IDENTIFIABILITY AND WEAK DEFECTIVITY INFORMAL NOTES FOR LUKECIN SCHOOL SEPTEMBER 2-6, 2013

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Abstract. Any tensor can be decomposed as a sum of decomposable tensors. In the symmetric case, this is the Waring decomposition of a polynomial as a sum of powers. This decomposition of a tensor is particularly useful in applications when it is unique, in this case we say the tensor is identifiable. We study results and criteria which guarantee that a tensor is identifiable. One of the most important is related to the geometric notion of weak defectiveness introduced by Chiantini and Ciliberto. If time allows, I will continue with applications of vector bundles (nonabelian apolarity).

Suggested preparatory reading: from J. Landsberg's book "Tensors: Geometry and Applications": chapter 4, chapter 5 from 5.1 to 5.4, chapter 6 from 6.1 to 6.4

These notes describe the content of the lectures, intended as a tool for participants.

1. Lecture 1

Abstract Lecture 1: rank, secant variety, Terracini Lemma, matrices of bounded rank. Chow_λ *and consequent filtration of the space of binary forms by partitions* λ. *Dual varieties, Biduality Theorem, the case of binary forms again. Abstract secant variety, secant degree, identifiability. Flattenings and equations of Segre varieties. Local structure of Segre varieties.*

1.1. **Rank and secant variety.** Definition of X-rank for a point $p \in \mathbb{P}^n$, relative to a projective variety $X \subset \mathbb{P}^n$.

$$\mathrm{rk}_X(p) = \min\{t | \exists x_1, \ldots, x_t \in X \text{ such that } p \in \langle x_1, \ldots, x_k \rangle\}$$

Definition of k-secant variety $\sigma_k(X)$ and of X-border rank.

The k-th secant variety $\sigma_k(X)$ of a projective irreducible variety X is the Zariski closure of the union of the projective span $< x_1, \ldots x_k >$ where $x_i \in X$. We have a chain of inclusions

$$X = \sigma_1(X) \subset \sigma_2(X) \subset \ldots$$

p has border rank (with respect to X) given by $\min\{t | p \in \sigma_t(X)\}$ we write $\underline{\mathrm{rk}}_X(p)$ to denote *border rank.*

Obviously

$$\underline{\mathrm{rk}}_X(p) \leq \mathrm{rk}_X(p).$$

The first example where strict inequality holds is x^2y, where

$$\underline{\mathrm{rk}}_X(x^2y) = 2 \qquad \mathrm{rk}_X(x^2y) = 3$$

1.2. **The Terracini Lemma.** Let's state also, for future reference, the celebrated "Terracini Lemma" (see e.g. [Zak]), whose proof is straightforward by a local computation.

Theorem 1.1 (Terracini Lemma)**.** *Let X be a projective irreducible variety and let $z \in < x_1, \ldots, x_k >$ be a general point in $\sigma_k(X)$. Then*

$$T_z\sigma_k(X) = < T_{x_1}X, \ldots, T_{x_k}X >$$

The tangent spaces $T_{x_i}X$ appearing in the Terracini lemma are the projective tangent spaces. Sometimes, we will denote by the same symbol the affine tangent spaces, this abuse of notation should not create any serious confusion.

1.3. **Variety of matrices of bounded rank.** Let

$$D_r = \{f \in V_0 \otimes V_1 | \mathrm{rk}\ f \leq r\} \tag{1}$$

We have that $D_r \setminus D_{r-1}$ are exactly the orbits for the action of $GL(V_0) \times GL(V_1)$ on the space $V_0 \otimes V_1$ of matrices, and in particular the maximal rank matrices form the dense orbit.

Note that D_1 is isomorphic to the Segre variety $\mathbb{P}(V_0) \times \mathbb{P}(V_1)$ and that it coincides with the set of decomposable tensors, which have the form $v_0 \otimes v_1$ for $v_i \in V_i$.

The first remark is

Lemma 1.2. *The rank of f (with respect to the Segre variety), as just defined, coincides with the usual rank of f as linear map (matrix). In other terms, we have*

$$\sigma_k(D_1) = D_k \tag{2}$$

Note that in this case, the Zariski closure is superfluous in the definition of σ_k.

Proof. Acting with the group $GL(V_0) \times GL(V_1)$, f takes the form $f = \sum_{i=1}^r v_0^i \otimes v_1^i$, where $\{v_0^i\}$ is a basis of V_0 and $\{v_1^i\}$ is a basis of V_1, corresponding to the matrix

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

In this form the statement is obvious. □

Example:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

can be decomposed in infinitely many ways.

Example: Tangent spaces and normal spaces to varieties of matrices of bounded rank (following is borrowed from 2009 Nordfjordeid notes).

Parametric description of tangent space at D_r

Proposition 1.3. *The tangent space of D_1 at $v \otimes w$ is given by*

$$v \otimes W + V \otimes w = \{v \otimes w' + v' \otimes w, \forall v' \in V, w' \in W\}$$

Proof. Consider any curve $v(t) \otimes w(t) \in D_1$ such that $v(0) = v$, $w(0) = w$.

The derivative for $t = 0$ is given by $v'(0) \otimes w + v \otimes w'(0)$. As $v'(0)$ and $w'(0)$ are arbitrary vectors, the thesis follows. □

The previous proposition is the case $r = 1$ of the following more general

Proposition 1.4. *The tangent space of D_r at $\sum_{i=1}^r v_i \otimes w_i \in D_r$ is given by*

$$\sum_{i=1}^r v_i \otimes W + V \otimes w_i$$

The proof is exactly the same. This can be seen also as the first display of the basic *Terracini lemma*.

If $f = \sum_{i=1}^r v_i \otimes w_i$ with minimal r, we get that both v_i and w_i are independent, otherwise we can express f as a sum of fewer r. For higher way tensors this is no more possible.

Cartesian description of tangent space at D_r

Theorem 1.5. *Let $f \in D_r \subseteq Hom(V^\vee, W)$. The tangent space to D_r at f is given by $\{g \in Hom(V^\vee, W) | g(\ker f) \subseteq \mathrm{Im}\ f\}$*

There are several proofs of this theorem, we propose the following one which is natural in the setting of tensor decomposition.

Proof. By assumption there are $v_i \in V$ and $w_i \in W$ such that $f = \sum_{i=1}^r v_i \otimes w_i$ Note that $\ker f =< v_1, \ldots, v_r >^\perp$ and $\mathrm{Im}\ f =< w_1, \ldots w_r >$

If $g \in \sum_{i=1}^r v_i \otimes W + V \otimes w_i$ then $g(\ker f) \subseteq \mathrm{Im}\ f$. This proves one inclusion. The second inclusion follows by a dimensional count. □

Corollary 1.6. *The normal space at $f \in D_r$ is given by*

$$Hom\ (\ker f, W/Im\ f)$$

The conormal space (it is the dual of the normal space) at $f \in D_r$ is given by

$$(\ker f) \otimes (Im\ f)^\perp \subseteq V^\vee \otimes W^\vee$$

The conormal space is quite useful because it coincides with $T_f^\perp$, the orthogonal of the tangent space.

Exercise For $f \in \mathrm{Hom}\ (V^\vee, W)$ denote by $f^t \in \mathrm{Hom}\ (W^\vee, V)$ the *transpose* of f, defined by $f^t(w)(v) = f(v)(w)$ for any $w \in W^\vee$, $v \in V^\vee$. Prove that $(\ker f)^\perp = \mathrm{Im}\ f^t$. Let $V = W$,prove that f is symmetric if and only if $f = f^t$, f is skew-symmetric if and only if $f = -f^t$

Corollary 1.7. *Symmetric case In the symmetric case ($V = W$ and $f \in S^2V$) we have that $(\ker f)^\perp = \mathrm{Im}\ f$ and the conormal space to DS_r at f is given by*

$$S^2(\ker f) \subseteq S^2(V^\vee)$$

1.4. $\mathrm{Chow}_\lambda(\mathbb{P}^n)$**.** Let's consider any partition $\lambda = (\lambda_1, \ldots, \lambda_s)$ of d, that is $d = \lambda_1 + \ldots + \lambda_s$, we may assume $\lambda_1 \geq \ldots \geq \lambda_s$. For any partition λ of d define $Chow_\lambda(\mathbb{P}^n)$ as the closure in $\mathbb{P}(S^d\mathbb{C}^{n+1})$ of the set of polynomials of degree d which are expressible as $l_1^{\lambda_1} \cdots l_s^{\lambda_s}$ where l_i are linear forms. In case $\lambda = 1^d$, we have the polynomials that are completely reducible as product of linear forms, this is sometimes called the split variety.

The dimension is $(\#\lambda)n$ where $\#\lambda$ is the number of summands in λ.

Moreover $Chow_\lambda(\mathbb{P}^n) \subset Chow_\mu(\mathbb{P}^n)$ precisely when $\lambda < \mu$.

Weyman [Wey] finds Hilbert function of these strata in several cases, namely when we have $n = 1$ an $(p, 1^{d-p})$.

On $\mathbb{P}^n$, $\lambda = d$ gives the d-Veronese variety and $\lambda = (d-1,1)$ gives its tangent developable. On $\mathbb{P}^1$, $\lambda = (2, 1^{d-2})$ gives the discriminant, of degree $2(d-1)$.

