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Pierścienie Coxa i odwzorowania algebraiczne

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Abstract

Given a morphism $F : X \to Y$ from a $\mathbb{Q}$-factorial Mori Dream Space X to a smooth Mori Dream Space Y and quasicoherent sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, we describe the inverse image of $\mathcal{G}$ by F and the direct image of $\mathcal{F}$ by F in terms of the corresponding modules over the Cox rings graded in the class groups.

Słowa kluczowe

Cox ring, Mori Dream Space, graded module, quasicoherent sheaf, direct image, inverse image

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.1 Matematyka

Klasyfikacja tematyczna

14-xx Algebraic geometry
14Axx Foundations
14A10 Varieties and morphisms
Secondary:
14C20 Divisors, linear systems, invertible sheaves
14L30 Group actions on varieties or schemes (quotients)

Tytuł pracy w języku angielskim

Cox rings and algebraic maps

Contents

Introduction

In this thesis, we will be interested in quasicoherent sheaves on Mori Dream Spaces. In [Cox95b], Cox described the so called **homogeneous coordinate ring** of a toric variety X. It is a $\mathrm{Cl}(X)$-graded ring, where $\mathrm{Cl}(X)$ is the divisor class group of X. The construction was later generalized to more general varieties and is now known as the **Cox ring**. It generalizes the homogeneous coordinate ring of a projective variety but it is intrinsic, i.e. it does not depend on the choice of embedding in any affine or projective variety. For toric varieties, the Cox ring is always a graded polynomial ring. In general, a variety admitting a finitely generated Cox ring is called a **Mori Dream Space** (MDS).

There is a correspondence between quasicoherent sheaves on a MDS X and $\mathrm{Cl}(X)$-graded modules over the Cox ring of X. Suppose X and Y are MDSes with Cox rings R and S, respectively, and $\mathcal{F}$ is a quasicoherent sheaf on X, $\mathcal{G}$ is a quasicoherent sheaf on Y. Assume $F : X \to Y$ is a morphism. Let $\mathcal{F}$ correspond to a $\mathrm{Cl}(X)$-graded R-module M and let $\mathcal{G}$ correspond to a $\mathrm{Cl}(Y)$-graded S-module N. One may ask, to which $\mathrm{Cl}(X)$-graded R-module corresponds the inverse image sheaf $F^*\mathcal{G}$ and to which $\mathrm{Cl}(Y)$-graded S-module corresponds the direct image sheaf $F_*\mathcal{F}$. We answer these two questions in Theorems 3.1 and 3.2, respectively.

This problem is standard for the morphism of affine schemes. There is an equivalence of categories:

$$\{R - modules\} \leftrightarrow \{\text{quasicoherent sheaves on } X = \operatorname{Spec} R\}$$

Let $F : \operatorname{Spec} R \to \operatorname{Spec} S$ be a morphism of affine schemes and let $F^* : S \to R$ be the corresponding homomorphism of coordinate rings. If $\mathcal{F} \in QCoh_X$ corresponds to an R-module M, then the direct image $F_*\mathcal{F}$ corresponds to ${}_SM$ - the S-module obtained from M by the restriction of scalars, i.e. as a group it is M and the structure of an S-module comes from the map $F^* : S \to R$. If $\mathcal{G} \in QCoh_Y$ corresponds to an S-module N, then the inverse image $F^*\mathcal{G}$ corresponds to $N \otimes_S R$ - the module obtained by the extension of scalars. These are all classical. See for example Chapter II.5 in [Har77].

Since the Cox ring R is $\mathrm{Cl}(X)$-graded, $\operatorname{Spec} R$ comes with an action of a quasitorus $H_X = \operatorname{Spec}(k[Cl(X)])$, where k is a fixed algebraically closed base field of characteristic zero. In [Cox95b], Cox proved also, that every normal toric variety X can be obtained as the good quotient for the action of H_X of an invariant open subset of $\operatorname{Spec} R$. Analogous result holds true for any MDS. Under some additional assumptions, every morphism of toric varieties can be lifted to affine varieties associated with their coordinate rings ([Cox95a]). We will use a similar result for MDSes.

We begin the first chapter with introducing the notion of a Cox ring. Then we describe the quotient construction of MDSes. Subsequently, we introduce an affine open cover of MDSes that we will use for local study of quasicoherent sheaves. The chapter ends with the description of the correspondence between $\mathrm{Cl}(X)$-graded R-modules and quasicoherent sheaves on X. In the second chapter we present the proof from [HM16] that, under some additional assumptions, every morphism of Mori Dream Spaces can be lifted to a map of the corresponding Cox rings. In the next chapter we present the proofs of the main results: Theorem 3.1 and Theorem 3.2. In the last chapter we give two examples.

Acknowledgments

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Chapter 1

Cox rings and Mori Dream Spaces

All the spaces that we will consider will be varieties over a fixed algebraically closed field k of characteristic zero. All constructions and propositions in this chapter are from [ADHL15]. We will skip most of the proofs.

1.1. Cox rings

Let X be a normal variety over k with finitely generated class group $\mathrm{Cl}(X)$. We will define the Cox sheaf $\mathcal{R}$ on X and the Cox ring R of X in two different settings using in each of them a different additional assumption. We will first assume additionaly that $\mathrm{Cl}(X)$ has no torsion as the construction is then less technical and therefore gives more insight into the idea.

Construction 1.1 (Construction of the Cox ring. Version 1)**.** *Let X be a normal variety over k with free finitely generated class group. Pick any subgroup $K \subset WDiv(X)$ such that the quotient map $WDiv(X) \to Cl(X)$ induces an isomorphism $c : K \to Cl(X)$. We define the **Cox sheaf** $\mathcal{R}$ as the quasicoherent sheaf of $\mathcal{O}_X$-algebras:*

$$\mathcal{R} = \bigoplus_{E \in K} \mathcal{O}_X(E),$$

where $\mathcal{O}_X(E)$ is the sheaf of $\mathcal{O}_X$-modules associated with the divisor E, i.e. for open $U \subset X$:

$$\Gamma(U, \mathcal{O}_X(E)) = \{f \in k(X)^* | (div(f) + E)|_U \geqslant 0\} \cup \{0\}.$$

$\mathcal{R}$ has a structure of an $\mathcal{O}_X$-algebra where the multiplication is defined by multiplying the homogeneous sections in $k(X)$. Since X is a noetherian topological space, the presheaf direct sum of $\mathcal{O}_X$-modules is a sheaf and hence for every open $U \subset X$:

$$\Gamma(U, \mathcal{R}) = \bigoplus_{E \in K} \Gamma(U, \mathcal{O}_X(E)).$$

*Therefore, we may define the **Cox ring** as*

$$R = \Gamma(X, \mathcal{R}) = \bigoplus_{E \in K} \Gamma(X, \mathcal{O}_X(E)).$$

We will skip the proof that up to isomorphism $\mathcal{R}$ does not depend on the choice of the subgroup K.

We will now present two easy examples of this construction.

Example 1.1. *Let X be a normal affine variety with trivial class group. Then $\mathcal{R} = \mathcal{O}_X$ and the Cox ring of X is the same as the coordinate ring of X.*

Example 1.2. *Let $X = \mathbb{P}^n$. Then Cl(X) $\cong \mathbb{Z}H$ where H is any hyperplane in $\mathbb{P}^n$. We have $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(dH)) \cong k[x_0, ..., x_n]_d$ - the group of homogeneous polynomials of degree d and this isomorphisms combine to give an isomorphism of the Cox ring of $\mathbb{P}^n$ with the homogeneous coordinate ring of $\mathbb{P}^n$. However the Cox ring is intrinsic wherereas the homogeneous coordinate ring of a projecive variety certainly depends on the particular embedding into a projective space. For instance, the image of the 2-uple embedding $\mathbb{P}^1 \to \mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$ but its homogeneous coordinate ring is isomorphic to $k[x, y, z]/(xz - y^2)$ which is not isomorphic to a polynomial ring of two variables.*

Requiring no torsion in $\mathrm{Cl}(X)$ is too restrictive. We will now remove this assumtion, requiring instead that there are no non-constant global invertible functions on X, i.e. $\Gamma(X, \mathcal{O}_X^*) = k^*$. This assumption is needed in the proof that the Cox sheaf (up to isomorphism) does not depend on the choices made in the following construction. Moreover, this additional assumption is easily satisfied, for instance if X is projective or complete.

Construction 1.2 (Construction of the Cox ring. Version 2)**.** *Take any subgroup of the Weil divisor group $K \subset$ WDiv(X) projecting onto Cl(X) under the quotient map WDiv(X) $\to$ Cl(X). Let K^0 be the kernel of $c : K \to$ Cl(X) and let $\chi : K^0 \to k(X)^*$ be a character of K^0 such that div$(\chi(E)) = E$, for every E in K^0. The existence of such a map is clear as K^0 is a free abelian group. Let $\mathcal{S}$ be the sheaf of $\mathcal{O}_X$-algebras associated with K:*

$$\mathcal{S} = \bigoplus_{E \in K} \mathcal{O}_X(E).$$

Let $\mathcal{I}$ be the sheaf of ideals of $\mathcal{S}$ locally generated by the sections $1 - \chi(E)$ where $E \in K^0$. Here 1 is a homogeneous element of degree 0 and $\chi(E)$ is a homogeneuos element of degree $-E$. Let $\pi : \mathcal{S} \to \mathcal{S}/\mathcal{I}$ be the projection map. The ***Cox sheaf*** *is the quotient sheaf $\mathcal{R} = \mathcal{S}/\mathcal{I}$ with the Cl(X)-grading given by:*

$$\mathcal{R} = \bigoplus_{[D] \in Cl(X)} \mathcal{R}_{[D]}, \text{ where } \mathcal{R}_{[D]} = \pi(\bigoplus_{E \in c^{-1}([D])} \mathcal{S}_E).$$

The ***Cox ring*** *of X is then given by:*

$$R = \Gamma(X, \mathcal{R}) = \bigoplus_{[D] \in Cl(X)} \Gamma(X, \mathcal{R}_{[D]}).$$

We will again skip the proof that the Cox sheaf does not depend, up to isomorphism, on the choices of K and χ.

In Lemma 1.4.3.4 in [ADHL15] it is proved that $\forall_{D \in K}\ \pi|_{\mathcal{S}_\mathcal{D}} : \mathcal{S}_\mathcal{D} \to \mathcal{R}_{[D]}$ is an isomorphism. Hence the Cox sheaf in either of the two constructions can be informally thought of as the direct sum of sheaves associated with each divisor class in $\mathrm{Cl}(X)$ with the appropriate $\mathcal{O}_X$-algebra structure.

From now on, we restrict ourselves to considering the Cox rings of varieties fitting into the setting of the second construction. A normal variety X with $\Gamma(X, \mathcal{O}_X^*) = k^*$ and a finitely generated class group and Cox ring will be called a **Mori Dream Space** (MDS). Note that this definition is not standard. For instance in [ADHL15] it is assumed also that X is projective but we do not need this assumption here.

We will later use the following important theorem.

Theorem 1.1 ([ADHL15], 1.5.1.1)**.** *Let X be a normal variety with only constant global invertible functions, finitely generated class group and a Cox sheaf $\mathcal{R}$. Then for every open $U \subset X$, the ring $\Gamma(U, \mathcal{R})$ is integral and normal.*

1.2. The quotient construction

Every MDS can be constructed as the good quotient of an open invariant subset of an affine variety by a quasitorus action. In this section we will recall this construction following Construction 1.6.3.1 in [ADHL15].

