implied by the first one. $\square$

Before proving the final theorem, we state a few more theorems, that enable us to control the canonical bundle of the variety obtained by using Bertini theorem 1.17.

Theorem 4.4. *[BN16, Theorem 1.1] Let (Z, L) be a normal polarized projective variety of dimension at least 4 and L be a very ample line bundle. Let $X \subset Z$ be a closed subscheme of codimension at least 3. Let us fix $s \in \mathbb{N}$, such that the sheaf $I \otimes L^{\otimes(s-1)}$ is globally generated. Then a general section of $I \otimes L^{\otimes s}$ cuts out a closed subscheme Y containing X and satisfying the following property:*

$$\mathrm{Cl}(Z) \cong \mathrm{Cl}(Y).$$

Theorem 4.5. *[Dol82, Theorem 3.3.4] Let $\mathbb{P}((1)_{j=0}^m, \mathbf{a})$ be a well-formed weighted projective space. Let U the M–regular locus of $\mathbb{P}((1)_{j=0}^m, \mathbf{a})$. Then the canonical divisor K_U is isomorphic to $\mathcal{O}_U(-\sum a_i - (m+1))$.*

Theorem 4.6. *[Har77, II Proposition 8.20] Let U be a smooth variety and let Y be a smooth subvariety of codimension 1. Let D be the Cartier divisor associated to Y. Then $K_Y \cong (K_U + D)|_Y$.*

Now we state and prove the final theorem.

Theorem 4.7. *There exists a smooth Fano variety Y with the Picard number equal to one, admitting a positive dimensional family of singular lines with respect to the anti–canonical polarization.*

Proof. Fix $m \in \mathbb{N}$ and denote $Z := Z_m$. Let us take $X \subset Z$ as in Lemma 4.3, note that X is contained in the M–regular locus of Z. The dimension of X is 2, and since it can be embedded in the M–regular locus of $\mathbb{P}(1,1,2,3)$, the embedding dimension of X is at most 3. By S denote the complement of M–regular locus of Z, by Lemma 3.14 the set S is finite. By Lemma 4.3, the tuple $(Z, S, X, \mathcal{O}(6))$ and $s = 3$, provided that $\dim(Z) = m+2 > 5 = \dim(X) + \dim(\mathbb{P}(1,1,2,3))$, satisfy the assumptions of Bertini Theorem 1.17. We also use Theorem 4.4 on the pair $(Z, \mathcal{O}(6))$ for $s = 3$, again by Lemma 4.3 the assumptions are satisfied. As a consequence, we get a smooth variety Y containing X with the property that

$$\mathrm{Cl}(Z) \cong \mathrm{Cl}(Y) \cong \mathrm{Pic}(Y).$$

The Weil divisor class group of weighted projective space $\mathrm{Cl}(Z)$ is isomorphic to $\mathbb{Z}$ ([Rob84, Theorem 2.7]). Thus, $\mathrm{Pic}(Y) \cong \mathbb{Z}$, in particular, the Picard number of Y is 1.

Now we use the adjunction formula 4.6 to compute the canonical divisor of Y

$$K_Y \cong (\mathcal{O}_Z(-5-(m+1)) \otimes \mathcal{O}_Z(18))|_Y \cong \mathcal{O}_Y(12-m).$$

If we put $m = 11$, we get that Y has $\mathrm{Pic}(Y) = \mathbb{Z}$, $-K_Y \cong \mathcal{O}_Y(1)$ and Y contains the surface X which is by Example 3.23 covered family of singular lines with respect to $\mathcal{O}_X(1) \cong K_Y|_X$. $\square$

Chapter 5

Summary

We have studied families of singular lines with respect to an ample polarization through a common point. Such a common point has to be unique for the chosen positive dimensional family, thanks to Bend and Break Theorem 2.8.

Following the article [Keb02, Theorem 2.4], we stated Theorem 2.12, saying that the common point of such a family has to be a special point (Definition 2.10). Then we have constructed such families through a common singular point, in Theorem 3.21, and through a common smooth special point, in Theorem 3.22. The main idea for constructions is to take a weighted projective space and consider cones over points; see Definition 3.16. If the weights of weighted projective space are chosen carefully, then the cone over a general point is a line on the M–regular locus of higher dimensional weighted projective space; see Theorem 3.20. The M–regular locus is not a projective variety, but if we choose weights carefully it is a co-finite subset of weighted projective space; see Lemma 3.14. Therefore, a general hypersurface is contained in it.