There is a formula due to Hilbert, for the degree of $Chow_\lambda(\mathbb{P}^1)$, reported by Chipalkatti[Chi]. Write $\lambda = 1^{e_1}2^{e_2}\ldots d^{e_d}$, then $\#\lambda = \sum e_i$ and the degree is

$$\binom{\#\lambda}{e_1,\ldots,e_d}\prod_r r^{e_r}$$

The normalization of $Chow_\lambda(\mathbb{P}^1)$ is $\prod_r \mathbb{P}(\mathrm{Sym}^{e_r}\mathbb{P}^1)$.

1.5. **The dual variety, the biduality Theorem.** The projective space $\mathbb{P}(V)$ consists of linear subspaces of dimension one of V. The dual space $\mathbb{P}(V^\vee)$ consists of linear subspaces of codimension one (hyperplanes) of V. Hence the points in $\mathbb{P}(V^\vee)$ are exactly the hyperplanes of $\mathbb{P}(V)$.

Let's recall the definition of dual variety. Let $X \subset \mathbb{P}(V)$ be a projective irreducible variety . A hyperplane H is called *tangent* to X if H contains the tangent space to X at some nonsingular point $x \in X$.

Definition 1.8. If $X \subset \mathbb{P}(V)$, then the *dual variety* is defined as

$$X^* := \overline{\{y \in \mathbb{P}(V^*) | y \text{ is tangent to } X \text{ at some } x \in X_{smooth}\}}$$

where the overline means Zariski closure.

Part of the biduality theorem below says that $X^{\vee\vee} = X$, but more is true. Consider the incidence variety W given by the closure of the set

$$\{(x,H) \in X \times \mathbb{P}(V^\vee) | x \text{ is a smooth point and } T_xX \subset H\}$$

W is identified in a natural way with the projective bundle $\mathbb{P}(N(-1)^\vee)$ where N is the normal bundle to X (see Remark 2.3).

Theorem 1.9. *(Biduality Theorem) Let $X \subset \mathbb{P}(V)$ be an irreducible projective variety. We have*

$$X^{\vee\vee} = X \tag{3}$$

Moreover if x is a smooth point of X and H is a smooth point of $X^\vee$, then H is tangent to X at x if and only if x, regarded as a hyperplane in $\mathbb{P}(V^\vee)$, is tangent to $X^\vee$ at H. In other words the diagram

$$\begin{tikzcd} & W \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X^\vee \end{tikzcd} \tag{4}$$

is symmetric.

For a proof, in the setting of symplectic geometry, we refer to [GKZ], Theorem 1.1 .

Note, as a consequence of the biduality theorem, that the fibers of both the projections of V over smooth points are linear spaces. This is trivial for the left projection, but it is not trivial for the right one. Let's record this fact

Corollary 1.10. *Let X be smooth and let H be a general tangent hyperplane (corresponding to a smooth point of $X^\vee$). Then $\{x \in X | T_xX \subseteq H\}$ is a linear subspace (this is called the contact locus of H in X).*

Theorem 1.11. [Oed] *The dual variety $Chow_\lambda(\mathbb{P}^n)^\vee$ is a hypersurface except for the two cases*
(i) $n = 1$ and $\lambda_s = 1$
(ii) $n \geq 2$ and $\lambda = (d-1, 1)$.

Example 1.12. When $X \subset \mathbb{P}^n$ is the rational normal curve, $\sigma_k(X)$ consists of polynomials which are sums of k powers, while $\sigma_k(X)^\vee$ consists of polynomials having k double roots. We get that $\sigma_k(X)^\vee = Chow_{2^k,1^{n-2k}}(\mathbb{P}^1)$.

1.6. **Abstract secant variety, secant degree, identifiability.** The abstract secant variety, with the symmetric product. Fibers of the map from abstract secant variety to secant variety. Secant degree and identifiability. Let $X \subset \mathbb{P}^n$. We consider the *abstract secant variety* $A\sigma_k(X)$, as defined in [CC2], i.e. the Zariski closure in $\mathrm{Sym}^k(X) \times \mathbb{P}^n$ of the variety of pairs $((p_1, \ldots, p_k), p)$ where $p \in \langle p_1, \ldots, p_k\rangle$, and the natural projection $\pi_k : A\sigma_k(X) \to \sigma_k(X)$, then we say that X is (generically) k-identifiable if the fibers of π_k consist of one point over general points of $\sigma_k(X)$.

Identifiability for specific tensors and for general tensor of a given rank.

Definition 1.13. Let $p_1, \ldots, p_k \in X$ be general, and let H be a general hyperplane tangent to $p_1, \ldots, p_k$, i.e. $\langle T_{p_1}X, \ldots, T_{p_k}X\rangle \subset H$. The k-contact locus of X (with respect to $p_1, \ldots, p_k, H$ is given by the points where H is tangent, namely by $\{p \in X | T_pX \subset H\}$.

Note that the k-contact locus contains, by definition, the points $p_1, \ldots, p_k$. It will be crucial, in the sequel, to understand the dimension of the k-contact locus.

Definition 1.14. The k-secant order of a variety, according to [CC2], is the number of irreducible components in the general fiber of the map $\pi_k : A\sigma_k(X) \to \sigma_k(X)$

1.7. **Flattenings and equations of Segre varieties.** We borrow from [Ot1] the following part. Let V_i be complex vector spaces of dimension $k_i + 1$ for $i = 0, \ldots, p$.

We are interested in the tensor product $V_0 \otimes \ldots \otimes V_p$, where the group $GL(V_0) \times \ldots \times GL(V_p)$ acts in a natural way.

Once a basis is fixed in each V_i, the tensors can be represented as multidimensional matrices of format $(k_0 + 1) \times \ldots \times (k_p + 1)$.

There are $p+1$ ways to cut a matrix of format $(k_0+1) \times \ldots \times (k_p+1)$ into parallel slices, generalizing the classical description of rows and columns for $p = 1$.

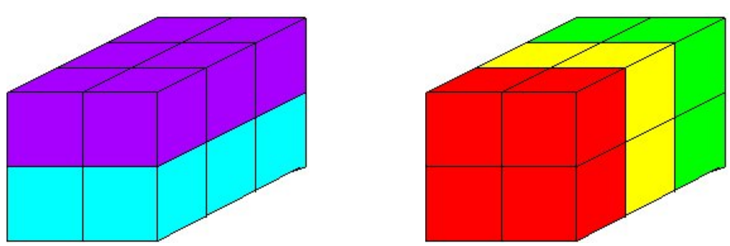


FIGURE 1. Two ways to cut a $3 \times 2 \times 2$ matrix into parallel slices

We illustrate a few properties of the Segre variety $\mathbb{P}(V_0) \times \ldots \times \mathbb{P}(V_p)$.

It is, in a natural way, a projective variety according to the Segre embedding

$$\begin{array}{ccc} \mathbb{P}(V_0) \times \ldots \times \mathbb{P}(V_p) & \longrightarrow & \mathbb{P}(V_0 \otimes \ldots \otimes V_p) \\ (v_0, \ldots v_p) & \mapsto & v_0 \otimes \ldots \otimes v_p \end{array}$$

In this embedding, the Segre variety coincides with the projectivization of the set of *decomposable* tensors. The proof of the following proposition is straightforward (by induction on p) and we omit it.

Proposition 1.15. *Every $\phi \in V_0 \otimes \ldots \otimes V_p$ induces, for any $i = 0, \ldots, p$ the contraction map*

$$C_i(\phi)\colon V_1^\vee \otimes \ldots \widehat{V_i^\vee} \ldots \otimes V_p^\vee \longrightarrow V_i$$

where the i-th factor is dropped from the source space. The tensor ϕ is decomposable if and only if $rk(C_i(\phi)) \leq 1$ for every $i = 0, \ldots, p$.

The previous proposition gives equations of the Segre variety as 2×2 minors of the contraction maps $C_i(\phi)$. These maps are called flattenings, because they are represented by bidimensional matrices obtained like in Figure 2.

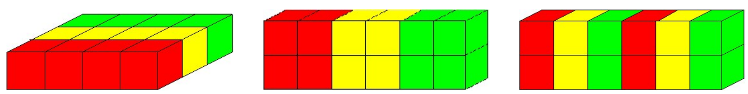


FIGURE 2. The three flattenings of the matrix in Figure 1. If the 2-minors of two of them vanish, then the matrix corresponds to a decomposable tensor (a point in the Segre variety)

Remark 1.16. In Prop. 1.15 it is enough that the rank conditions are satisfied for all $i = 0, \ldots, p$ except one .

Remark 1.17. Representation-theoretic structure of equations defining Segre varieties and their secant varieties is nontrivial. Note that, already for quadratic equations defining Segre variety $S^2(A \otimes B \otimes C) = \left(S^2 A \otimes S^2 B \otimes S^2 C\right) \oplus \left(S^2 A \otimes \wedge^2 B \otimes \wedge^2 C\right) \oplus \left(\wedge^2 A \otimes S^2 B \otimes \wedge^2 C\right) \oplus \left(\wedge^2 A \otimes \wedge^2 B \otimes S^2 C\right)$,

which can be proved by iterating the Cauchy identities

$$S^2(A \otimes B) = \left(S^2 A \otimes S^2 B\right) \oplus \left(\wedge^2 A \otimes \wedge^2 B\right),$$
$$\wedge^2(A \otimes B) = \left(S^2 A \otimes \wedge^2 B\right) \oplus \left(\wedge^2 A \otimes S^2 B\right).$$

The 2×2 of the three flattenings we have seen, give respectively the three summands

$$\left(S^2 A \otimes \wedge^2 B \otimes \wedge^2 C\right), \qquad \left(\wedge^2 A \otimes S^2 B \otimes \wedge^2 C\right), \qquad \left(\wedge^2 A \otimes \wedge^2 B \otimes S^2 C\right).$$

1.8. **Multidimensional matrices and the local geometry of Segre varieties.** A feature of the Segre variety is that it contains a lot of linear subspaces.