Construction 1.3. *Let X be a normal variety over k with finitely generated class group $Cl(X)$ and no non-constant global invertible functions. Assume that the Cox ring R is finitely generated. From Theorem 1.1 it follows that $\mathcal{R}$ is a sheaf of reduced $\mathcal{O}_X$-algebras and from Proposition 1.6.1.1 in [ADHL15] it follows that it is locally of finite type. Hence the relative spectrum of the Cox sheaf* ***Spec***$(\mathcal{R})$ *is a variety (see Exercise 5.17 in the second chapter of [Har77] for the definition and basic properties of a relative spectrum). We will denote it by $\widehat{X}$ and call it the* ***characteristic space*** *of X. It comes with an action of the quasitorus H_X associated with $Cl(X)$ (i.e. $H_X = \mathrm{Spec}(k[Cl(X)])$) and a good quotient for this action $\pi_X : \widehat{X} \to X$. Let $\overline{X}$ be the spectrum of the Cox ring. Since R is integral and finitely generated as a k-algebra, $\overline{X}$ is a variety. We will call it the* ***total coordinate space*** *of X. Since R is $Cl(X)$-graded, the total coordinate space comes with an action of H_X. There is an equivariant open embedding $i_X : \widehat{X} \to \overline{X}$ with the complement of the image of codimension at least two. The homogeneous ideal of R defining the complement $\overline{X} \setminus \widehat{X}$ will be denoted by $\mathcal{J}_{irr}(X)$ and will be called the* ***irrelevant ideal*** *of X.*

1.3. The [D]-divisor and the [D]-localization

In the study of local behaviour of sheaves on MDSes we will use the notions of a $[D]$-divisor and a $[D]$-localization from Section 1.5.2 in [ADHL15]. In the notation from Construction 1.2, take any divisor $D \in K$ and a non-zero $f \in \Gamma(X, \mathcal{R}_{[D]})$. Then by Lemma 1.4.3.3 in [ADHL15] there exists a unique $\tilde{f} \in \Gamma(X, \mathcal{S}_D)$ such that $\pi(\tilde{f}) = f$. We define the [**D**]**-divisor** of f as $div_{[D]}(f) = div(\tilde{f}) + D$. Note that this divisor is always effective. The $[D]$-divisor does not depend on the choice of a representative $D \in K$ and the choices made in Construction 1.2. It follows easily from the definition that for $0 \neq f \in \Gamma(X, \mathcal{R}_{[D_1]})$ and $0 \neq g \in \Gamma(X, \mathcal{R}_{[D_2]})$ we have:

$$div_{[D_1]+[D_2]}(fg) = div_{[D_1]}(f) + div_{[D_2]}(g). \tag{1.1}$$

For $0 \neq f \in \Gamma(X, \mathcal{R}_{[D]})$ we define the [**D**]**-localization** of X by f as the complement of the support of the [D]-divisor of f, that is:

$$X_{[D],f} = X \setminus Supp(div_{[D]}(f)).$$

We will later need the following lemma.

Lemma 1.1. *Suppose X is a MDS with the Cox ring R. Then for all divisor classes $[D], [E] \in Cl(X)$ and for all non-zero $f \in R_{[D]}$ and $g \in R_{[E]}$ we have $X_{[D],f} \cap X_{[E],g} = X_{[D]+[E],fg}$.*

Proof. Let f, g be as in the statement. We have:

$X_{[D],f} \cap X_{[E],g} = (X \backslash Supp(div_{[D]}f)) \cap (X \backslash Supp(div_{[E]}g)) = X \backslash (Supp(div_{[D]}f) \cup Supp(div_{[E]}g)).$

From equation (1.1) it follows that $div_{[D]}f + div_{[E]}g = div_{[D]+[E]}fg$. Since both $div_{[D]}f$ and $div_{[E]}g$ are effective it follows that:

$$Supp(div_{[D]+[E]}fg) = Supp(div_{[D]}f) \cup Supp(div_{[E]}g)$$

which implies that $X_{[D],f} \cap X_{[E],g} = X \setminus Supp(div_{[D]+[E]}fg) = X_{[D]+[E],fg}$. □

The importance of the $[D]$-localizations comes from the following proposition.

Proposition 1.1 ([ADHL15], 1.6.3.3)**.** *Let X be a Mori Dream Space with the Cox ring R. Let $0 \neq f \in R_{[D]}$. If $X_{[D],f}$ is affine then $\pi_X^{-1}(X_{[D],f}) = \widehat{X}_f = \overline{X}_f$. In particular, X can be covered by open affine sets of the form $X_{[D],f}$.*

Observe that if $X_{[D],f}$ is affine then it is isomorphic to $\operatorname{Spec} R_{(f)}$ as it is a good quotient of the affine set $\widehat{X}_f = \overline{X}_f \cong \operatorname{Spec} R_f$.

1.4. Quasicoherent sheaves on Mori Dream Spaces

As in the case of toric varieties, there is a correspondence between quasicoherent sheaves on MDSes and modules over their Cox rings graded in the class group. Moreover, coherent sheaves correspond to the finitely generated modules. We will need not only the statement of the following proposition but also some facts that appear in the proof. Therefore we will sketch it here.

Proposition 1.2 ([ADHL15], 4.2.1.11)**.** *Let X be a Mori Dream Space with the Cox ring R. There is a functor:*

$$\{Cl(X)\text{-graded } R\text{-modules}\} \to QCoh_X \text{ given by } M \mapsto (\pi_{X*} i_X{}^* \mathbb{M})_0$$

where $\mathbb{M}$ is the quasicoherent $\mathcal{O}_{\overline{X}}$-module associated with the R-module M. This functor is exact and essentialy surjective. Moreover, it induces an exact and essentialy surjective functor:

$$\{\text{finitely generated } Cl(X)\text{-graded } R\text{-modules}\} \to Coh_X.$$

Proof. (Following the proof in [ADHL15])
1) **Exactness** Take an exact sequence of $\mathrm{Cl}(X)$-graded R-modules:

$$0 \to L \to M \to N \to 0.$$

The functor $M \mapsto \mathbb{M}$ is exact ([Har77], Proposition II.5.2). Since i_X is an open immersion and exactness of a sequence of sheaves can be checked on the level of stalks, $i_X{}^*$ is exact. Take an element $X_{[D],f}$ of an affine open cover of X and an exact sequence of quasicoherent $\mathcal{O}_{\widehat{X}}$-modules:

$$0 \to i_X{}^*\mathbb{L} \to i_X{}^*\mathbb{M} \to i_X{}^*\mathbb{N} \to 0.$$

Restricting it to $\pi_X^{-1}(X_{[D],f}) \cong \operatorname{Spec} R_f$ we obtain a corresponding exact sequence of R_f-modules (we use the equivalence between the category of quasicoherent sheaves on $\operatorname{Spec} R_f$ and the category of R_f-modules):

$$0 \to L_f \to M_f \to N_f \to 0.$$

Applying the direct image functor by the morphism $\pi_X|_{\operatorname{Spec} R_f} : \operatorname{Spec} R_f \to \operatorname{Spec} R_{(f)}$, we obtain the same exact sequence but now treated as $R_{(f)}$-modules. Since the maps of modules we began with were graded, after taking the degree zero part we obtain an exact sequence of $R_{(f)}$-modules:

$$0 \to L_{(f)} \to M_{(f)} \to N_{(f)} \to 0.$$

which proves the exactness of the functor given in the statement of the propositon.

2)**Restricted functor** From Proposition 1.1 it follows that for an element of the affine open cover of the form $X_{[D],f} \cong \operatorname{Spec} R_{(f)}$, $\Gamma(X_{[D],f}, (\pi_{X*} {i_X}^* \mathbb{M})_0)$ is isomorphic to $M_{(f)}$. If M is a finitely generated R-module, this module is a finitely generated $R_{(f)}$-module. Hence the stated functor sends finitely generated modules to coherent sheaves.

3)**Essential surjectivity** Let $\mathcal{N}$ be a quasicoherent sheaf on X. Since $\widehat{X}$ is the relative spectrum of the Cox sheaf $\mathcal{R}$, we have $\pi_{X*}\mathcal{O}_{\widehat{X}} = \mathcal{R}$. Let $\mathcal{N}' = \mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{N}$. It is a sheaf of $\mathrm{Cl}(X)$-graded $\mathcal{R}$-modules with:

$$\mathcal{N}'_0 = \mathcal{N}. \tag{1.2}$$

We also have:

$$\pi_{X*}{\pi_X}^*\mathcal{N} = \mathcal{N}'. \tag{1.3}$$

This can be checked locally on the affine open cover of X. Take affine $X_{[D],f}$ and let $\mathcal{N}|_{X_{[D],f}}$ be isomorphic to the sheaf associated with the $R_{(f)}$-module M. Then $\pi_{X*}{\pi_X}^*(\mathcal{N}|_{X_{[D],f}})$ is isomorphic to the sheaf associated with the $R_{(f)}$-module obtained by first taking the extension of scalars of M and then taking the restriction of scalars. That is, it is isomorphic to the sheaf associated with the $R_{(f)}$-module $M \otimes_{R_{(f)}} R_f$. This sheaf is isomorphic to $(\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{R})|_{X_{[D],f}}$. This proves equation (1.3). Let $\mathcal{M} = {\pi_X}^*\mathcal{N}$ and set $M = \Gamma(\widehat{X}, \mathcal{M})$. $\overline{X}$ is normal and $\overline{X} \setminus \widehat{X}$ is of codimension at least two. Hence restricting functions gives an isomorphism $\Gamma(\overline{X}, \mathcal{O}_{\overline{X}}) \to \Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}})$. Therefore M is a $\mathrm{Cl}(X)$-graded R-module and

$${i_X}^*\mathbb{M} = {\pi_X}^*\mathcal{N}.$$

Hence by the above three equations, we have $\mathcal{N} = (\pi_{X*}{i_X}^*\mathbb{M})_0$. This proves the essential surjectivity.

4) **Essential surjectivity of the restricted functor** In the notation from part 3): if $\mathcal{N}$ is coherent, then $\mathcal{M}$ is coherent. Cover $\widehat{X}$ by finitely many open affine subsets $\widehat{X}_{f_1}, ..., \widehat{X}_{f_k}$ where $f_1, ..., f_k$ are homogeneos elements of R (it can be done by Proposition 1.1). Let $g_{i,1}, ...g_{i,r(i)}$ be homogeneous sections of $\mathbb{M} = i_{X*}\mathcal{M}$ over $\widehat{X}_{f_i}$ that generate $\Gamma(\widehat{X}_{f_i}, \mathbb{M})$ as an R_{f_i}-module, $i = 1, .., k$. By Lemma II.5.3 in [Har77] for every i there exist $n_{i,1}, ...n_{i,r(i)}$ such that $(f_i)^{n_{i,s}} g_{i,s}$ extends to global sections of $\mathbb{M}$, $s \in \{1, ..., r(i)\}$. Let M' be the $\mathrm{Cl}(X)$-graded submodule of M generated by $(f_i)^{n_{i,s}} g_{i,s}$ where $i \in \{1, .., k\}$ and $s \in \{1, ..., r(i)\}$. Then $\mathcal{M} = {i_X}^*\mathbb{M}'$. This proves the essential surjectivity of the restricted functor. □

By $\widetilde{M}$ we will denote the quasicoherent sheaf on X corresponding to the $Cl(X)$-graded R-module M via the functor from the previous proposition. A graded R-module M also defines a quasicoherent sheaf on the total coordinate space $\overline{X}$. To make it clear which sheaf we are considering, we will adopt a non-standard convention of calling the latter $\mathbb{M}$.