In Chapter 4 we constructed an example of a polarized smooth projective surface admitting two lines intersecting at two points. Finally, we have established that there exists a smooth Fano variety with the Picard number equal to 1 admitting a positive dimensional family of lines with respect to the anti–canonical polarization.

Bibliography

[BKK20] Jarosław Buczyński, Michał Kapustka, and Grzegorz Kapustka. Special lines on contact manifolds. to appear in Annales de l'Institut Fourier, arXiv: 1405.7792, 2020.

[BM19] Jarosław Buczyński and Giovanni Moreno. Complex contact manifolds, varieties of minimal rational tangents, and exterior differential systems. In *Geometry of Lagrangian Grassmannians and nonlinear PDEs*, volume 117 of *Banach Center Publ.*, pages 145–176. Polish Acad. Sci. Inst. Math., Warsaw, 2019.

[BN16] John Brevik and Scott Nollet. Grothendieck-Lefschetz theorem with base locus. *Israel J. Math.*, 212(1):107–122, 2016.

[BR86] Mauro Beltrametti and Lorenzo Robbiano. Introduction to the theory of weighted projective spaces. *Exposition. Math.*, 4(2):111–162, 1986.

[Dol82] Igor Dolgachev. Weighted projective varieties. In *Group actions and vector fields (Vancouver, B.C., 1981)*, volume 956 of *Lecture Notes in Math.*, pages 34–71. Springer, Berlin, 1982.

[FH09] Baohua Fu and Jun-Muk Hwang. Minimal rational curves on complete toric manifolds. arXiv: 0912.1638, 2009.

[GW20] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I. Schemes—with examples and exercises.* Springer Studium Mathematik—Master. Springer Spektrum, Wiesbaden, [2020] ©2020. Second edition [of 2675155].

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[HK05] Jun-Muk Hwang and Stefan Kebekus. Geometry of chains of minimal rational curves. *J. Reine Angew. Math.*, 584:173–194, 2005.

[HL21] Jun-Muk Hwang and Qifeng Li. Characterizing symplectic Grassmannians by varieties of minimal rational tangents. *J. Differential Geom.*, 119(2):309–381, 2021.

[HL22] Jun-Muk Hwang and Qifeng Li. Minimal rational curves and 1-flat irreducible G-structures. *J. Geom. Anal.*, 32(6):Paper No. 179, 37, 2022.

[HM04] Jun-Muk Hwang and Ngaiming Mok. Birationality of the tangent map for minimal rational curves. *Asian J. Math.*, 8(1):51–63, 2004.

[Hwa22] Jun-Muk Hwang. Varieties of minimal rational tangents of unbendable rational curves subordinate to contact structures. *J. Math. Soc. Japan*, 74(2):571–590, 2022.

[IF00] A. R. Iano-Fletcher. Working with weighted complete intersections. In *Explicit birational geometry of 3-folds*, volume 281 of *London Math. Soc. Lecture Note Ser.*, pages 101–173. Cambridge Univ. Press, Cambridge, 2000.

[Keb01] Stefan Kebekus. Lines on contact manifolds. *J. Reine Angew. Math.*, 539:167–177, 2001.

[Keb02] Stefan Kebekus. Families of singular rational curves. *J. Algebraic Geom.*, 11(2):245–256, 2002.

[Keb05] Stefan Kebekus. Lines on complex contact manifolds. II. *Compos. Math.*, 141(1):227–252, 2005.

[KK04] Stefan Kebekus and Sándor J. Kovács. Are rational curves determined by tangent vectors? *Ann. Inst. Fourier (Grenoble)*, 54(1):53–79, 2004.

[Kol96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.

[KPSW00] Stefan Kebekus, Thomas Peternell, Andrew J. Sommese, and Jarosław A. Wiśniewski. Projective contact manifolds. *Invent. Math.*, 142(1):1–15, 2000.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[Mor75] Shigefumi Mori. On a generalization of complete intersections. *J. Math. Kyoto Univ.*, 15(3):619–646, 1975.

[PS14] Thomas Peternell and Florian Schrack. Contact Kähler manifolds: symmetries and deformations. In *Algebraic and complex geometry*, volume 71 of *Springer Proc. Math. Stat.*, pages 285–308. Springer, Cham, 2014.

[Rob84] Lorenzo Robbiano. Factorial and almost factorial schemes in weighted projective spaces. In *Complete intersections (Acireale, 1983)*, volume 1092 of *Lecture Notes in Math.*, pages 62–84. Springer, Berlin, 1984.

[Sta22] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`, 2022.

[Vak17] Ravi Vakil. The rising sea — foundations of algebraic geometry., 2017. a book in preparation, November 18, 2017 version, `http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf`.