For any point $x = v_0 \otimes \ldots \otimes v_p$, the linear space $v_0 \otimes \ldots V_i \ldots \otimes v_p$ passes through x for $i = 0, \ldots, p$; it can be identified with the fiber of the projection

$$\pi_i\colon \mathbb{P}^{k_0} \times \ldots \times \mathbb{P}^{k_p} \longrightarrow \mathbb{P}^{k_0} \times \ldots \widehat{\mathbb{P}^{k_i}} \ldots \times \mathbb{P}^{k_p}$$

We will denote the projectivization of the linear subspace $v_0 \otimes \ldots V_i \ldots \otimes v_p$ as $\mathbb{P}^{k_i}_x$.

These linear spaces have important properties described by the following proposition.

Proposition 1.18. *Let $x \in X = \mathbb{P}^{k_0} \times \ldots \times \mathbb{P}^{k_p}$.*

(i) The tangent space at x is the span of the $p+1$ linear spaces $\mathbb{P}^{k_i}_x$, that is T_pX is the projectivization of $\oplus_i v_0 \otimes \ldots V_i \ldots \otimes v_p$

(ii) The tangent space at x meets X in the union of the $p+1$ linear spaces $\mathbb{P}^{k_i}_x$.

(iii) Any linear space in X passing through x is contained in one of the $p+1$ linear spaces $\mathbb{P}^{k_i}_x$.

Proof. The tangent vector to a path $v_0(t) \otimes \ldots \otimes v_p(t)$ for $t = 0$ is $\sum_{i=0}^{p} v_0(0) \otimes \ldots v_i'(0) \ldots \otimes v_p(0)$. Since $v_i'(0)$ may be chosen as an arbitrary vector, the statement (i) is clear.

(ii) Fix a basis $\{e_j^0, \ldots, e_j^{k_j}\}$ of V_j for $j = 0, \ldots, p$ and let $\{e_{j,0}, \ldots, e_{j,k_j}\}$ be the dual basis. We may assume that x corresponds to $e_0^0 \otimes \ldots \otimes e_p^0$. Consider a decomposable tensor ϕ in the tangent space at x, so $\phi = v_0 \otimes \ldots \otimes e_p^0 + \ldots + e_0^0 \otimes \ldots \otimes v_p$ for some v_i. We want to prove that v_i and e_i^0 are linearly independent for at most one index i. Otherwise we may assume $\dim(v_0, e_0^0) = 2$, $\dim(v_1, e_1^0) = 2$. Consider the contraction

$$\begin{array}{rcl} C_0(\phi)(e_{1,0} \otimes e_{2,0} \otimes \ldots \otimes e_{p,0}) & = & v_0 \quad + \qquad\qquad (\ldots)e_0^0 \\ C_0(\phi)(e_{1,1} \otimes e_{2,0} \otimes \ldots \otimes e_{p,0}) & = & \left(e_{1,1}(v_1) + \sum_{j=2}^{p} e_{j,0}(v_j)\right) e_0^0 \end{array}$$

Since we may assume also $e_{1,1}(v_1) \neq 0$, by replacing $e_{1,1}$ with a scalar multiple we have also $\left(e_{1,1}(v_1) + \sum_{j=2}^{p} e_{j,0}(v_j)\right) \neq 0$. This implies that rank $C_0(\phi) \geq 2$ which is a contradiction.

(iii) A linear space in X passing through x is contained in the tangent space at x, hence the statement follows from (ii). $\square$

1.9. **Exercises for Lecture 1.**

(1) Find the tangent space of $D_r \subseteq \mathrm{Hom}\ (K^{n+1}, K^{m+1})$ at the following matrices of rank r

i)

$$\begin{bmatrix} & & \\ & 1 & \\ & & \\ & & \end{bmatrix} \text{ where 1 is at place } (i,j)$$

ii)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Answer $a_{i,j} = x_i + y_j$

iii)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

iv)The tangent space to the Segre variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$ at the point corresponding to $a_{ijk} = 1\ \forall i, j, k$.

(2) Compute dimension and degree of the split variety $T \subset \mathbb{P}^9$ of triangles in the plane. What are its equations ?

Answer: Dimension is 6, it is birational to $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. To compute the degree, cut with six special hyperplanes, so that you have to compute how many triangles contain six given points. There are $\binom{6}{2} = 15$ triangles, so that the degree is 15.

Equations can be computed by noting that a triangle contains only flexes, and a cubic is a triangle iff it is proportional to its Hessian. The Hessian is again a cubic, with entries of degree three. Get 2×2 minors of 2×15 matrix, obtain quartic equations.

(3) Compute the locus in $\mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$ where the minors of one flattening vanishes, precisely, if x_i for $i = 1, 2, 3$ are the three 2×2 slices, compute

$$\{x \in \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) | \, (x_1|x_2|x_3)_2 = 0$$

where $(\ldots)_2$ means the set of all 2×2-minors.

2. Lecture 2

Abstract of Lecture 2: contact loci of Segre varieties, dual of Segre variety, triangular inequality, its geometric explanation. Dual of secant varieties, interpretation with the Terracini Lemma. Weak defectivity and its interpretation with dual varieties. Equations of σ_2 of a Segre variety, again by flattenings, by Landsberg and Manivel. Hamming distance, singular locus of 2*-secant variety of a Segre variety. The secant order is the same for a variety and for its secant locus. Examples of non identifiable Veronese varieties, explanation with contact loci which contains elliptic curves. Classification of weakly defective Veronese varieties, by [Chiantini-Ciliberto], [Mella], [Ballico]. Examples of non identifiable Segre varieties, explanation with contact loci which contains elliptic curves. The "biggest" example* $3 \times 6 \times 6$.

2.1. Contact loci of Segre varieties, dual of Segre varieties.

As a first application we compute the dimension of the dual to a Segre variety.

Theorem 2.1. *[Contact loci in Segre varieties] Let $X = \mathbb{P}^{k_0} \times \ldots \times \mathbb{P}^{k_p}$.*

(i) If $k_0 \geq \sum_{i=1}^p k_i$ then a general hyperplane tangent at x is tangent along a linear space of dimension $k_0 - \sum_{i=1}^p k_i$ contained in the fiber $\mathbb{P}^{k_0}_x$. In this case the codimension of $X^\vee$ is $1 + k_0 - \sum_{i=1}^p k_i$.

(ii) If $k_0 \leq \sum_{i=1}^p k_i$ then a general hyperplane tangent at x is tangent only at x. In this case $X^\vee$ is a hypersurface.

(iii) The dual variety $X^\vee$ is a hypersurface if and only if the following holds

$$\max k_i = k_0 \leq \sum_{i=1}^p k_i$$

Proof. We remind that, by Proposition 1.18 (i), a hyperplane H is tangent at x if and only if it contains the $p+1$ fibers through x. By Corollary 1.10 a general hyperplane is tangent along a linear variety. By Prop. 1.18 (iii) a linear variety in X it is contained in one of the fibers. Let H be a general hyperplane tangent at x. We inspect the fibers through y when $y \in \mathbb{P}^{k_0}_x$. The locus where H contains the fiber $\mathbb{P}^{k_i}_y$ is a linear space in $\mathbb{P}^{k_0}_x$ of codimension k_i, indeed the fibers can be globally parametrized by y

plus other k_i independent points. This description proves (i), because the variety W in (4) has the same dimension of a hypersurface in $\mathbb{P}(V^\vee)$ and we just computed the general fibers of p_2. Also (ii) follows by the same argument because the conditions are more than the dimension of the space. (iii) is a consequence of (i) and (ii). □

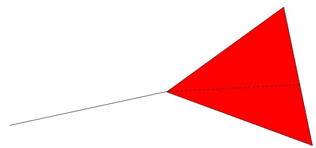


FIGURE 3. The tangent space at a point $x \in X = \mathbb{P}^1 \times \mathbb{P}^2$ cuts X into two linear spaces meeting at x, the general hyperplane tangent at x is tangent along a line (dotted in the figure)

Definition 2.2. A format $(k_0+1) \times \ldots \times (k_p+1)$ with $k_0 = \max_j k_j$ is called a boundary format if $k_0 = \sum_{i=1}^p k_i$. In other words, the boundary format corresponds to the equality in (iii) of Theorem 2.1

Remark 2.3. According to [L], Theorem 2.1 says that for a Segre variety with normal bundle N, the twist $N(-1)$ is ample if and only if the inequality $\max k_i = k_0 \leq \sum_{i=1}^p k_i$ holds.

Definition 2.4. Let

$$\max k_i = k_0 \leq \sum_{i=1}^p k_i$$

The equation of the dual variety to $\mathbb{P}^{k_0} \times \ldots \times \mathbb{P}^{k_p}$ is called the hyperdeterminant.

2.2. **Dual to the variety of matrices of bounded rank. Dual to secant varieties, interpretation with Terracini Lemma.** Note that for $p=1$ the dual variety to $D_1 = \mathbb{P}^{k_0} \times \mathbb{P}^{k_1}$ is a hypersurface if and only if $k_0 = k_1$ (square case). This is better understood by the following result.