We collect for further reference some facts that follow immediately from the proof of the above proposition.

Proposition 1.3. *Let X be a Mori Dream Space with the Cox sheaf $\mathcal{R}$ and the Cox ring R. Let $\mathcal{F}$ be a quasicoherent sheaf on X. Denote by M, the $Cl(X)$-graded R module $\Gamma(\widehat{X}, {i_X}^*\mathcal{F})$. Then the following statements hold true:*
(A) ${\pi_X}^*\widetilde{M} \cong {i_X}^*\mathbb{M}$
(B) ${\pi_X}_*{\pi_X}^*\mathcal{F} \cong \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{R}$.

Chapter 2

Lifting morphisms of Mori Dream Spaces to their Cox rings

The main tool that will be used in the proof of Theorems 3.1 and 3.2 is the following result from [HM16].

Theorem 2.1. *Let X and Y be Mori Dream Spaces with the Cox rings R and S, respectively. Assume that X is $\mathbb{Q}$-factorial and Y is smooth. Let $F : X \to Y$ be a morphism. Then there exists a morphism $\overline{F} : \overline{X} \to \overline{Y}$ such that:*
(1) the induced map on coordinate rings $\overline{F}^* : S \to R$ is a graded homomorphism, and
(2) the following diagram is commutative:

$$\begin{tikzcd}
\overline{X} \arrow[r, dotted, "\overline{F}"] & \overline{Y} \\
\widehat{X} \arrow[u, hook, "i_X"] \arrow[r, dotted, "\widehat{F}"] \arrow[d, two heads, "\pi_X"] & \widehat{Y} \arrow[u, hook, "i_Y"] \arrow[d, two heads, "\pi_Y"] \\
X \arrow[r, "F"] & Y
\end{tikzcd}$$

where $\widehat{F}$ is the restriction of $\overline{F}$ to $\widehat{X}$.

In the proof of the above theorem we will use the following two lemmas.

Lemma 2.1 ([ADHL15] 1.6.3.5)**.** *Let Z be a $\mathbb{Q}$-factorial MDS with the Cox ring T and let $0 \neq f \in T_{[D]}$. Then $Z_{[D],f}$ is affine if and only if f belongs to the irrelevant ideal $\mathcal{J}_{irr}(Z)$ of Z.*

Lemma 2.2. *Suppose Z is a $\mathbb{Q}$-factorial MDS with a Cox ring T. Then the affine sets of the form $Z_{[D],f}$ with $[D] \in Cl(Z)$ and $0 \neq f \in T_{[D]}$ form a basis for the topology of Z.*

Proof. Let $W \subset Z$ be any open subset of Z. Then $(\pi_Z)^{-1}(W)$ is an open invariant subset of $\widehat{Z}$. In particular there is a homogeneous ideal $I \lhd T$ such that $(\pi_Z)^{-1}(W)$ is the complement in $\overline{Z}$ of $V(I)$. Let $f_1, ..., f_n$ be the homogeneous generators of I of degrees $[D_1], ..., [D_n]$, respectively. Then $(\pi_Z)^{-1}(W) = \bigcup\limits_{i=1}^{n} \overline{Z}_{f_i}$. Since $(\pi_Z)^{-1}(W) \subset \widehat{Z}$ we have $\overline{Z}_{f_i} = \widehat{Z}_{f_i}$. Since Z is $\mathbb{Q}$-factorial and $\widehat{Z}_{f_i}$ affine, we have by Lemma 2.1 that $Z_{[D_i],f_i}$ is affine. By Proposition 1.1 $\pi_Z(\widehat{Z}_{f_i}) = Z_{[D_i],f_i}$. Therefore $W = \bigcup\limits_{i=1}^{n} Z_{[D_i],f_i}$. □

Proof of Theorem 2.1 following [HM16] Lemma 3.2. Since we are now working with two MDSes, we will change the notation from Construction 1.2 by introducing subscripts X, Y to the symbols $\mathcal{S}$, $\mathcal{I}$, $\mathcal{R}$ and χ.

Let $C_Y = \bigoplus_{i=1}^{l} \mathbb{Z}D_i$ be a subgroup of the Cartier divisor group $CDiv(Y)$ of Y projecting onto $Pic(Y) = Cl(Y)$. Let C_Y^0 be the kernel of this restricted projection map. That is we have the following exact sequence:

$$C_Y^0 \hookrightarrow C_Y \twoheadrightarrow Pic(Y).$$

We can assume that $im(F)$ is not contained in any $SuppD_i$ for $i \in \{1, ...l\}$. Indeed, suppose that $im(F) \subset SuppD_{i_0}$ for some $i_0 \in \{1, ..., l\}$. Pick any point $y \in im(F)$. Since D_{i_0} is a Cartier divisor, there exists an open neighborhood U of y such that $D_{i_0}|_U = div(s)|_U$ for a rational function s. Define $\widetilde{D_{i_0}}$ to be $D_{i_0} - div(s)$. Then $U \cap Supp(\widetilde{D_{i_0}}) = \emptyset$. In particular $im(F) \not\subset Supp(\widetilde{D_{i_0}})$. Therefore by changing D_i's to linearly equivalent divisors we may assume that $im(F) \not\subset Supp(D_i)$ for all i.

Therefore we have a well defined map $F^* : C_Y \to WDiv(X)$ given by pulling back D_i's via F. Let K_X be a subgroup of $WDiv(X)$ containing $F^*(C_Y)$ and such that it projects onto $Cl(X)$. Since pullback of a principal divisor is principal we have the following commutative diagram:

$$\begin{tikzcd}
C_Y^0 \arrow[r, hook] \arrow[d] & C_Y \arrow[r, two heads] \arrow[d, "F^*"] & Pic(Y) \arrow[d, "F^*"] \\
K_X^0 \arrow[r, hook] & K_X \arrow[r, two heads] & Cl(X)
\end{tikzcd}$$

Since $im(F)$ is irreducible and is not contained in any $Supp(D_i)$ for $i \in \{1, ..., l\}$, we have $im(F) \not\subset SuppD$ for every divisor $D \in C_Y$. Take any $D \in C_Y$ and a rational function $f \in \Gamma(Y, \mathcal{O}_Y(D))$. Then f is regular on a nonempty open subset of $im(F)$. Hence its pullback $f \circ F$ exists. Moreover, if $f \circ F \neq 0$, then $div(f \circ F) + F^*D \geqslant 0$. Hence we have a homomorphism given by pulling back rational functions:

$$\Gamma(Y, \mathcal{S}_Y) = \bigoplus_{D \in C_Y} \Gamma(Y, \mathcal{O}_Y(D)) \xrightarrow{\alpha} \bigoplus_{E \in K_X} \Gamma(X, \mathcal{O}_X(E)) = \Gamma(X, \mathcal{S}_X).$$

Moreover, it is graded with respect to $F^* : C_Y \to K_X$. We claim that this homomorphism induces a graded homomorphism of the Cox rings. Let χ_X be any homomorphism $K_X^0 \to k(X)^*$ such that $div(\chi_X(E)) = E$ for every $E \in K_X^0$. Let χ_Y be any homomorphism $C_Y^0 \to k(Y)^*$ such that $div(\chi_Y(D)) = D$ for every $D \in C_Y^0$. We will modify χ_Y so that $\alpha(\chi_Y(D)) = \chi_X(F^*D)$. Then it will follow that:

$$\alpha(\Gamma(Y, \mathcal{I}_Y)) \subset \Gamma(X, \mathcal{I}_X).$$

Recall that $\Gamma(Y, \mathcal{I}_Y)$ is a homogeneous ideal of $\Gamma(Y, \mathcal{S}_Y)$ generated by $1 - \chi_Y(D)$ where $D \in C_Y^0$. Here 1 is of degree 0 and $\chi_Y(D)$ is of degree $-D$. Fix a basis D_1, .., D_s of C_Y^0. Then by Remark 1.4.3.2 in [ADHL15], $\Gamma(Y, \mathcal{I}_Y)$ is generated by $1 - \chi_Y(D_i)$ where $1 \leqslant i \leqslant s$. Hence it is enough to show that $\alpha(1 - \chi_Y(D_i)) \in \Gamma(X, \mathcal{I}_X)$ for $1 \leqslant i \leqslant s$. Observe that $div(\alpha(\chi_Y(D_i))) = F^*(div(\chi_Y(D_i))) = F^*(D_i) = div(\chi_X(F^*(D_i)))$. Since $\Gamma(X, \mathcal{O}_X^*) = k^*$ it follows that there exists $a_i \in k^*$ such that $\alpha(a_i \chi_X(D_i)) = \chi_X(F^*(D_i))$. Since D_i's form a basis of C_Y^0 we can modify χ_Y by requesting $\chi'_Y(D_i) = a_i \chi_Y(D_i)$ for $1 \leqslant i \leqslant s$. Then

$\alpha(1-\chi'_Y(D_i)) = 1-\chi_X(F^*D_i)$. Hence we have a well defined homomorphism, graded with respect to $F^* : C_Y \to K_X$:

$$\Gamma(Y,\mathcal{S}_Y)/\Gamma(Y,\mathcal{I}_Y) \xrightarrow{\alpha} \Gamma(X,\mathcal{S}_X)/\Gamma(X,\mathcal{I}_X)$$

By Lemma 1.4.3.5 in [ADHL15] we have an isomorphism:

$$\Gamma(X,\mathcal{S}_X)/\Gamma(X,\mathcal{I}_X) \cong \Gamma(X,\mathcal{S}_X/\mathcal{I}_X) = \Gamma(X,\mathcal{R}_X) = R$$

We have a similar isomorphism for the Cox ring of Y. Recall that:

$$(\mathcal{R}_X)_{[E]} = \pi(\bigoplus_{F\in c^{-1}([E])} \mathcal{S}_F)$$

Here π is the projection $\mathcal{S}_X \to \mathcal{R}_X$, and c is the quotient map $c : K_X \to Cl(X)$. Therefore α induces a graded map of the Cox rings (coming from the data C_Y, χ'_Y and K_X, χ_X, respectively):

$$S = \bigoplus_{D\in Cl(Y)} \Gamma(Y,(\mathcal{R}_Y)_{[D]}) \xrightarrow{\overline{F}^*} \bigoplus_{E\in Cl(X)} \Gamma(X,(\mathcal{R}_X)_{[E]}) = R.$$

In the construction of the morphism of the Cox rings we have made a few choices. By composing that morphism with isomorphisms of the Cox rings coming from different choices, we may obtain analogous morphisms for all other choices.

We will now prove that $\overline{F} : \overline{X} \to \overline{Y}$ induced by $\overline{F}^* : S \to R$ restricts to a map $\widehat{F} : \widehat{X} \to \widehat{Y}$. By Proposition 1.1 we may cover Y by open affine sets of the form $Y_{[D],f}$. By Lemma 2.2 we may cover the preimage of $Y_{[D],f}$ by affine open sets of the form $X_{[E],g}$. Restricting F to $X_{[E],g} \to Y_{[D],f}$ we obtain a diagram where vertical arrows are good quotients and all spaces are affine:

$$\begin{tikzcd} \widehat{X}_g \arrow[d, two heads, "\pi_X"] & \widehat{Y}_f \arrow[d, two heads, "\pi_Y"] \\ X_{[E],g} \arrow[r, "F"] & Y_{[D],f} \end{tikzcd}$$

We have the corresponding diagram of ring homomorphisms:

$$\begin{tikzcd} R_g & S_f \arrow[l, dotted] \\ R_{(g)} \arrow[u, hook] & S_{(f)} \arrow[l, "F^*"'] \arrow[u, hook] \end{tikzcd}$$

By the choice of $X_{[E],g}$, we have $F^{-1}(Supp(div_{[D]}f)) \subset Supp(div_{[E]}g)$. Observe that:

$$Supp(div_{[F^*D]}f\circ F) = Supp(F^*(div_{[D]}f)) \subset F^{-1}(Supp(div_{[D]}f)) \subset Supp(div_{[E]}g).$$

Let $\tilde{g}$ be the unique element of $\Gamma(X,(\mathcal{S}_X)_{(F^*D)})$ such that $\pi(\tilde{g}) = f\circ F \in R_{[F^*D]}$. It exists by Lemma 1.4.3.3 in [ADHL15]. Then $\frac{1}{\tilde{g}} \in \Gamma(X_{[E],g},\mathcal{O}_X(-F^*D))$ since $(div(\frac{1}{\tilde{g}}) - F^*D)|_{X_{[E],g}} = (-div_{[F^*D]}f\circ F)|_{X_{[E],g}} = 0$. Hence $\overline{F}^*(f)$ is an invertible element of $\Gamma(X_{[E],g},\mathcal{R}_X) = R_g$. Hence $\overline{F}^*$ induces the dotted arrow. The diagram is commutative since both horizontal arrows are given by pulling back functions via F. Since all $X_{[E],g}$ for different f's cover X, we have that $\widehat{X}_g$'s cover $\widehat{X}$. This finishes the proof that $\overline{F}$ restricts to $\widehat{X} \to \widehat{Y}$ and the diagram from the statement of the theorem commutes. $\square$

Remark 2.1. *In the proofs of Theorems 3.1 and 3.2 we will not explicitly use the assumptions that X is $\mathbb{Q}$-factorial and Y is smooth. We require only the existence of a lifting as in Theorem 2.1 and $\mathbb{Q}$-factoriality of Y.*

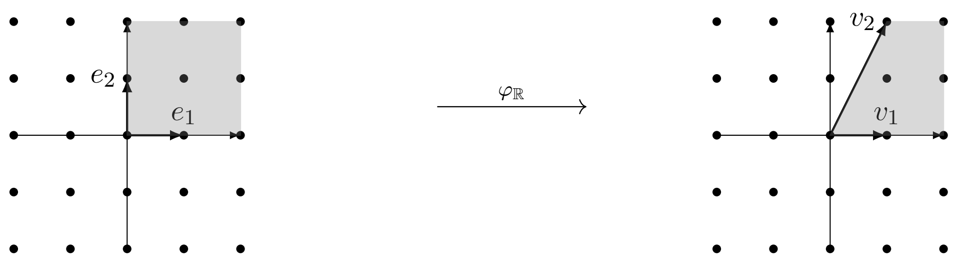
Remark 2.2. *There are similar results on existence of a lift of a map of MDSes to Cox rings. In [Cox95a] for a morphism from a complete toric variety into a smooth toric variety without torus factors. In [BB13] there are considered rational maps of toric varieties. In this case the lift to the total coordinate spaces is a multi-valued function. It is generalized further in [BK16] by considering rational maps of MDSes. See also [HM16].*

In general, for a morphism of $\mathbb{Q}$-factorial MDSes $X \to Y$, there does not exist a lift to the total coordinate spaces as in Theorem 2.1. The following simple example ilustrates this.

Example 2.1. *We will consider a map $\phi : X \to Y$ of affine toric varieties. Y will not be smooth but both X and Y will be $\mathbb{Q}$-factorial Mori Dream Spaces. We will show that there is no lift of ϕ to the Cox rings of X and Y giving a commutative diagram as in Theorem 2.1. Let $\varphi : \mathbb{Z}^2 \to \mathbb{Z}^2$ be given by a matrix:*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

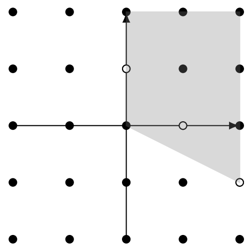
Let σ_1 be the cone in $\mathbb{R}^2$ given by $\sigma_1 = cone(e_1, e_2)$ where $e_1 = (1,0)$ and $e_2 = (0,1)$. Let X be the affine toric variety associated with σ_1. That is $X \cong \mathbb{A}^2$. Let σ_2 be the cone in $\mathbb{R}^2$ given by $\sigma_2 = cone(v_1, v_2)$ where $v_1 = (1,0)$ and $v_2 = (1,2)$. We will denote by Y the affine toric variety associated with σ_2. Let $\varphi_{\mathbb{R}}$ be the tensored map $\varphi \otimes_{\mathbb{Z}} id_{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}^2$. That data is illustrated by the diagram:



Observe that $\varphi_{\mathbb{R}}(\sigma_1) \subset \sigma_2$. Hence φ induces a map $\phi : X \to Y$.

We will recall the Cox ring construction for toric varieties from [Cox95b]. Let X be a toric variety associated with a fan Σ in a vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ obtained from a lattice N. Assume further that the ray generators of Σ generate $N_{\mathbb{R}}$ as a vector space. Then the Cox ring of X is $\mathbb{C}[x_{\rho_1}, ..., x_{\rho_n}]$ where $\rho_1, \dots, \rho_n$ are the rays of Σ. We set $deg(\rho_i) = [D_i]$, where D_i is the divisor associated with ρ_i, i.e. the closure of the orbit coresponding to ρ_i.

We want to calculate the affine coordinate ring of Y and describe the map ϕ^* on the level of rings. The affine coordinate ring of Y is given by $\mathbb{C}[\sigma_2^\vee \cap M]$ where M is a lattice dual to N. Let $x_1 = \chi^{e_1^*}$ and $x_2 = \chi^{e_2^*}$. We have the following picture of the dual cone $\sigma_2^\vee$:



The white dots represent generators of the affine semigroup $\sigma_2^\vee \cap M$*. Hence as a subring of* $\mathbb{C}[x_1^\pm, x_2^\pm]$*,* $\mathbb{C}[\sigma_2^\vee \cap M]$ *is given by* $\mathbb{C}[x_1^2 x_2^{-1}, x_1, x_2]$*. We will describe the map:*

$$\phi^* : \mathbb{C}[x_1^2 x_2^{-1}, x_1, x_2] \to \mathbb{C}[x_1, x_2].$$

It is given by the matrix of the map dual to φ*. In particular it maps* x_2 *to* x_2*.*

Let $\eta : \mathbb{C}[x_1^2 x_2^{-1}, x_1, x_2] \to \mathbb{C}[y_1^2, y_1 y_2, y_2^2]$ *be the isomorphism given by* $\eta(x_1^2 x_2^{-1}) = y_1^2$*,* $\eta(x_1) = y_1 y_2$ *and* $\eta(x_2) = y_2^2$*. Via this isomorphism we may identify* Y *with* $\operatorname{Spec}(\mathbb{C}[y_1, y_2]^{C_2})$ *where* $C_2 = \langle \epsilon \rangle$ *is a cyclic group of order two acting on* $\mathbb{A}^2$ *via* $\epsilon(a, b) = (-a, -b)$*. This is precisely the quotient construction of* Y *as in Construction 1.3. Suppose there exists a lift* $\overline{\phi}$ *of* ϕ *to the Cox rings of* X *and* Y*. Then we have the following diagram:*

$$\begin{tikzcd}
\mathbb{C}[x_1, x_2] & \mathbb{C}[y_1, y_2] \arrow[l, dotted, "\overline{\phi}^*"'] \\
\mathbb{C}[x_1, x_2] \arrow[u, "id"] & \mathbb{C}[y_1^2, y_1 y_2, y_2^2] \arrow[l, "\phi^* \circ \eta^{-1}"] \arrow[u, hook]
\end{tikzcd}$$

As $\phi^* \circ \eta^{-1}(y_2^2) = x_2$ *we have* $\overline{\phi}^*(y_2^2) = x_2$*. This gives a contradiction.*

Chapter 3

Main results

In this chapter, we are in the following setup. Let X, Y be Mori Dream Spaces. We assume that Y is $\mathbb{Q}$-factorial. The Cox sheaves of X and Y are $\mathcal{R}$ and $\mathcal{S}$, respectively. The Cox rings of X and Y are R and S, respectively. We have a morphism $F : X \to Y$. We assume that there exists a lift $\overline{F}$ of F as in Theorem 2.1. The homomorphism $Cl(Y) \to Cl(X)$ that is a part of the data of the graded homomorphism $\overline{F}^* : S \to R$ will be denoted by φ.

Let Z be a MDS with the Cox ring T. The following simple example shows that non-isomorphic $Cl(Z)$-graded T-modules can determine isomorphic sheaves on Z.

Example 3.1. *Let $Z = \mathbb{P}^1$. Then the Cox ring is $T = k[x, y]$, and $\widehat{Z} = \mathbb{A}^2 \setminus \{(0, 0)\}$. Let M be the base field k with the structure of a $\mathbb{Z}$-graded $k[x, y]$-module given by $x\alpha = y\alpha = 0\ \forall_{\alpha\in k}$. Then $\mathbb{k}$ is a skyscraper sheaf on $\mathbb{A}^2$ supported at the origin. Hence ${i_Z}^*\mathbb{k} = 0$ and therefore $\widetilde{k} \cong \widetilde{0}$.*

The above example suggests that we should make a choice of a particular module describing a given quasicoherent sheaf $\mathcal{F}$ on Z. We will denote by $\Gamma_*(\mathcal{F})$ the $Cl(Z)$-graded T-module $\Gamma(\widehat{Z}, {\pi_Z}^*\mathcal{F})$.

3.1. The inverse image

Let N be a $Cl(Y)$-graded S-module. Then $N \otimes_S R$ has a structure of an R-module. We will define its $Cl(X)$-grading as follows: for homogeneous $n \in N$ and homogeneous $r \in R$ define $deg(n \otimes r) = \varphi(deg(n)) + deg(r)$. It is straightforward to verify that this grading is well defined and gives $N \otimes_S R$ a structure of a $Cl(X)$-graded R-module.

Theorem 3.1. *In the setup from the begining of the chapter, let $\mathcal{G}$ be a quasicoherent sheaf on Y. Assume that $\Gamma_*(\mathcal{G}) \cong N$ for a $Cl(Y)$-graded S-module N. Then $F^*\mathcal{G} \cong \widetilde{N \otimes_S R}$.*

Proof. We are interested only in sheaves up to isomorphism, so it is enough to prove the theorem for $\mathcal{G} = \widetilde{N}$. From the commutativity of the diagram in Theorem 2.1 we have ${\pi_X}^*F^*\widetilde{N} = \widehat{F}^*{\pi_Y}^*\widetilde{N}$. Proposition 1.3(A) implies that $\widehat{F}^*{\pi_Y}^*\widetilde{N} \cong \widehat{F}^*{i_Y}^*\mathbb{N}$. Using once more the diagram in Theorem 2.1 we obtain $\widehat{F}^*{i_Y}^*\mathbb{N} = {i_X}^*\overline{F}^*\mathbb{N}$. From the description of the inverse image of a quasicoherent sheaf by a morphism of affine schemes we obtain ${i_X}^*\overline{F}^*\mathbb{N} \cong {i_X}^*(N \otimes_S R)$ (here of course by $N \otimes_S R$ we mean in fact the sheaf of $\mathcal{O}_{\overline{X}}$-modules associated with this module). Hence we have:

$$(F^*\widetilde{N}) \otimes_{\mathcal{O}_X} \mathcal{R} \overset{1.3(B)}{\cong} {\pi_X}_*{\pi_X}^*F^*\widetilde{N} \cong {\pi_X}_*{i_X}^*(N \otimes_S R).$$

Taking the zeroth gradation we obtain an isomorphism $F^*\widetilde{N} \cong \widetilde{N \otimes_S R}$. □

3.2. The direct image

As we have seen in the beginning of Section 3.1, the extension of scalars of a graded module by a graded homomorphism of graded rings gives a graded module. For the restriction of scalars it is not the case. Moreover, even if taking the restriction of scalars of a $\mathrm{Cl}(X)$-graded R-module M gives a $\mathrm{Cl}(Y)$-graded S-module, it may be the case that it does not correspond to the direct image of $\widetilde{M}$ as the following example shows.