Theorem 2.5. *Let $k_0 \geq k_1$. In the projective spaces of $(k_0+1) \times (k_1+1)$ matrices the dual variety to the variety D_r (defined in formula (1)) is D_{k_1+1-r}.*

When $k_0 = k_1$ (square case) the determinant hypersurface is the dual of D_1.

The above Theorem is important because it gives a geometric interpretation of the determinant, as the dual of the Segre variety. This is the notion that better generalizes to multidimensional matrices.

In order to prove the Theorem 2.5 we need the following proposition

Proposition 2.6. *Let X be a irreducible projective variety. For any k*

$$(\sigma_{k+1}(X))^\vee \subset (\sigma_k(X))^\vee$$

Proof. The proposition is almost a tautology after the Terracini Lemma. The dual to the $(k+1)$-th secant variety $(\sigma_{k+1}(X))^\vee$ is defined as the closure of the set of hyperplanes H containing $T_z\sigma_{k+1}(X)$ for z being a smooth point in $\sigma_{k+1}(X)$, so $z \in < x_1, \ldots, x_{k+1} >$ for general $x_i \in X$. By the Terracini Lemma (Prop. 1.1) H contains $T_{x_1}, \ldots T_{x_{k+1}}$, hence H contains $T_{z'}\sigma_k(X)$ for the general $z' \in < x_1, \ldots, x_k >$ (removing the last point). □

Corollary 2.7.

$$\dim\left(\sigma_k(X)\right)^\vee \leq N-k$$

Proof of Theorem 2.5. Due to Proposition 2.6 and Corollary 2 we have the chain of inclusions

$$D_1^\vee \supset D_2^\vee \supset \ldots \supset D_{k_1}^\vee$$

By the biduality Theorem any inclusion must be strict. Since the $D_i^\vee$ are $GL(V_0) \times GL(V_1)$-invariant, and the finitely many orbit closures are given by D_i, the only possible solution is that the above chain coincides with

$$D_{k_1} \supset \ldots \supset D_1$$

. □

Remark 2.8. A common misunderstanding after Theorem 2.5 is that $X \subset Y$ implies the converse inclusion $X^\vee \supset Y^\vee$. This is in general false. The simplest counterexample is to take X to be a point of a smooth (plane) conic Y. Here $X^\vee$ is a line and $Y^\vee$ is again a smooth conic.

Remark 2.9. The proof of Theorem 2.5 is short, avoiding local computations, but rather indirect.

We point out the elegant proof of Theorem 2.5 given by Eisenbud in Prop. 1.7 of [E88], which gives more information. Eisenbud considers $V_0 \otimes V_1$ as the space of linear maps $Hom(V_0^\vee, V_1)$ and its dual $Hom(V_1, V_0^\vee)$. These spaces are dual under the pairing $< f, g >:= tr(fg)$ for $f \in Hom(V_0^\vee, V_1)$ and $g \in Hom(V_1, V_0^\vee)$. Eisenbud proves that if $f \in D_r \setminus D_{r-1}$ then the tangent hyperplanes at f to D_r are exactly the g such that $fg = 0$, $gf = 0$. These conditions force the rank of g to be $\leq k_1 + 1 - r$. Conversely any g of rank $\leq k_1 + r - 1$ satisfies these two conditions for some f of rank r, proving Theorem 2.5 .

Interesting examples: the dual variety of $\sigma_5(v_4(\mathbb{P}^2))$ (defective case of Clebsch quartics, see Alessandra lecture) is given by quartics with five double points, they are double conics.

The dual variety of $\sigma_9(v_6(\mathbb{P}^2))$ is given by double cubics.

2.3. **Weak defectivity and interpretation with dual varieties.**

Proposition 2.10. *Let $X \subset \mathbb{P}^N$. The following are equivalent*

(i) The general hyperplane H containing $\langle T_{x_1}X, \ldots, T_{x_k}X\rangle$ for general $x_1, \ldots, x_k$ is tangent to X only at $x_1, \ldots, x_k$, i.e. the k-contact locus with respect to $x_1, \ldots, x_k, H$ consists exactly of the points $x_1, \ldots, x_k$,

(ii) The general hyperplane containing $\langle T_{x_1}X, \ldots, T_{x_k}X\rangle$ for general $x_1, \ldots, x_k$ is tangent to X only at finitely many points, i.e. i.e. the k-contact locus with respect to $x_1, \ldots, x_k, H$ is zero-dimensional.

(iii) $\dim\left[\sigma_k(X)\right]^\vee = N-k$, that is a general hyperplane tangent to $\sigma_k(X)$ is tangent along a linear space of projective dimension $k-1$.

Definition 2.11. X is called k-weakly defective if the conditions of previous proposition are violated, i.e. if $\dim\left[\sigma_k(X)\right]^\vee \leq N-k-1$

Theorem 2.12 (Terracini)**.** *A k-defective variety is also k-weakly defective.*

Proof. k-defectivity has been defined in Alessandra's lectures. The proof is a straightforward application of Terracini Lemma. □

The converse to previous Theorem does not hold. The simplest counterexample is $\mathbb{P}^1 \times \mathbb{P}^2$ which is 1-weakly defective.

In conclusion we have the basic

Theorem 2.13 (Chiantini-Ciliberto)**.** *If X is not k-weakly defective, then it is k-identifiable.*

2.4. **Equations of σ_2 of a Segre variety, again by flattenings, by Landsberg and Manivel.** The vanishing of 3×3 minors of the flattenings give equations of 2-secant variety.

For $p \geq 3$, the $(p+1) \times (p+1)$ minors of the flattening vanish on the p-secant variety, but they are not sufficient in general to define it. The first interesting case is $\sigma_3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$, which is an hypersurface of degree 9. It is analogous to the Aronhold invariant in the symmetric case.

2.5. **The Hamming distance.** Notion of distance on Segre variety (and on Grassmann variety), borrowed from [CGG1].

Proposition 2.14. *Let $X = \mathbb{P}(A_1) \times \ldots \times \mathbb{P}(A_k)$ be a Segre variety. Let $x, y \in X$. The following are equivalent*

- *(a) The minimum length of a sequence of lines on X, joining x and y is s*
- *(b) The minimum degree of a rational normal curve on X, joining x and y is s.*
- *(c) $x = p_1 \otimes \ldots \otimes p_k$ and $y = q_1 \otimes \ldots \otimes q_k$ where the sets $\{p_1, \ldots, p_k\}, \{q_1, \ldots, q_k\}$ of projective points have exactly s different elements.*

This defines a distance, which is like the Hamming distance

Precisely, setting $d(x, y) = s$ as above, then d is a distance on X, in the sense that it satisfies the usual properties

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$

Proof. (i) and (ii) are trivial. In order to prove (iii), use property (a). □

As observed in [CGG1], this is equivalent to play with rooks on a k-dimensiponal chessboard. The points at distance ≤ 1 from x are exactly the points attacked from a rook placed at x. The singular locus of 2-secant variety to a Segre variety.

Theorem 2.15 (Michałek, Oeding, Zwiernik)**.** [MOZ] *Let X be a Segre variety with at leat three factors. The singular locus of $\sigma_2(X)$ is equal to* $\displaystyle\bigcup_{\{x,y|d(x,y)\leq 2\}} < x, y >$

Proposition 2.16. *Let $x \in X$ a Segre variety. Then $X \cap T_xX = \{y \in X | d(x, y) \leq 1\}$ If a linear subspace L is contained in X, then it is contained in $X \cap T_xX$ $\forall x \in L$.*

2.6. **The secant order is the same for a variety and for its secant locus.** We have to distinguish between X of first type (k-contact locus is irreducible, one interesting example is $\mathbb{P}(\mathbb{C}^2 \times \mathbb{C}^3 \times \mathbb{C}^3)$, where the 2-contact locus is given by a twisted cubic) and X of second type (k-contact locus has k irreducible components, one interesting example is $\mathbb{P}(\mathbb{C}^{17} \times \mathbb{C}^{18} \times \mathbb{C}^{240})$, where the 269-tangentially contact locus is given by 269 distinct linear spaces of projective dimension 140).

Theorem 2.17 (Chiantini-Ciliberto)**.** [CC2] *If the k-contact locus C of X, with respect to general $p_1, \ldots, p_k, H$ is irreducible, then the secant degree of X is equal to the secant degree of C.*

2.7. **Examples of non identifiable Veronese varieties, explanation with contact loci which contains elliptic curves.** All defective cases appearing in AH classification are also weakly defective.

Beyond these, there are the following two:

Through 9 points of $\mathbb{P}^2$ there is a (cubic) elliptic curve C, and $v_6(C)$ embeds in $v_6(\mathbb{P}^2) \subset \mathbb{P}S^6\mathbb{C}^3$.

$v_6(\mathbb{P}^2)$ is 9-weakly defective, the 6-contact locus is $v_6(C)$. Note that $\left[\sigma_9(v_6(\mathbb{P}^2))\right]^\vee$ consists of double cubics, and has dimension $9 < 27 - 9 = 18$.

Through 8 points of $\mathbb{P}^3$ there is a elliptic normal curve C of degree 4, which is the intersection of the pencil of quadrics $\langle Q_1, Q_2 \rangle$ through the 8 points and $v_4(C)$ embeds in $v_4(\mathbb{P}^3) \subset \mathbb{P}S^4\mathbb{C}^4$.