Example 3.2. *Let $X = \mathbb{P}^1$, $Y = \mathbb{A}^1$ with coordinates x, y and t, respectively. Let $F([x : y]) = 0$. Consider $\mathcal{O}_{\mathbb{P}^1} \cong \widetilde{k[x,y]}$ and two sheaves on Y.*

1) Let $\mathcal{F} = \overline{F}_* \widetilde{k[x,y]} = \widetilde{k[x,y]}$, where $k[x,y]$ is treated as a $k[t]$ module via $\overline{F}^*$ (that is $tf = 0\ \forall f \in k[x,y]$).

2) Let $\mathcal{G} = F_*\mathcal{O}_{\mathbb{P}^1}$.

We have $\Gamma(\mathbb{A}^1, \mathcal{F}) = k[x,y]$ and $\Gamma(\mathbb{A}^1, \mathcal{G}) = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$. Hence $\mathcal{F} \ncong \mathcal{G}$.

Let M be a $Cl(X)$-graded R-module. Let:

$$M_S^* = \bigoplus_{[E] \in Cl(Y)} M_{\varphi([E])}.$$

The graded ring homomorphism $\overline{F}^* : S \to R$ gives M_S^* a structure of a $Cl(Y)$-graded S-module: $\forall_{s\in S}\forall_{m \in M_S^*}\ m \cdot s = \overline{F}^*(s) \cdot m$.

Theorem 3.2. *In the setup from the begining of the chapter, let $\mathcal{F}$ be a quasicoherent sheaf on X with $\Gamma_*(\mathcal{F}) = M$. Then $F_*\mathcal{F} \cong \widetilde{M_S^*}$.*

In the proof we will define isomorphisms of sections of these two sheaves on a basis for the topology of Y. In order to be able to glue these isomorphisms to an isomorphism of sheaves we will carefully show that all isomorphisms considered on the way are natural. Before giving the proof of this theorem we will establish a few lemmas.

Lemma 3.1. *Let Z be a MDS with the Cox ring T. Let $\mathcal{F}$ be a quasicoherent sheaf on Z with $\Gamma_*(\mathcal{F}) = M$ and let g, h be homogeneous elements of T. Then there are commutative diagrams:*

$$\begin{array}{ccccc} T_g & \to & \mathcal{O}_{\overline{Z}}(\overline{Z}_g) & \to & \mathcal{O}_{\widehat{Z}}(\widehat{Z}_g) = \mathcal{O}_{\overline{Z}}(\widehat{Z}_g) \\ \downarrow & & \downarrow & & \downarrow \\ T_{hg} & \to & \mathcal{O}_{\overline{Z}}(\overline{Z}_{hg}) & \to & \mathcal{O}_{\widehat{Z}}(\widehat{Z}_{hg}) = \mathcal{O}_{\overline{Z}}(\widehat{Z}_{hg}) \\ \uparrow & & \uparrow & & \uparrow \\ T_h & \to & \mathcal{O}_{\overline{Z}}(\overline{Z}_h) & \to & \mathcal{O}_{\widehat{Z}}(\widehat{Z}_h) = \mathcal{O}_{\overline{Z}}(\widehat{Z}_h) \end{array} \qquad \begin{array}{ccccc} M_g & \to & \Gamma(\overline{Z}_g, \mathbb{M}) & \to & \Gamma(\widehat{Z}_g, \mathbb{M}) \\ \downarrow & & \downarrow & & \downarrow \\ M_{hg} & \to & \Gamma(\overline{Z}_{hg}, \mathbb{M}) & \to & \Gamma(\widehat{Z}_{hg}, \mathbb{M}) \\ \uparrow & & \uparrow & & \uparrow \\ M_h & \to & \Gamma(\overline{Z}_h, \mathbb{M}) & \to & \Gamma(\widehat{Z}_h, \mathbb{M}) \end{array}$$

with all horizontal arrows isomorphisms. In particular, for every homogeneous $h \in T$ there are isomorphisms $\alpha_h : T_h \to \mathcal{O}_{\widehat{Z}}(\widehat{Z}_h)$ and $\beta_h : M_h \to \Gamma(\widehat{Z}_h, \mathbb{M})$.

Proof. $\overline{Z}$ is normal and the complement of $\widehat{Z}$ is of codimension at least two. It follows that restricting functions gives an isomorphism $\mathcal{O}_{\overline{Z}}(\overline{Z}) \xrightarrow{\cong} \mathcal{O}_{\widehat{Z}}(\widehat{Z})$. Hence we have $\mathcal{O}_{\overline{Z}} \cong i_{Z*}\mathcal{O}_{\widehat{Z}}$. Therefore $\forall_{g\in T}$, restricting sections is an isomorphism $\mathcal{O}_{\overline{Z}}(\overline{Z}_g) \cong \mathcal{O}_{\widehat{Z}}(\widehat{Z}_g)$. This proves that the three right horizontal arrows of the left diagram are isomorphisms. It is well known that there exists three left horizontal arrows in this diagram, that are isomorphisms such that the left two squares commute ([Har77] Proposition II.2.2). The right two squares commute since all maps are restrictions of the sections of the structure sheaf $\mathcal{O}_{\overline{Z}}$.

Since $\Gamma_*(\mathcal{F}) = M$, $\mathbb{M} \cong i_{Z*}{\pi_Z}^*\mathcal{F}$ as sheaves of abelian groups. Therefore for every non zero homogeneous $f \in T$ the restriction of sections gives an isomorphism $\Gamma(\overline{Z}_f, \mathbb{M}) \to \Gamma(\widehat{Z}_f, \mathbb{M})$. Similar argument to the given above, shows that these isomorphisms give a commmutative diagram as in the statement of the lemma. $\square$

Given a surjective map of sets $G : Z_1 \to Z_2$, we say that a subset $U \subset Z_1$ is **saturated** with respect to G if $G^{-1}G(U) = U$.

Lemma 3.2. *Let Z be a Mori Dream Space with the Cox ring T. Let f, g be two (possibly zero) homogeneous elements of T such that $\widehat{Z}_f$ and $\widehat{Z}_g$ are saturated with respect to π_Z. Then we have the following commutative diagram with obvious vertical maps:*

$$\begin{tikzcd}
T_{(f)} \arrow[r, "\cong"] \arrow[d] & \mathcal{O}_Z(\pi_Z(\widehat{Z}_f)) \arrow[d] \\
T_{(fg)} \arrow[r, "\cong"] & \mathcal{O}_Z(\pi_Z(\widehat{Z}_{fg})) \\
T_{(g)} \arrow[r, "\cong"] \arrow[u] & \mathcal{O}_Z(\pi_Z(\widehat{Z}_g)) \arrow[u]
\end{tikzcd}$$

Proof. We have $(\pi_{Z*}\mathcal{O}_{\widehat{Z}})_0 \cong \mathcal{O}_Z$. Hence we have isomorphisms:

$$\Gamma(\pi_Z(\widehat{Z}_f), \mathcal{O}_Z) \cong \Gamma(\pi_Z(\widehat{Z}_f), \pi_{Z*}\mathcal{O}_{\widehat{Z}})_0 = \Gamma({\pi_Z}^{-1}(\pi_Z(\widehat{Z}_f)), \mathcal{O}_{\widehat{Z}})_0 = \Gamma(\widehat{Z}_f, \mathcal{O}_{\widehat{Z}})_0 \cong T_{(f)}.$$

Since the first isomorphism comes from the isomorphism of sheaves and the last comes from Lemma 3.1, they commute with restrictions and we have the commutative diagram from the statement. We have used here a fact that intersection of saturated sets is saturated. $\square$

Proof of Theorem 3.2. For all $[D] \in Cl(Y)$ and for all $0 \neq f \in S_{[D]}$ with $Y_{[D],f}$ affine we will define an isomorphism of $\mathcal{O}_Y(Y_{[D],f})$-modules $\Gamma(Y_{[D],f}, F_*\mathcal{F}) \xrightarrow{\chi_{[D],f}} \Gamma(Y_{[D],f}, \widetilde{M_S^*})$ such that $\forall_{[E]\in Cl(Y)}\ \forall_{0\neq g\in S_{[E]}}$ we have the following commutative diagram:

$$\begin{tikzcd}
\Gamma(Y_{[D],f}, F_*\mathcal{F}) \arrow[r, "\chi_{[D],f}"] \arrow[d] & \Gamma(Y_{[D],f}, \widetilde{M_S^*}) \arrow[d] \\
\Gamma(Y_{[D]+[E],fg}, F_*\mathcal{F}) \arrow[r, "\chi_{[D]+[E],fg}"] & \Gamma(Y_{[D]+[E],fg}, \widetilde{M_S^*}) \\
\Gamma(Y_{[E],g}, F_*\mathcal{F}) \arrow[r, "\chi_{[E],g}"] \arrow[u] & \Gamma(Y_{[E],g}, \widetilde{M_S^*}) \arrow[u]
\end{tikzcd}$$

where the vertical arrows are restriction maps. By Lemma 1.1 it will follow that such maps $\chi_{[D],f}$ define isomorphism of $\mathcal{O}_Y$-modules $F_*\mathcal{F} \to \widetilde{M_S^*}$. Note that the identifications that we have already done in the lemmas are all natural in the sense that they fit into similar diagrams.

Step 1. Pick any $[D] \in Cl(Y)$ and $0 \neq f \in S_{[D]}$ such that $Y_{[D],f}$ is affine. By Proposition 1.1 $\pi_Y^{-1}(Y_{[D],f}) = \widehat{Y}_f$ therefore, since π_Y is surjective, $\widehat{Y}_f$ is saturated with respect to π_Y. Hence by Lemma 3.2 we may assume that $\forall_{[D]\in Cl(Y)}\ \forall_{0\neq f\in S_{[D]}}$ such that $Y_{[D],f}$ is affine we have $\mathcal{O}_Y(Y_{[D],f}) = S_{(f)}$. We will describe $F_*\mathcal{F}|_{Y_{[D],f}}$. Since $Y_{[D],f}$ is affine it is enough to compute $\Gamma(Y_{[D],f}, F_*\mathcal{F})$ and describe its $S_{(f)}$-module structure.

Since $Y_{[D],f}$ is affine, $\pi_Y^{-1}(Y_{[D],f}) = \widehat{Y}_f = \overline{Y}_f$. We have also $\widehat{F}^{-1}(\widehat{Y}_f) = \widehat{X}_{f\circ\overline{F}}$. From this equality we obtain from the diagram in Theorem 2.1 that $\pi_X^{-1}(F^{-1}(Y_{[D],f})) = \widehat{X}_{f\circ\overline{F}}$. Hence from surjectivity of π_X it follows that $F^{-1}(Y_{[D],f}) = \pi_X(\widehat{X}_{f\circ\overline{F}})$ and we have the following commutative diagram:

$$\begin{tikzcd}
\overline{X}_{f\circ\overline{F}} \arrow[r, dotted, "\overline{F}"] & \overline{Y}_f \\
\widehat{X}_{f\circ\overline{F}} \arrow[u, hook, "i_X"] \arrow[r, dotted, "\widehat{F}"] \arrow[d, two heads, "\pi_X"] & \widehat{Y}_f \arrow[u, "="'] \arrow[d, two heads, "\pi_Y"] \\
\pi_X(\widehat{X}_{f\circ\overline{F}}) \arrow[r, "F"] & Y_{[D],f}
\end{tikzcd}$$

It follows that $\forall_{[D]\in Cl(Y)}\ \forall_{0\neq f\in S_{[D]}}$ such that $Y_{[D],f}$ is affine, $\widehat{X}_{f\circ\overline{F}}$ is saturated with respect to π_X and therefore by Lemma 3.2 we may assume that for such $[D]$ and f we have $\mathcal{O}_X(\pi_X(\widehat{X}_{f\circ\overline{F}})) = R_{(f\circ\overline{F})}$.

By Proposition 1.3(B) $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{R} = \pi_{X*}\pi_X{}^*\mathcal{F}$ so $\Gamma(\pi_X(\widehat{X}_{f\circ\overline{F}}),\mathcal{F})$ is the degree zero part of $\Gamma(\widehat{X}_{f\circ\overline{F}},\pi_X{}^*\mathcal{F})$ which by naturality of $\beta_h{}'s$ in Lemma 3.1 can be assumed to be equal to $M_{f\circ\overline{F}}$. We have established that $\Gamma(Y_{[D],f},F_*\mathcal{F}) = M_{(f\circ\overline{F})}$ therefore describing the group structure of $\Gamma(Y_{[D],f},F_*\mathcal{F})$.

Step 2. We will now describe the $S_{(f)}$-module structure of $\Gamma(Y_{[D],f},F_*\mathcal{F})$. Firstly we want to describe the module structure on $\Gamma(F^{-1}(Y_{[D],f}),\mathcal{F})$. From the diagram in Theorem 2.1 and the description of quasicoherent sheaves on Mori Dream Spaces from Proposition 1.2 we know that $\Gamma(F^{-1}(Y_{[D],f}),\mathcal{F})$ is the degree zero part of $\Gamma(F^{-1}(Y_{[D],f}),\pi_{X*}i_X{}^*\mathbb{M}) = \Gamma(\widehat{X}_{f\circ\overline{F}},i_X{}^*\mathbb{M})$. Hence it is $M_{(f\circ\overline{F})}$ with the $\mathcal{O}_X(\pi_X(\widehat{X}_{f\circ\overline{F}})) = R_{(f\circ\overline{F})}$-module structure coming from the map $\mathcal{O}_X(\pi_X(\widehat{X}_{f\circ\overline{F}})) \to \pi_{X*}\mathcal{O}_{\widehat{X}}(\pi_X(\widehat{X}_{f\circ\overline{F}})) = \mathcal{O}_{\widehat{X}}(\widehat{X}_{f\circ\overline{F}})$. This map is the inclusion $R_{(f\circ\overline{F})} \to R_{f\circ\overline{F}}$. Hence $\Gamma(F^{-1}(Y_{[D],f}),\mathcal{F})$ is $M_{(f\circ\overline{F})}$ not only as an abelian group but also as an $R_{(f\circ\overline{F})}$-module. Therefore $\Gamma(Y_{[D],f},F_*\mathcal{F}) = M_{(f\circ\overline{F})}$ with the $S_{(f)}$-module structure coming from the map $\mathcal{O}_Y(Y_{[D],f}) \to F_*\mathcal{O}_X(Y_{[D],f})$. Which is the map $S_{(f)} \to R_{(f\circ\overline{F})}$. Therefore, up to natural isomorphisms, $\Gamma(Y_{[D],f},F_*\mathcal{F}) = M_{(f\circ\overline{F})}$ as an $S_{(f)}$-module.

Step 3. We will describe the sections of $\widetilde{M_S^*}$ over affine sets of the form $Y_{[D],f}$. We will assume, using the naturality of $\alpha_h{}'s$ in Lemma 3.1, that $\mathcal{O}_{\widehat{Y}}(\widehat{Y}_f) = S_f$ and using Lemma 3.2 that $\mathcal{O}_Y(Y_{[D],f}) = S_{(f)}$. Then from the description of quasicoherent sheaves on Mori Dream Spaces we have $\Gamma(Y_{[D],f},\widetilde{M_S^*}) = (M_S^*)_{(f)}$ as an $S_{(f)}$-module. We have $deg(f) = [D]$ and $deg(f\circ\overline{F}) = \varphi([D])$ hence:

$$M_{(f\circ\overline{F})} = \{\frac{m}{(f\circ\overline{F})^k} | k\in\mathbb{N},\ m\in M \text{ and } deg(m) = k\varphi([D])\}$$

and

$$(M_S^*)_{(f)} = \{\frac{m}{f^k} | k\in\mathbb{N},\ m\in M_S^* \text{ and } deg(m) = k[D]\}.$$

From the definition of M_S^* it follows that we have $(M_S^*)_{k[D]} = M_{k\varphi([D])}$ as abelian groups. Therefore:

$$(M_S^*)_{(f)} = \{\frac{m}{f^k} | k\in\mathbb{N},\ m\in M \text{ and } deg(m) = k\varphi([D])\}$$

and we have an isomorphism of $S_{(f)}$-modules $\chi_{[D],f} : M_{(f\circ\overline{F})} \to (M_S^*)_{(f)}$ given by $\frac{m}{(f\circ\overline{F})^k} \mapsto \frac{m}{f^k}$ for $m \in M_{k\varphi([D])}$. This isomorphism is natural so isomorphisms of this type for all affine sets of the form $Y_{[D],f}$ will glue by Lemma 1.1 to an isomorphism of $\mathcal{O}_Y$-modules $F_*\mathcal{F} \to \widetilde{M_S^*}$. Observe that $\chi_{[D],f}$ is well defined. If $\frac{m}{(f\circ\overline{F})^k} = \frac{n}{(f\circ\overline{F})^l}$, then there exists $s \in \mathbb{N}$, such that $(f \circ \overline{F})^s((f \circ \overline{F})^l m - (f \circ \overline{F})^k n) = 0$. Then by definition of the module structure on $(M_S^*)_{(f)}$ we have $f^s(f^l m - f^k n) = 0$.

□

Chapter 4

Examples

In this chapter we will present two examples. The first will be a sort of a reality check. We will consider a morphism of affine MDSes. Then the direct image sheaf and the inverse image sheaf are well understood since both varieties are affine. We will check that the more complicated constructions given in the previous chapter yield the correct results. As expected, it will follow from pure commutative algebra. The interesting part will be the calculation of the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$. The second example will be an example of a toric map of toric surfaces. In the toric situation, the Cox rings are well known. We will focus on showing that for a given module M such that it determines the sheaf $\mathcal{F}$ that we will be interested in, we have $\Gamma_*(\mathcal{F}) \cong M$.

4.1. An affine example

We will recall the definitions of a strongly stable action and of the H-factorial variety from [ADHL15].

Definition 4.1. *Let H be an affine algebraic group and $\mathcal{Y}$ an irreducible H-prevariety. We say that the H-action on $\mathcal{Y}$ is* ***strongly stable*** *if there is an open invariant subset $W \subset \mathcal{Y}$ with the following properties:*

(i) The complement $\mathcal{Y} \setminus W$ is of codimension at least two in $\mathcal{Y}$.

(ii) The group H acts freely on W.

(iii) For every $y \in W$ the orbit Hy is closed in $\mathcal{Y}$.

Definition 4.2. *Let an algebraic group H act on an irreducible, normal prevariety $\mathcal{Y}$. We say that $\mathcal{Y}$ is* ***H-factorial*** *if every H-invariant Weil divisor on $\mathcal{Y}$ is a divisor of a rational function $f \in k(\mathcal{Y})$ that is regular and H-homogeneous on an open invariant subset of $\mathcal{Y}$. Such a function f will be called a* ***H-homogeneous rational function****.*

We will use the following theorem from [ADHL15].

Theorem 4.1 ([ADHL15], 1.6.4.3)**.** *Let a quasitorus H act on a normal quasiaffine variety $\mathcal{Y}$ with a good quotient $q : \mathcal{Y} \to Y$. Assume that $\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = k^*$ holds, $\mathcal{Y}$ is H-factorial and the H-action is strongly stable. Then Y is a normal prevariety of affine intersection, $\Gamma(Y, \mathcal{O}_Y) = k^*$ holds, $Cl(Y)$ is finitely generated, the Cox sheaf of Y is locally of finite type, and $\mathcal{Y}$ is equivariantly isomorphic to $\widehat{Y}$.*

Example 4.1. *Let $X = \mathbb{A}^1$ be the affine line. Let Y be $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$, where $\Delta = \{(x, x) \in \mathbb{P}^1 \times \mathbb{P}^1 | x \in \mathbb{P}^1\}$. We will first prove that Y is an affine smooth MDS.*

Consider the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ *given by* $([x_0, x_1], [y_0, y_1]) \mapsto [x_0y_0, x_0y_1, x_1y_0, x_1y_1]$. *We will denote the coordinates in* $\mathbb{P}^3$ *by* $z_0, ..., z_3$. *Then* $\mathbb{P}^1 \times \mathbb{P}^1$ *is given by* $z_0z_3 = z_1z_2$. *The complement of the diagonal* $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ *is given by:*

$$\begin{aligned} z_0z_3 &= z_1z_2 \\ z_1 &\neq z_2. \end{aligned} \tag{4.1}$$

Consider the change of coordinates given by the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We will denote by $\widetilde{z_i}$ *the image of* z_i *under this transformation. In the new coordinates equations (4.1) become:*

$$\begin{aligned} \widetilde{z_0}\widetilde{z_3} &= \widetilde{z_1}(\widetilde{z_1} - \widetilde{z_2}) \\ \widetilde{z_2} &\neq 0. \end{aligned}$$

Therefore $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$ *is affine. In the affine coordinates* $y_1 = \frac{\widetilde{z_0}}{\widetilde{z_2}}$, $y_2 = \frac{\widetilde{z_1}}{\widetilde{z_2}}$ *and* $y_3 = \frac{\widetilde{z_3}}{\widetilde{z_2}}$ *it is given by a single equation:*

$$y_1y_3 = y_2(y_2 - 1).$$

Let $\mathcal{Y} = \operatorname{Spec} S$ *where* $S := k[x, y, t, s]/(xs - yt + 1)$. *We will consider* S *as a* $\mathbb{Z}$*-graded ring by putting* $deg(x) = deg(y) = 1$ *and* $deg(s) = deg(t) = -1$. *This grading gives rise to an action of a quasitorus* $H := k^* = \operatorname{Spec} k[\mathbb{Z}]$ *on* $\mathcal{Y}$. *Observe that* $S_0 = k[xs, xt, ys, yt]/(xs - yt + 1)$. *Let* $\phi : k[y_1, y_2, y_3] \to k[xs, xt, ys, yt]$ *be given by:*

$$\begin{aligned} y_1 &\mapsto xt \\ y_2 &\mapsto yt \\ y_3 &\mapsto ys. \end{aligned}$$

It induces an isomorphism $\overline{\phi}$ *of* $k[y_1, y_2, y_3]/(y_1y_3 - y_2(y_2 - 1))$ *with* $k[xs, xt, ys, yt]/(xs - yt + 1)$. *Hence* $Y = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$ *is a good quotient for the action of* k^* *on* $\mathcal{Y}$.