$v_4(\mathbb{P}^3)$ is 8-weakly defective, because all quartics which are singular in the 8 points belong to the span $\langle Q_1^2, Q_1Q_2, Q_2^2 \rangle$, and they are singular on C. The 8-contact locus is $v_4(C)$. Note that $\left[\sigma_8(v_4(\mathbb{P}^3))\right]^\vee$ consists of reducible quartics, which are union of two quadrics, and has dimension $18 < 34 - 8 = 26$.

2.8. **Classification of weakly defective Veronese varieties, by [Chiantini-Ciliberto], [Mella], [Ballico].**

Theorem 2.18. [CC1, Mel, Bal] *The previous two example are the only weakly defective Veronese varieties (beyond the defective ones).*

2.9. **Examples of non identifiable Segre varieties, explanation with contact loci which contains elliptic curves. The "biggest" example $3 \times 6 \times 6$.**

$\mathbb{P}^2 \times \mathbb{P}^{2h} \times \mathbb{P}^{2h}$ is $(3h + 1)$-defective. In case $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, the 4-contact locus (and also the 4-tangentially contact locus) is $v_3(\mathbb{P}^2)$.

$\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3$ is 5-defective

$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n \times \mathbb{P}^n$ is $(2n + 1)$-defective

$\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$ is 6-weakly defective. In case $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$, the 6-contact locus (and also the 4-tangentially contact locus) is an elliptic curve of degree 12.

$\mathbb{P}^2 \times \mathbb{P}^5 \times \mathbb{P}^5$ is 8-weakly defective.

$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is 5-weakly defective.

Remark 2.19. In [AOP] there is a conjecture stating that the above ones are the only defective examples for Segre varieties, beyond certain cases where one of the factors has dimension much bigger than the others (unbalanced, these cases are well understood). In [BCO] there is a conjecture stating that the above ones are the only weakly defective examples for Segre varieties, beyond certain cases where one of the factors has dimension much bigger than the others (unbalanced, these cases are again well understood).

There are several defective examples for Segre-Veronese varieties, see [BBC], [LP] and the conjecture in [AB].

2.10. **Exercises for Lecture 2.**

(1) Describe the dual varieties $\left[\mathrm{Chow}_\lambda(\mathbb{P}^1)\right]^\vee$ for any partition λ of 4.
(2) Multilinear rank according to Carlini-Kleppe[CaKl]. Let $t \in V_1\otimes V_2\otimes V_3$ which induces $t_1\colon V_1^* \to V_2\otimes V_3$, $t_2\colon V_2^* \to V_1\otimes V_3$, $t_3\colon V_3^* \to V_1\otimes V_2$.
   Define $r_i := \mathrm{rk}\ t_i$. Prove that $r_1 \le r_2r_3$, $r_2 \le r_1r_3$, $r_3 \le r_1r_2$.

3. Lecture 3

Abstract for Lecture 3: Kruskal rank and Kruskal criterion. Tangencially weak defectivity. 1-tangentially weak defective varieties and Gauss map. Criteria for identifiability. Examples for Segre varieties. Strassen Theorem. Weierstrass canonical form. Kronecker canonical form (Fibonacci blocks ?).

3.1. **Kruskal rank and Kruskal criterion.** The most celebrated result about uniqueness of decomposition of specific tensors is due to Kruskal [K]. It is often quoted in terms of Kruskal's rank.

A collection of vectors $A = \{a_1, \ldots, a_p\}$ in a vector space V is said to have Kruskal rank r, if r is the minimal such that all subsets of r vectors of A , are linearly independent. We denote the Kruskal rank by Krk.

Theorem 3.1 (Kruskal criterion)**.** *Consider a tensor* $t = \sum_{i=1}^r a_i^{(1)} \otimes a_i^{(2)} \otimes a_i^{(3)}$. *Denote* $A = \{a_1^{(1)}, \ldots, a_r^{(1)}\}$, $B = \{a_1^{(2)}, \ldots, a_r^{(2)}\}$, $C = \{a_1^{(3)}, \ldots, a_r^{(3)}\}$.

If

$$r \le \frac{1}{2}\left[\mathrm{Krk}(A) + \mathrm{Krk}(B) + \mathrm{Krk}(C)\right] - 1$$

then t has rank r and it is identifiable, i.e. the decomposition displayed is the unique one (up to reorder the summands)

The statement generalizes to d factors, the relevant inequality becomes

$$r \le \frac{1}{2}\left[\sum_{i=1}^d \mathrm{Krk}(A_i) - d + 1\right]$$

A consequence of Kruskal's criterion is the following statement, which applies to general tensors (see Corollary 3 in [AMR]).

Proposition 3.2. *(***Kruskal's criterion for general tensors***) The general tensor $t \in A\otimes B\otimes C$ has a unique decomposition if*

$$k \le \frac{1}{2}\left[\min(a,k) + \min(b,k) + \min(c,k) - 2\right]$$

In the cubic case, the general tensor $t \in A\otimes A\otimes A$ of rank k has a unique decomposition if

$$k \le \frac{3a-2}{2}$$

Kruskal's result is so important in the literature, that recently there have been published (at least!) three different proofs [Land1, R, SS].

3.2. **Tangentially weakly defective varieties, criterion for identifiability.** Let $p_1, \ldots, p_k \in X$ be general. The k-tangentially contact locus of a variety X is given by the points $\{p \in X | T_pX \subset \langle T_{p_1}X, \ldots, T_{p_k}X\rangle\}$.

A variety X is said to be k-tangentially weakly defective if the k-tangentially contact locus has positive dimension.

We have the chain of implications

$$k\text{-defective} \implies k\text{-tangentially weakly defective} \implies k\text{-weakly defective}$$

where no of these implications can be reversed.

Proposition 3.3. *Let $X \subset \mathbb{P}^N$ be a non-degenerate, irreducible variety of dimension n. Consider the following statements:*

(i) X is k-identifiable

(ii) Given k general points $x_1, \ldots x_k \in X$, the span $< \mathbb{T}_{x_1}X, \ldots, \mathbb{T}_{x_k}X >$ contains $\mathbb{T}_xX$ only if $x = x_i$ for some $i = 1, \ldots k$, i.e. X is not k-tangentially weakly defective.

(iii) there exists a set of k particular points $x_1, \ldots x_k \in X$, such that the span $< \mathbb{T}_{x_1}X, \ldots, \mathbb{T}_{x_k}X >$ contains $\mathbb{T}_xX$ only if $x = x_i$ for some $i = 1, \ldots k$.

Then we have (iii) $\implies$ (ii) $\implies$ (i).

Proof. (iii) $\implies$ (ii) follows at once by semicontinuity.

Let us prove that (ii) $\implies$ (i). Take a general point $u \in S_k(X)$ and assume that u belongs to the span of points $x_1, \ldots, x_k \in X$. By the generality of u, we may assume that $x_1, \ldots, x_k$ are general points of X. If u also belongs to the span of points $y_1, \ldots, y_k \in X$, with at least one of them, say y_1, not among the x_i's, then, by Terracini's Lemma, the span of the tangent spaces to X at the points x_i's, which is the tangent space to $S_k(X)$ at u, also contains the tangent space to X at y_1. This contradicts (ii). □

Proposition 3.4. *1-weakly defective $\iff$ the dual of X is not a hypersurface*

1-tangentiall weakly defective $\iff$ the Gauss map sending each point to its tangent space sitting in a Grassmannian is degenerate $\iff$ X is a developable scroll, i.e. a scroll such that along the general generating space, the tangent space to X is constant (in particular it is singular).

Remind that developable scroll surfaces are cones or tangent surfaces to curves.

Criterion of generic identifiability.

Proposition 3.5. *If there exists a set of k particular points $x_1, \ldots x_k \in X$, such that the span $\langle \mathbb{T}_{x_1}X, \ldots, \mathbb{T}_{x_k}X\rangle$ contains $\mathbb{T}_xX$ only if $x = x_i$ for some $i = 1, \ldots k$, then X is k-identifiable.*

3.3. **Strassen Theorem about weak defectivity.** In Strassen Theorem (see main lemma 3.1 of [BCO]) we need a technical fact, we have to work with r-dimensional subspaces and we compare X and $X \times \mathbb{P}^m$.

Lemma 3.6. *Let X be a smooth non-degenerate projective subvariety of $\mathbb{P}^N$, of dimension n. Let Y denote the canonical Segre embedding of $X \times \mathbb{P}^m$ into $\mathbb{P}^M$, $M =$*

$mN+m+N$. Fix k with $(n+1)k < N+1$ and $r < N$ such that $r+1 \geq (n+m+1)k$. Assume that a general linear subspace of $\mathbb{P}^N$, of dimension r, which is tangent to X at k general points, is not tangent to X elsewhere.

Then the general linear subspace of $\mathbb{P}^M$, of dimension $mr+m+r$, which is tangent to Y at $(m+1)k$ general points, is not tangent to Y elsewhere.

Proof. First of all, notice that $\dim(Y) = (m+n)$, and $(m+1)(r+1) \geq (m+n+1)(m+1)k$. Thus, by an obvious parameter count, there are linear subspaces of dimension $mr+m+r$ which are tangent to Y at $(m+1)k$ general points.

Fix $m+1$ independent points $p_0, \dots, p_m$ of $\mathbb{P}^m$ and for $j = 0, \dots, m$ take k general points q_{ij} of the fiber $X \times \{p_j\}$. Call π_j the natural projection of $X \times \{p_j\}$ to X.