We claim that S *is the Cox ring of* Y. *Since* Y *is affine, we have the equality of the characteristic space* $\widehat{Y}$ *and the total coordinate space* $\overline{Y}$. *Hence if the assumptions of Theorem 4.1 are satisfied, we have a graded isomorphism of the Cox ring of* Y *with the ring* S. *By the Jacobian criterion* $\mathcal{Y}$ *is smooth. In particular, it is normal. Let* $\overline{s}$ *be the class of* s *in* S. *Let* $U \subset \mathcal{Y}$ *be the subset where* $\overline{s}$ *does not vanish. Then* $U \cong k^* \times k \times k$. *Hence* $\Gamma(\mathcal{Y}, \mathcal{O}^*_{\mathcal{Y}}) = k^*$ *since global invertible functions are constant on an open dense subset* U. *Moreover, we have an exact sequence:*

$$\mathbb{Z}div(\overline{s}) \to Cl(\mathcal{Y}) \to Cl(U) \to 0$$

We have $Cl(U) = 0$ *hence* $Cl(\mathcal{Y})$ *is generated by the image of* $div(\overline{s})$ *which is* 0 *by definition. Thus* $Cl(\mathcal{Y}) = 0$. *Hence* S *is a UFD ([Har77] Proposition II.6.2). Therefore, Proposition 1.5.3.3 in [ADHL15] implies that* $\mathcal{Y}$ *is* H*-factorial.*

We are left with the proof that the action of H on $\mathcal{Y}$ is strongly stable. As W in Definition 4.1 we will take $\mathcal{Y}$. Observe that for every point z of $\mathcal{Y}$ there are functions of both positive and negative degree not vanishing at z. Indeed, if $\overline{x}(z) = \overline{y}(z) = 0$ then we get a contradiction with $\overline{xs} - \overline{yt} + 1 = 0$. Similarily, at least one of $\overline{s}$ and $\overline{t}$ must be non zero at z. Hence by Proposition 1.2.2.8 in [ADHL15] the H-action on $\mathcal{Y}$ is free. In particular there are no 0-dimensional orbits. Let z be any point of $\mathcal{Y}$. The closure of its orbit Hz is a union of Hz and lower dimensional orbits. As there are no lower dimensional orbits, Hz is closed. It finishes the proof that the Cox ring of Y is S.

Let $F : X = \mathbb{A}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta = Y$ be the morphism given by $z \mapsto (0, 1, z)$, where Y is treated as $V(y_1y_3 - y_2(y_2 - 1)) \subset \mathbb{A}^3$. Let $\psi : k[y_1, y_2, y_3]/(y_1y_3 - y_2(y_2 - 1)) \to k[x, y, s, t]/(xs - yt + 1)$ be given by the composition:

$$k[y_1, y_2, y_3]/(y_1y_3 - y_2(y_2 - 1)) \xrightarrow{\overline{\phi}} k[xs, xt, ys, yt]/(xs - yt + 1) = S_0 \hookrightarrow S.$$

Since Y is a smooth MDS, and X is a $\mathbb{Q}$-factorial MDS we have a lift of

$$F^* : k[y_1, y_2, y_3]/(y_1y_3 - y_2(y_2 - 1)) \to k[z]$$

to a map $\overline{F}^* : k[x, y, s, t]/(xs - yt + 1) \to k[z]$ such that $\overline{F}^* \circ \psi = F^*$. In other words, we can extend the homomorphism F^* from the degree zero part to the whole Cox ring. For instance, we can set:

$$\overline{F}^*(\overline{x}) = 0$$
$$\overline{F}^*(\overline{y}) = 1$$
$$\overline{F}^*(\overline{s}) = z$$
$$\overline{F}^*(\overline{t}) = 1.$$

Let $\mathcal{G}$ be a quasicoherent sheaf on Y associated with a S_0-module N. Then $F^*\mathcal{G}$ is a quasicoherent sheaf on X associated with a $k[z]$-module $N \otimes_{S_0} k[z]$. On the other hand, $\Gamma_*(\mathcal{G}) = \Gamma(\overline{Y}, {\pi_Y}^*\mathcal{G}) = N \otimes_{S_0} S$. Hence by Theorem 3.1, $F^*\mathcal{G}$ is isomorphic to the sheaf associated with a $k[z]$-module $(N \otimes_{S_0} S) \otimes_S k[z] = N \otimes_{S_0} k[z]$ as expected.

Let $\mathcal{F}$ be a quasicoherent sheaf on X associated with a $k[z]$-module M. Since $Cl(X) = 0$, we have $M_S^* = \bigoplus_{k\in\mathbb{Z}} M$ where each direct summand has a structure of a S-module given by $\overline{F} : S \to k[z]$. Hence by Theorem 3.2, $F_*\mathcal{F}$ is isomorphic to the sheaf associated with a S_0-module $(M_S^*)_0 = M$. The structure of an S_0 module is given by $\overline{F}^*\psi : S_0 \to k[z]$. Since $\overline{F}^*\psi = F^*$, it agrees with the usual construction of a direct image sheaf of a quasicoherent sheaf by a map of affine varieties.

The concrete F played no role. The check was a pure commutative algebra independent of particular choices of $\mathcal{F}$, $\mathcal{G}$ and F.

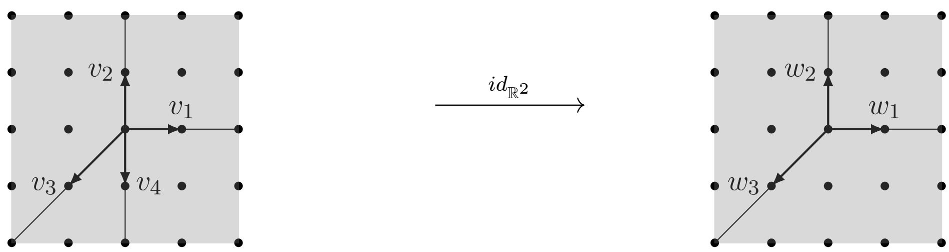
4.2. A toric example

We will first prove the following lemma.

Lemma 4.1. *Let X be a smooth MDS with a Cox ring R. Given a Cartier divisor $D \in Pic(X)$ we have $\Gamma_*(\mathcal{O}_X(D)) \cong R([D])$.*

Proof. In the notation from Construction 1.3 we have $\Gamma_*(\mathcal{O}_X(D)) = \Gamma(\widehat{X}, {\pi_X}^*(\mathcal{O}_X(D)))$. Since π_X is surjective and D is Cartier we have ${\pi_X}^*(\mathcal{O}_X(D)) = \mathcal{O}_{\widehat{X}}(\pi^*D)$ where π^*D is the pullback of the divisor D. Write D as $D = E_1 - E_2$ where both E_1 and E_2 are effective. From Proposition 1.5.2.2 in [ADHL15] there exists $[D_1]$, $[D_2] \in Cl(X)$ and $f_1 \in R_{[D_1]}$, $f_2 \in R_{[D_2]}$ such that $E_i = div_{[D_i]}(f_i)$. From Lemma 1.5.3.6 in [ADHL15] it follows that $\pi_X{}^* D = div(\frac{f_1}{f_2})$. Hence $\Gamma_*(\mathcal{O}_X(D)) \cong R([D_1]-[D_2])$. By the definition of the $[D]$-divisor, $[D_1]-[D_2] = [D]$. $\square$

Example 4.2. *Let M, N be dual lattices of rank two. Let X be the Hirzebruch surface $\mathbb{F}_1$ given by the unique complete fan $\Sigma_1 \subset N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^2$ determined by the vectors $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (-1,-1)$ and $v_4 = (0,-1)$. Let Y be the projective plane over $\mathbb{C}$ considered as a toric variety given by the unique complete fan $\Sigma_2 \subset \mathbb{R}^2$ determined by the vectors $w_1 = (1,0)$, $w_2 = (0,1)$ and $w_3 = (-1,-1)$. We will consider the toric morphism induced by the identity map $\varphi : \mathbb{Z}^2 \to \mathbb{Z}^2$. This is the blow-up of the distinguished point of $\mathbb{P}^2$ associated with the cone $\sigma = cone(w_1, w_3)$ (Proposition 3.3.15 in [CLS11]). Observe that the tensored map $\varphi_{\mathbb{R}} = id_{\mathbb{R}^2}$ is compatible with the fans Σ_1 and Σ_2 in the sense that for every cone $\sigma \in \Sigma_1$ there exists a cone $\tau \in \Sigma_2$ such that $\varphi_{\mathbb{R}}(\sigma) \subset \tau$. Hence φ induces a (toric) map $F : X \to Y$. The data of the fans is represented by the following picture:*



We will denote the Cox ring of X by R and the Cox ring of Y by S. We will calculate the class group $Cl(X)$ of X. It is generated by D_1, ... D_4 where D_i is the closure of the orbit associated with the ray v_i, $i = 1, ..., 4$. The relations are:

$$D_1 - D_3 = 0$$
$$D_2 - D_3 - D_4 = 0$$

Hence it is the free abelian group generated by D_3 and D_4. The Cox ring R is thus $\mathbb{C}[x_1, ..., x_4]$ where x_i is associated with v_i, $i = 1, ..., 4$. That is:

$$deg(D_1) = (1,0)$$
$$deg(D_2) = (1,1)$$
$$deg(D_3) = (1,0)$$
$$deg(D_4) = (0,1)$$

Similarily, the Cox ring S of Y is $\mathbb{C}[y_1, y_2, y_3]$, where $deg(y_i) = 1$ $i = 1, ..., 3$. From Section 5.1 of [CLS11] it follows that the irrelevant ideal of a toric variety coming from a fan Σ is $\langle \prod_{\rho \notin \sigma(1)} x_\rho | \sigma \in \Sigma_{max} \rangle$. Here Σ_{max} is the set of cones of Σ maximal with respect to inclusion and $\sigma(1)$ is the set of rays of the cone σ. Hence for X we have that the irrelevant ideal is generated by x_1x_2, x_2x_3, x_3x_4 and x_1x_4. The action of the quasitorus $(\mathbb{C}^*)^2$ associated with the class group $Cl(X) \cong \mathbb{Z}^2$ is determined by the grading of the Cox *ring. That is* $(\lambda, \mu)(a, b, c, d) = (\lambda a, \lambda\mu b, \lambda c, \mu d)$*. Of course for* Y *we obtain the well known quotient construction* $(\mathbb{C}^3 \setminus (0,0,0))/\mathbb{C}^*$*. We will check that the map* $\mathbb{C}^4 \xrightarrow{\overline{F}} \mathbb{C}^3$ *given by* $(a, b, c, d) \mapsto (ad, b, cd)$ *satisfies the property of the lift from Theorem 2.1. Indeed,* $\overline{F}^*(y_1) = x_1x_4$*,* $\overline{F}^*(y_2) = x_2$ *and* $\overline{F}^*(y_3) = x_3x_4$*. Since* $deg(x_1x_4) = deg(x_2) = deg(x_3x_4) = (1,1)$*,* $\overline{F}^*$ *is a graded homomorphism. Suppose that* $(ad, b, cd) = (0,0,0)$*. Then* $b = 0$*. If, moreover,* $d = 0$ *then* $(a, b, c, d) \notin \widehat{X}$*. If* $d \neq 0$*, then* $a = b = c = 0$ *and* $(a, b, c, d) \notin \widehat{X}$*. Hence* $\overline{F}$ *restricts to a map* $\widehat{F} : \widehat{X} \to \widehat{Y}$*.*