For $h = 0, \dots, m$, fix a general linear subspace $R_h \subset \mathbb{P}^N \times \{p_h\}$, of dimension r, which is tangent to $X \times \{p_h\}$ at the k points $q_{1h}, \dots, q_{kh}$ and passes through the points $\pi_j(q_{ij}) \times \{p_h\}$, for $j \neq h$. Since $r+1 \geq k(n+1)+km$, such spaces R_h exist. Moreover R_h it is tangent to $X \times \{p_h\}$ only at the point $q_{1h}, \dots, q_{kh}$, by our assumption on X.

Let R be the span of all the R_h's. We claim that R, which is a linear subspace of dimension $mr+m+r$, is tangent to Y at all the points q_{ij}, and it is not tangent to Y elsewhere. This will conclude the proof of the lemma, by semicontinuity.

First notice that for all i, j, R contains $m+1$ general points of $\{\pi_j(q_{ij})\} \times \mathbb{P}^m$, hence it contains these fibers. Since R also contains the tangent spaces to $X \times \{p_h\}$ at the points q_{ih}'s for all h, then it is tangent to Y at all the points q_{ij}'s.

Assume now that there exists a point $x \in Y$, different from the q_{ih}'s, such that R is tangent to Y at x. Call x' the projection of x to $\mathbb{P}^m$, so that in some coordinate system, we can write $x' = a_0p_0 + \cdots + a_mp_m$. There is at least one of the a_i's, say a_0, which is non-zero. Assume that also $a_1 \neq 0$. Then, the projection of R to $\mathbb{P}^N \times \{p_0\}$, which by construction coincides with R_0, is also tangent to $X \times \{p_0\}$ at the projection of q_{k1}. Indeed, we have a splitting $\mathbb{C}^{N+1} \otimes \mathbb{C}^{m+1} = \big(\mathbb{C}^{N+1} \otimes \langle p_0 \rangle\big) \oplus \big(\mathbb{C}^{N+1} \otimes \langle p_1, \dots, p_m \rangle\big)$ and the projection comes from the restriction of the first (linear) projection in this splitting. By the generality of the choice of the q_{ij}'s, q_{k1} cannot coincide with any of the points $q_{10}, \dots, q_{k0}$. Thus we get a contradiction.

So, we conclude that $a_1 = 0$. Similarly we get that $a_2 = \cdots = a_m = 0$. It follows that $x = x'$ belongs to $X \times \{p_0\}$ and since R_0 is tangent to $X \times \{p_0\}$ at x, then x must coincide with some point q_{i0}. $\square$

Remark 3.7. It is worthy of spending one Remark to point out that, by semicontinuity, if a general linear subspace of $\mathbb{P}^N$, of dimension r, which is tangent to X at k general points, is not tangent to X elsewhere, then the same phenomenon occurs for general linear subspaces of dimension $r-1$, $r-2$, and so on.

The Lemma, together with Theorem 3.5, produces the following general principle:

Corollary 3.8. *With the same assumptions on X of Lemma 3.6, then $Y = X \times \mathbb{P}^m$ is $(m+1)k$-identifiable.*

Thus we will prove the identifiability of Segre products, starting with a X who is a Segre product for which we know that the assumptions of Lemma 3.6 hold (by computer-aided specific computations or by Theorem 3.9 below) and then extending the number of factors of X, and using Lemma 3.6 inductively.

The following Theorem is due to Strassen in case c odd.

Theorem 3.9. *Let X be the product of three projective spaces $X = \mathbb{P}^a \times \mathbb{P}^b \times \mathbb{P}^c$, $2 < a \leq b \leq c$, naturally embedded in $\mathbb{P}^N$, with $N = (a+1)(b+1)(c+1) - 1$. Then a general linear subspace L of codimension $a+b+2$ in $\mathbb{P}^N$, that contains the span of the tangent spaces to X at k general points, with:*

$$k \leq \frac{(a+1)(b+1)(c+1)}{a+b+c+1} - c - 1,$$

is not tangent to X elsewhere.

Proof. Let $\mathbb{P}^c = \mathbb{P}(C)$, where C is a vector space of dimension $c+1$. Fix one vector $v_0 \in C$ and split C in a direct sum $C = \langle v_0 \rangle \oplus C'$, where C' is a supplementary subspace of dimension c. From the geometric point of view, this is equivalent to split the product X in two products

$$X' = \mathbb{P}^a \times \mathbb{P}^b \times \mathbb{P}^{c-1} \text{ and } X'' = \mathbb{P}^a \times \mathbb{P}^b \times \{P_0\} = \mathbb{P}^a \times \mathbb{P}^b.$$

Fix general points $P_1, \ldots, P_k \in X'$, with $P_i = v_i \otimes w_i \otimes u_i$ and let $Q_1, \ldots, Q_k$, $Q_i = v_i \otimes w_i$, be the corresponding points of X''. The linear span of the Q_i's is a space of dimension $k-1$ in $\mathbb{P}^{N''}$, where $N'' = ab + a + b$.

By assumption $k - 1 \leq N'' - \dim(X'') = N'' - a - b$. Indeed if $c + 1 \geq a + b$ then

$$k - 1 \leq \frac{(a+1)(b+1)(c+1)}{a+b+c+1} - a - b - 1 \leq (a+1)(b+1) - a - b - 1.$$

If $c + 1 < a + b$ then $k < (a+1)(b+1)/2$ and $(a+1)(b+1)/2 > a + b$.

Fix a linear space L'' of codimension $a+b+1$ in $\mathbb{P}^{N''}$, which contains the span of the Q_i's. Since the points Q_i's are general in X'', it follows from the Theorem 2.6 in [CC1] (it is a generalization of the "trisecant lemma") that the linear space L'' does not meet X'' in other points. Moreover L'' is not tangent to X'' at any of the points Q_i's.

Let L' be a hyperplane in $\mathbb{P}^{N'}$, $N' = (a+1)(b+1)c - 1$, which is tangent to X' at the points P_i's. The hyperplane L' exists, since by assumption

$$k(\dim(X') + 1) < (a+1)(b+1)(c+1) - c(a+b+c) < N - 1.$$

Let L be the linear span of L' and L''. L has codimension $a+b+2$ and it is tangent to X at the k points $P_1, \ldots, P_k$, since it contains the tangent spaces to X' at the P_i's, moreover it contains the points Q_i's, so it contains the fiber $\mathbb{P}^c$ passing through each P_i.

We want to exclude that L is tangent to X at any other point $P \neq P_i$. Call Q the projection of P to X''. If L is tangent to X at P, then it must contain the fiber $\mathbb{P}^c$ passing through P, thus it contains Q. This proves that Q is one of the Q_i's (say $Q = Q_1$), since L does not meet X'' elsewhere. But then L contains the fibers $\mathbb{P}^a$ and $\mathbb{P}^b$ at two points P, P_1 with the same projection to X''. Thus it contains these fibers at any point of the line ℓ joining P, P_1. As ℓ contains Q_1, we get a contradiction, since $L'' = L \cap \mathbb{P}^{N''}$ is not tangent to X'' at Q_1. □

Corollary 3.10. *Let X be the product of three projective spaces $X = \mathbb{P}^a \times \mathbb{P}^b \times \mathbb{P}^c$, $2 < a \leq b \leq c$, naturally embedded in $\mathbb{P}^N$, with $N = (a+1)(b+1)(c+1) - 1$. Then for*

$$k \leq \frac{(a+1)(b+1)(c+1)}{a+b+c+1} - c - 1,$$

X is k-identifiable.

Proof. Follows immediately from the previous Theorem and the criterion of identifiability. □

3.4. **Weierstrass canonical form and Kac's Theorem.** Note that the only format $2\times b\times c$ where the hyperdeterminant exists (so that the triangular inequality is satisfied) are $2 \times k \times k$ and $2 \times k \times (k + 1)$.

The $2 \times k \times k$ case has an interesting behaviour. We record the main classification result in the nondegenerate case. We denote by Det the hyperdeterminant, that is the equation of the dual to the Segre variety.

Theorem 3.11 (Weierstrass)**.** *Let A be a tensor of size $2 \times k \times k$ and let A_0, A_1 be the two slices. Assume that $Det(A) \neq 0$. Under the action of $GL(k) \times GL(k)$ A is equivalent to a matrix where A_0 is the identity and $A_1 = diag(\lambda_1, \dots, \lambda_k)$. In this form the hyperdeterminant of A is equal to $\prod_{i<j}(\lambda_i - \lambda_j)^2$.*

Corollary 3.12. *The general tensor of size $2 \times k \times k$ has rank k and it has a unique decomposition, that, in suitable basis $\{a_0, a_1\}$, $\{b_1, \dots, b_k\}$, $\{c_1, \dots, c_k\}$, is*

$$\sum_{i=1}^{k} (a_0 + \lambda_i a_1) \otimes b_i \otimes c_i$$

Note that the group $SL(2)\times SL(k)\times SL(k)$ acts with a dense orbit on $\mathbb{P}(\mathbb{C}^2\otimes\mathbb{C}^k\otimes\mathbb{C}^k)$, and the isotropy subgroup at a general element has dimension 2.

The other case $2 \times k \times (k + 1)$ has boundary format and it was also solved by Weierstrass.

Theorem 3.13 (Weierstrass)**.** *All nondegenerate matrices of type $2 \times k \times (k + 1)$ are $GL(k) \times GL(k + 1)$ equivalent to the identity matrix having the two slices*

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix} \qquad \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Proof. Let A, A' two such matrices. Since they are nondegenerate they define two exact sequences on $\mathbb{P}^1$

$$0 \to \mathcal{O}(-k) \longrightarrow \mathcal{O}^{k+1} \xrightarrow{A} \mathcal{O}(1)^k \to 0$$
$$0 \to \mathcal{O}(-k) \longrightarrow \mathcal{O}^{k+1} \xrightarrow{A'} \mathcal{O}(1)^k \to 0$$

We want to show that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-k) \arrow[r] \arrow[d, "1"] \arrow[dr] & \mathcal{O}^{k+1} \arrow[r, "A"] \arrow[d, "f"] & \mathcal{O}(1)^k \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}(-k) \arrow[r] & \mathcal{O}^{k+1} \arrow[r, "A'"] & \mathcal{O}(1)^k \arrow[r] & 0
\end{tikzcd}$$

In order to show the existence of f we apply the functor $Hom(-, \mathcal{O}^{k+1})$ to the first row. We get

$$Hom(\mathcal{O}^{k+1}, \mathcal{O}^{k+1}) \xrightarrow{g} Hom(\mathcal{O}(-k), \mathcal{O}^{k+1}) \to Ext^1(\mathcal{O}(1)^k, \mathcal{O}^{k+1}) \simeq H^1(\mathcal{O}(-1)^{k(k+1)}) = 0$$

Hence g is surjective and f exists. Now it is straightforward to complete the diagram with a morphism $\phi\colon \mathcal{O}(1)^k \to \mathcal{O}(1)^k$, which is a isomorphism by the snake lemma. □

Let (x_0, x_1) be homogeneous coordinates on $\mathbb{P}^1$. The identity matrix appearing in Theorem 3.13 corresponds to the morphism of vector bundles given by

$$I_k(x_0, x_1) := \begin{pmatrix} x_0 & x_1 & & \\ & \ddots & \ddots & \\ & & x_0 & x_1 \end{pmatrix}$$

in suitable basis $\{a_0, a_1\}$, $\{b_0, \ldots, b_{k-1}\}$, $\{c_0, \ldots, c_k\}$, it decomposes as $\sum_{i=0}^{1} \sum_{j=0}^{k-1} a_i b_j c_{i+j}$, with $2k$ summands. It is interesting that the rank is smaller than $2k$, as shown by the following.

Proposition 3.14. *The general tensor of size $2 \times k \times (k+1)$ has rank $k+1$. There are infinitely many decomposition. After fixing the $(k+1)$-th root of unity $\tau = e^{\frac{2\pi\sqrt{-1}}{k+1}}$, one of them can be expressed as*

$$\frac{1}{k+1} \sum_{j=0}^{k} \left[\underbrace{(\sum_{i=0}^{1} a_i \tau^{ij})(\sum_{i=0}^{k-1} b_i \tau^{ij})(\sum_{i=0}^{k} c_i \tau^{-ij})}_{j\text{-th summand}} \right] = \sum_{i=0}^{1} \sum_{j=0}^{k-1} a_i b_j c_{i+j} \tag{5}$$

in suitable basis $\{a_0, a_1\}$, $\{b_0, \ldots, b_{k-1}\}$, $\{c_0, \ldots, c_k\}$.

The expression coming from (5) is closely related to the *Discrete Fourier Transform.* The link is provided by the fact that this tensors corresponds to the multiplication of homogeneous polynomials in two variables, of degree respectively 1 and $k-1$. The resulting polynomial, of degree k, can be found by evaluating it on the $(k+1)$-th roots of unity, which are τ^i for $i = 0, \ldots, k$. The other decompositions can be found by fixing $(k+1)$ distinct values, and evaluating the resulting polynomial on them, by solving an interpolation problem. Note that the group $SL(2) \times SL(k) \times SL(k+1)$ acts with a dense orbit on $\mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^{k+1})$, and the isotropy subgroup at a general element is exactly $SL(2)$.

It is interesting, and quite unexpected, that the format $2 \times k \times (k+1)$ is a building block for all the other formats $2 \times b \times c$. The canonical form illustrated by the following Theorem is called the Weierstrass canonical form (there is an extension in the degenerate case that we do not pursue here).

Theorem 3.15 (Kronecker, 1890)**.** *Let $2 \le b < c$. There exist unique $n, m, q \in \mathbb{N}$ satisfying*

$$\begin{cases} b = nq + m(q+1) \\ c = n(q+1) + m(q+2) \end{cases}$$

such that the general tensor $t \in \mathbb{C}^2 \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ decomposes under the action of $GL(b) \times GL(c)$ as n blocks $2 \times q \times (q+1)$ and m blocks $2 \times (q+1) \times (q+2)$ in Weierstrass form.

Note that the group $SL(2)\times SL(b)\times SL(c)$ acts with a dense orbit on $\mathbb{P}(\mathbb{C}^2\otimes\mathbb{C}^b\otimes\mathbb{C}^c)$, and the isotropy subgroup at a general element has dimension $(c-b)^2+2$.

Kac has generalized this statement to the format $2\le w\le s\le t$ satisfying the inequality $t^2-wst+s^2\ge 1$. Note that in these cases the hyperdeterminant does not exist (for $w\ge 3$). The result is interesting because it gives again a canonical form.

Given w, define by the recurrence relation $a_0=0$, $a_1=1$, $a_j=wa_{j-1}-a_{j-2}$

For $w=2$ get $0,1,2,\ldots$ and Kronecker's result.

For $w=3$ get $0,1,3,8,21,55,\ldots$ (odd Fibonacci numbers)

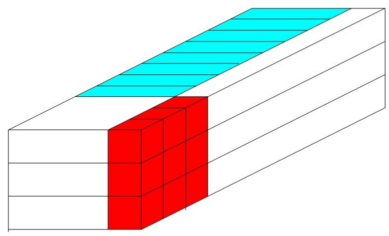


Figure 4. A decomposition in two Fibonacci blocks

Theorem 3.16 (Kac, 1980)**.** *Let $2\le w\le s\le t$ satisfying the inequality $t^2-wst+s^2\ge 1$. Then there exist unique $n,m,j\in\mathbb{N}$ satisfying*

$$\begin{cases} s &= & na_j &+& ma_{j+1}\\ t &=& na_{j+1} &+& ma_{j+2}\end{cases}$$

such that the general tensor $t\in\mathbb{C}^w\otimes\mathbb{C}^s\otimes\mathbb{C}^t$ decomposes under the action of $GL(s)\times GL(t)$as n blocks $w\times a_j\times a_{j+1}$ and m blocks $w\times a_{j+1}\times a_{j+2}$ which are denoted "Fibonacci blocks". They can be described by representation theory (see [Br]*).*

The original proof of Kac ([Kac]) uses representations of quivers. In [Br] there is an independent proof in the language of vector bundles.

4. Lecture 4

Abstract for Lecture 4: Criterion of identifiability for specific tensors, beyond Kruskal bound. Equations for secant varieties. Nonabelian apolarity. Young flattening. Criterion of smoothness for equations given by bundles.

4.1. **Criterion of identifiability for specific tensors, by the infinitesimal dimension of tangentially contact locus.** The trick is to compute the tangent dimension of the (tangentially) contact locus at the point p_1. We borrow from section 9 of [BCO] the following Algorithm to check if a Segre variety X is s-identifiable.

The steps are the following.

(1) We choose s random points $p_1,\ldots,p_s$ on the Segre variety X, working on an affine chart. The point p_1 can be chosen as $(1,0,\ldots)$ on each factor.
(2) We compute the equations of the span of tangent spaces $< T_{p_1},\ldots,T_{p_s} >$. This is just the solution of a linear system.
(3) For any of the cartesian equations we compute its partial derivatives, the common locus is the s-tangentially contact locus C of points p such that $T_pX\subset\langle T_{p_1},\ldots,T_{p_s}\rangle$.

(4) We compute the rank of the jacobian matrix of C at p_1. If it is equal to the dimension of X then X is s-identifiable. If it is smaller than the dimension of X, then a further analysis is required.

The above algorithm allows to detect if a specific decomposition $t = \sum_{i=1}^{s} \lambda_i p_i$ is unique, only if t corresponds to a smooth point of $\sigma_s(X)$, so that Terracini Lemma applies and we have $T_t\sigma_s(X) = \langle T_{p_1}X, \ldots, T_{p_s}X\rangle$.

In order to check if t corresponds to a smooth point of $\sigma_s(X)$, the cartesian equations of $\sigma_s(X)$ are wished.

The rest of this lecture is devoted to this question.

This technique allows to give a test that in certain cases goes beyond the Kruskal bound. For example for $5 \times 5 \times 5$ tensors, with this technique it is possible to check the identifiability of specific tensors of rank ≤ 7, while with Kruskal criterion we can arrive up to $\lfloor \frac{3\cdot 5-2}{2} \rfloor = 6$.

4.2. **Equations for secant varieties. Nonabelian apolarity. Young flattening.** Let $X \subset \mathbb{P}^n$ be a projective variety, embedded by the very ample line bundle L. A technique to find equations of $\sigma_k(X)$ is to express the equations of X as minors of a certain matrix, whose entries are coordinates in $\mathbb{P}^n$. A convenient way to achieve this goal is the following construction.

Let E be a vector bundle on X embedded, by the very ample line bundle L, in $\mathbb{P}(H^0(L)^\vee)$.