We will check that $\overline{F}$ *describes* F*, that is we have a commutative diagram as in Theorem 2.1. Let* $\sigma_1 = cone(v_1, v_2)$*,* $\sigma_2 = cone(v_2, v_3)$*,* $\sigma_3 = cone(v_3, v_4)$ *and* $\sigma_4 = cone(v_1, v_4)$ *be the cones of maximal dimension in* Σ_1*. Let* $\tau_1 = cone(w_1, w_2)$*,* $\tau_2 = cone(w_2, w_3)$ *and* $\tau_3 = cone(w_1, w_3)$ *be the cones of maximal dimension in* Σ_2*. By* U_σ *we will denote the affine toric variety corresponding to the cone* σ*. In section 5.1 in [CLS11] it is proved that the isomorphism of* $\mathbb{C}[\sigma_1^\vee \cap M]$ *- the coordinate ring of* U_{σ_1} *with* $(R_{x^{\hat{\sigma}_1}})_0$ *is given by:*

$$\chi^m \mapsto \prod_j x_j^{\langle m, v_j \rangle} \tag{4.2}$$

We will denote by $s = \chi^{e_1^*}$ *and* $t = \chi^{e_2^*}$*.*

Consider first the map $F : U_{\sigma_1} \to U_{\tau_1}$*. In order to determine the map on the level of rings we take the dual cones and dual of the matrix that determines the morphism* F*. Observe that* s *and* t *generates the semigroup* $\sigma_1^\vee \cap M = \sigma_2^\vee \cap M$*. Hence we obtain that* F^* *is the identity map* $\mathbb{C}[s, t] \to \mathbb{C}[s, t]$*. The isomorphisms of the form (4.2) are given by:*

$$\begin{aligned} s &\mapsto \frac{x_1}{x_3} \\ t &\mapsto \frac{x_2}{x_3x_4} \end{aligned} \tag{4.3}$$

for U_{σ_1} *and by:*

$$\begin{aligned} s &\mapsto \frac{y_1}{y_3} \\ t &\mapsto \frac{y_2}{y_3} \end{aligned} \tag{4.4}$$

for U_{τ_1}*. Hence the map obtained by composing* F^* *with isomorphisms (4.3) and (4.4):*

$$\mathbb{C}[y_1, y_2, y_3]_{(y_3)} \to \mathbb{C}[s, t] \to \mathbb{C}[s, t] \to \mathbb{C}[x_1, x_2, x_3, x_4]_{(x_3x_4)}$$

is given by:

$$\begin{aligned} \frac{y_1}{y_3} &\mapsto s \quad \mapsto s \mapsto \frac{x_1}{x_3} \\ \frac{y_2}{y_3} &\mapsto t \quad \mapsto t \mapsto \frac{x_2}{x_3x_4} \end{aligned}$$

This map clearly agrees with the map $\mathbb{C}[y_1, y_2, y_3]_{y_3} \to \mathbb{C}[x_1, x_2, x_3, x_4]_{x_3x_4}$ *induced by* $\overline{F} : \mathbb{C}[y_1, y_2, y_3] \to \mathbb{C}[x_1, x_2, x_3, x_4]$*.*

We will now consider $F : U_{\sigma_2} \to U_{\tau_2}$*. We have the equality:*

$$\mathbb{C}[\sigma_2^\vee \cap M] = \mathbb{C}[\tau_2^\vee \cap M] = \mathbb{C}[\frac{t}{s}, \frac{1}{s}]$$

The map $F^* : \mathbb{C}[\tau_2^\vee \cap M] \to \mathbb{C}[\sigma_2^\vee \cap M]$ *is the identity. The isomorphisms of the form (4.2) are given by:*

$$\begin{aligned} \frac{t}{s} &\mapsto \frac{x_2}{x_1x_4} \\ \frac{1}{s} &\mapsto \frac{x_3}{x_1} \end{aligned} \tag{4.5}$$

for U_{σ_2} and by:

$$\begin{aligned} \frac{t}{s} &\mapsto \frac{y_2}{y_1} \\ \frac{1}{s} &\mapsto \frac{y_3}{y_1} \end{aligned} \tag{4.6}$$

for U_{τ_2}. Hence the map obtained by composing F^* with isomorphisms (4.5) and (4.6):

$$\mathbb{C}[y_1, y_2, y_3]_{(y_1)} \to \mathbb{C}[\frac{t}{s}, \frac{1}{s}] \to \mathbb{C}[\frac{t}{s}, \frac{1}{s}] \to \mathbb{C}[x_1, x_2, x_3, x_4]_{(x_1x_4)}$$

is given by:

$$\begin{aligned} \frac{y_2}{y_1} &\mapsto \frac{t}{s} &&\mapsto \frac{t}{s} \mapsto \frac{x_2}{x_1x_4} \\ \frac{y_3}{y_1} &\mapsto \frac{1}{s} &&\mapsto \frac{1}{s} \mapsto \frac{x_3}{x_1} \end{aligned}$$

That homomorphism agrees with $\overline{F}$.

Now we will consider $F : U_{\sigma_3} \to U_{\tau_3}$. We have $\mathbb{C}[\sigma_3^\vee \cap M] = \mathbb{C}[\frac{s}{t}, \frac{1}{s}]$ and $\mathbb{C}[\tau_3^\vee \cap M] = \mathbb{C}[\frac{s}{t}, \frac{1}{t}]$. The map $F^* : \mathbb{C}[\frac{s}{t}, \frac{1}{t}] \to \mathbb{C}[\frac{s}{t}, \frac{1}{s}]$ is the inclusion. The isomorphisms of the type (4.2) are determined by:

$$\begin{aligned} \frac{s}{t} &\mapsto \frac{x_1x_4}{x_2} \\ \frac{1}{s} &\mapsto \frac{x_3}{x_1} \end{aligned}$$

for U_{σ_3} and by:

$$\begin{aligned} \frac{s}{t} &\mapsto \frac{y_1}{y_2} \\ \frac{1}{t} &\mapsto \frac{y_3}{y_2} \end{aligned}$$

for U_{τ_3}. Hence the map $\mathbb{C}[y_1, y_2, y_3]_{(y_2)} \to \mathbb{C}[x_1, x_2, x_3, x_4]_{(x_1x_2)}$ is given by:

$$\begin{aligned} \frac{y_1}{y_2} &\mapsto \frac{s}{t} &&\mapsto \frac{x_1x_4}{x_2} \\ \frac{y_3}{y_2} &\mapsto \frac{1}{t} &&\mapsto \frac{x_3x_4}{x_2} \end{aligned}$$

This map also agrees with $\overline{F}^*$.

We are left with the map $F : U_{\sigma_4} \to U_{\tau_3}$. We have $\mathbb{C}[\sigma_4^\vee \cap M] = \mathbb{C}[s, \frac{1}{t}]$. The isomorphism of the form (4.2) for U_{σ_4} is given by:

$$\begin{aligned} s &\mapsto \frac{x_1}{x_3} \\ \frac{1}{t} &\mapsto \frac{x_3x_4}{x_2} \end{aligned}$$

Therefore the map $\mathbb{C}[y_1, y_2, y_3]_{(y_2)} \to \mathbb{C}[x_1, x_2, x_3, x_4]_{(x_2x_3)}$ *is given by:*

$$\begin{aligned} \frac{y_1}{y_2} &\mapsto \frac{s}{t} &&\mapsto \frac{x_1x_4}{x_2} \\ \frac{y_3}{y_2} &\mapsto \frac{1}{t} &&\mapsto \frac{x_3x_4}{x_2} \end{aligned}$$

Since this map agrees with $\overline{F}^*$ *we have proved that the guessed map* $\overline{F}$ *is the lift of* F*.*

We will consider the pullback of the canonical sheaf on Y *and the pushforward of the ideal sheaf of* D_2*, i.e.* $\mathcal{O}_X(-D_2)$*.*

From Proposition 8.2.7 in [CLS11] it follows that the canonical sheaf ω_Y *on* Y *is the sheaf associated with the module* $N := S(-deg(y_1y_2y_3)) = S(-3)$ *and the canonical sheaf on* X *is the sheaf associated with the module* $R(-deg(x_1x_2x_3x_4)) = R((-3,-2))$*. By Lemma 4.1 assumptions for Theorem 3.1 are satisfied. Therefore* $F^*\omega_Y$ *is isomorphic to the sheaf associated with the* $Cl(X)$*-graded* R*-module* $N \otimes_S R$*. The map* $\varphi : Cl(Y) \cong \mathbb{Z} \to \mathbb{Z}^2 \cong Cl(X)$ *determined by the graded homomorphism* $\overline{F}^*$ *is given by* $a \mapsto (a, a)$*. Hence* $N \otimes_S R \cong S(-3) \otimes_S R \cong R((-3,-3))$*. Therefore, using again Lemma 4.1,* $F^*\omega_Y \cong \mathcal{O}_X(-D_1 - D_2 - D_3 - 2D_4) \cong \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D_4)$*.*

Let $\mathcal{I}_{D_2} = \mathcal{O}_X(-D_2)$*. From Lemma 4.1 it follows that* $M := \Gamma_*(\mathcal{I}_{D_2}) = R((-1,-1))$*. Hence* $F_*\mathcal{I}_{D_2}$ *is isomorphic to the sheaf associated with the* $\mathbb{Z}$*-graded* S*-module:*

$$M_S^* = \bigoplus_{k \in \mathbb{Z}} R_{(k-1,k-1)} = \mathbb{C}[x_1x_4, x_2, x_3x_4](-1) \cong \mathbb{C}[y_1, y_2, y_3](-1) = S(-1)$$

Hence by Theorem 3.2, we have $F_*\mathcal{I}_{D_2} \cong \mathcal{O}_Y(-1)$*.*
We will obtain this result differently. Let $\mathcal{E} = \mathcal{O}_Y(1)$*. From Lemma 4.1 the following equality of* S*-modules holds* $\Gamma_*(\mathcal{E}) = S(1)$*. Hence Theorem 3.1 implies that* $F^*\mathcal{E}$ *is isomorphic to the sheaf associated with* $S(1) \otimes_S R \cong R((1,1))$*. Thus* $\mathcal{I}_{D_2} \otimes_{\mathcal{O}_X} F^*\mathcal{E} \cong \mathcal{O}_X$*. From the Projection Formula from Exercise 5.1 in the second chapter of [Har77] we obtain* $F_*\mathcal{O}_X \cong F_*\mathcal{I}_{D_2} \otimes_{\mathcal{O}_Y} \mathcal{E}$*. Hence* $F_*\mathcal{I}_{D_2} \cong F_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-1)$*. From the proof of Corollary III.11.4 in [Har77] it follows that* $F_*\mathcal{O}_X \cong \mathcal{O}_Y$*. Therefore we have* $F_*\mathcal{I}_{D_2} \cong \mathcal{O}_Y(-1)$ *which agrees with the result obtained using Theorem 3.2.*

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