Consider the natural morphism

$$H^0(E) \quad \otimes \quad H^0(L)^\vee \quad \xrightarrow{A} \quad H^0(E^\vee \otimes L)^\vee$$

which induces $\forall f \in H^0(L)^\vee$ the linear map

$$\begin{array}{rccc} A_f\colon & H^0(E) & \to & H^0(E^\vee \otimes L)^\vee \\ & s & \mapsto & A(s \otimes f) \end{array}$$

Think at this construction as a linear map depending (linearly) on $f \in H^0(L)^\vee$, so as a matrix with entries are linear combination of the coordinates of the ambient space. All catalecticant matrices have this form, they correspond to the case $X = v_d(\mathbb{P}^n)$, $L = \mathcal{O}_{\mathbb{P}^n}(d)$, $E = \mathcal{O}_{\mathbb{P}^n}(e)$.

In the special case $E = \mathcal{O}_X$, the previous construction just express the construction that a globally generated line bundle L defines a morphism from X to $\mathbb{P}(H^0(L)^\vee)$.

Lemma 4.1 (Nonabelian Apolarity Lemma)**.**

$$H^0(I_Z \otimes E) \subseteq \ker A_f$$

Proof. Assume $f = [x]$ with $x \in \mathrm{Cone}(X) \subset H^0(L)^\vee$

We have

$H^0(L)^\vee \otimes H^0(I_Z \otimes E) \otimes H^0(E^\vee \otimes L) \to H^0(L)^\vee \otimes H^0(I_Z \otimes L)$

the latter pairing associates to (x, s) the values of $s(x)$, which vanishes if $x \in X$.

By linearity $A_{\sum_i [x_i]} = \sum A_{[x_i]}$ and the same argument works in general. □

Remark 4.2. In the special case $X = v_d(\mathbb{P}^n)$, $L = \mathcal{O}_{\mathbb{P}^n}(d)$, $E = \mathcal{O}_{\mathbb{P}^n}(e)$, we get the Classical Apolarity Lemma of Kristian's lecture, indeed

$$H^0(I_Z \otimes E) = (I_Z)_e \qquad \ker A_f = \left(f^\perp\right)_e$$

Note that the vanishing of a section of E at a point imposes rk E conditions.

Theorem 4.3 (Landsberg-O)**.** [LO]

- *Let $Z = \{x_1, \dots x_k\} \subset X$ be a set of points such that $H^0(E^\vee \otimes L) \to H^0(E^\vee \otimes L_{|Z})$ is surjective.*
  Let $f = \sum_{i=1}^k x_i \in H^0(L)^\vee$.
- *Then*

$$H^0(I_Z \otimes E) = \ker A_f$$
$$Z \subseteq \text{ base locus of } \ker A_f$$

In particular

$$rk\ A_f = rk\ E \cdot rk\ f$$

Remark 4.4. In the special case $k = 1$, $E = \mathcal{O}_X$, the previous Theorem just express the construction that a globally generated line bundle L defines a morphism from X to $\mathbb{P}(H^0(L)^\vee)$.

Steps of the proof of Theorem 4.3

(1) We apply the Nonabelian Apolarity Lemma, getting
$$\ker A_f \supseteq H^0(I_Z \otimes E)$$
(2) The surjectivity assumption allows a local computation to check the dimension of the kernel.

Example: non abelian apolarity for plane cubics
Consider $X = \mathbb{P}^2$, $L = \mathcal{O}(3)$, $f \in S^3\mathbb{C}^3$.
Let $E = Q(1)$ where Q is the quotient bundle of rank two on $\mathbb{P}^2$. Consider

$$A_f \colon H^0(E) = \text{ad } \mathbb{C}^3 \to \text{ad } \mathbb{C}^3 = H^0(E^\vee \otimes L)^\vee$$

cubics of rank $\leq k$ are defined by the condition rk $A_f \leq 2k$ for $k \leq 3$.

The equation of the hypersurface $\sigma_3(v_3(\mathbb{P}^2))$ is given by the *Pfaffian* of A_f, *(Aronhold invariant).*

4.3. **Explicit construction of the minors of A_f from a presentation of E.** On $\mathbb{P}^n$ we have the presentation

$$\begin{array}{ccccccc}
\dots \longrightarrow L_2 \longrightarrow L_1 & & \xrightarrow{p_E} & & L_0 \longrightarrow L_{-1} \longrightarrow \dots \\
& \searrow & & \nearrow & \\
& & E & & \\
& \nearrow & & \searrow & \\
0 & & & & 0
\end{array}$$

the matrix A_f can be obtained by differentiating with respect to p_E the catalecticant matrix. The presentation of $E = Q(1)$ on $\mathbb{P}^2$ is

$$\begin{bmatrix} & x_2 & -x_1 \\ -x_2 & & x_0 \\ x_1 & -x_0 & \end{bmatrix}$$

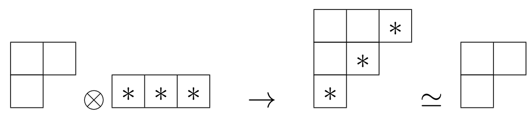
Let $f \in S^3\mathbb{C}^3$, note that the partial derivatives $\frac{\partial f}{\partial x_i} = f_i$ are quadrics, and that the catalecticant of a quadric coincides with its Hessian matrix, that we denote by $H(f_i)$.

Explicit form of Aronhold invariant of f Let $f \in S^3\mathbb{C}^3$

The minors of A_f coincide with the minors of the following map $P_f \colon End(V) \to End(V)$ (regardless the trace):

$$\begin{bmatrix} & H(f_2) & -H(f_1) \\ -H(f_2) & & H(f_0) \\ H(f_1) & -H(f_0) & \end{bmatrix}$$

Such constructin is calleed a *Young flattening*, it can be generalized with Young diagrams. The corresponding picture in this case is



All the subPfaffians of size 8 extracted by P_f coincide, up to scalar multiple, with the *Aronhold invariant.*

4.4. **Criterion of smoothness for equations given by bundles.** The following theorem gives a useful criterion to find local equations of secant varieties.

Theorem 4.5. [LO] *Theorem 5.4.3.*
Let $v = \sum_{i=1}^{r} x_i \in V$ and let $Z = \{[x_1], \ldots, [x_r]\}$, where $[x_j] \in X$. If

$$H^0(I_Z \otimes E) \otimes H^0(I_Z \otimes E^* \otimes L) \longrightarrow H^0(I_{Z^2} \otimes L) \tag{6}$$

is surjective, then $\sigma_r(X)$ is an irreducible component of the vanishing locus of the minors of size $r \cdot \mathrm{rk}(E) + 1$ of $A_f \colon H^0(E) \to H^0(E^\vee \otimes L)^\vee$.

Note that the target space of the map in (6) corresponds to the space of hyperplanes (in the embedding space of X) which are tangent to X at $[x_1], \ldots, [x_r]$, and it can be interpreted, by Terracini Lemma, as the conormal space to $\sigma_k(X)$ at $[v]$.

4.5. **Exercises for Lectures** 3 **and** 4**.** The exercises consist on three different point of view about the

Theorem 4.6 (Sylvester Pentahedral Theorem)**.** *The general cubic surface g has a unique pentahedral given by $\{l_1, \ldots, l_5\}$ such that*

$$g = \sum_{i=1}^{5} \lambda_i l_i^3$$

for suitable λ_i. In other terms the 5*-secant degree of $v_3(\mathbb{P}^3)$ is* 1*, that is $v_3(\mathbb{P}^3)$ is* 5*-identifiable.*

The pentahedral of the five planes with equation given respectively by the vanishing of $\{l_1, \ldots l_5\}$ is called the Sylvester pentahedral of g. It defines a configuration of 10 lines and 10 points.

(1) Prove, by using Kruskal criterion, that in $\mathbb{P}^3$

$$g = \sum_{i=0}^{3} x_i^3 - 27(\sum_{i=0}^{3} x_i)^3$$

is the unique Waring decomposition of g.

(2) Define, for any $g \in S^3\mathbb{C}^3$

$$V(g) := \{y \in \mathbb{P}^3 | \text{the polar quadric } \sum_{i=0}^{3} y_i \frac{\partial g}{\partial x_i} \text{ has rank} \leq 2\}$$

Prove the Sylvester Pentahedral Theorem by the following steps.

- Prove that $V(\sum_{i=1}^{5} \lambda_i l_i^3)$ contains the 10 points $\{l_i = l_j = l_k = 0\}$.
- Prove that, for general g, $V(g)$ consists of 10 points. *Hint: it is allowed to use that* $\deg \sigma_2(v_2(\mathbb{P}^3)) = 10$.
- Compare the previous two points.

(3) Let $X = \mathbb{P}^3$, $L = \mathcal{O}(3)$, Q be the quotient bundle on $\mathbb{P}^3$ and $E = Q^\vee \otimes \mathcal{O}(2)$.

- Compute the dimension of $H^0(E)$ (*answer: 20*) and of $H^0(E^\vee \otimes L)^\vee$ (*answer: 15*).
- Show that the general section of $H^0(E)$ vanishes on five points (*Hint: compute* $c_3(Q^\vee \otimes \mathcal{O}(2)) = 5$).

Indeed, Nonabelian Apolarity can be used to give an alternative proof of Sylvester Pentahedral Theorem, and to compute effectively the Pentahedral by the common zero locus of sections in the 5-dimensional $\ker A_g$, [OO].

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