A note on families of saturated ideals

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Abstract

Under minor homogeneity assumptions we show that the set of saturated ideals in a flat family of multihomogeneous ideals is open. Moreover, we show that a general configuration of points in a smooth complete toric variety has the expected multigraded Hilbert function. We work over the base field $\Bbbk$ of arbitrary characteristics. This is a part of the project on advanced border apolarity techniques. These working notes are provided as a temporary reference for other authors, while we are preparing the details of the complete paper.

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1 Introduction

Let $\Bbbk$ be any base field (not necessarily algebraically closed). We work with a polynomial ring $S = \Bbbk[\alpha_1, \dots, \alpha_n]$. Suppose A is a finitely generated abelian group, that is $A \simeq \mathbb{Z}^q \oplus A^{\mathrm{tor}}$, where $q \geqslant 0$, A^{tor} is a finite abelian group. An A-grading of S is a homomorphism (called the *degree*) from the semigroup of monomials in S to A. An A-grading is *positive* if the only monomial of degree 0 is $1 \in S$. An ideal $I \subset S$ is homogeneous (with respect to the A-grading) if I is generated by homogeneous elements, that is polynomials, whose all monomials are of the same degree. Whenever A and the degree map are clear from the context, we will simply say "homogeneous".

Fix an A-grading of S. We consider a *flat family of homogeneous ideals* in $S = \Bbbk[\alpha_1, \dots, \alpha_n]$. That is, let U be a $\Bbbk$-scheme, and consider the free sheaf $S_U := \mathcal{O}_U \otimes_{\Bbbk} S$ of A-graded algebras. Let $\mathcal{I} \subset S_U$ be a homogeneous ideal sheaf flat over U. For each point $u \in U$ we consider the fibre ideal $\mathcal{I}_u \subset \kappa(u)[\alpha_1, \dots, \alpha_n] = \kappa(u) \otimes_{\Bbbk} S$. Let $J \subset S$ be a fixed homogeneous ideal, and for each u let $J_u := \kappa(u) \otimes_{\Bbbk} J \subset \kappa(u) \otimes_{\Bbbk} S$ be the extension of J to the coefficients in $\kappa(u)$. We say that an ideal $I \subset \kappa(u) \otimes_{\Bbbk} S$ is saturated with respect to J if $(I : J_u) = I$.

Our first claim is that (under some minor assumptions) the subset of those $u \in U$ that $\mathcal{I}_u$ is saturated with respect to J is Zariski open in X.

think: more is true, it makes sense to have it as an open sub-scheme.

Theorem 1.1. *Suppose S is positively graded polynomial ring and $\mathcal{I}$ is a flat family of homogeneous ideals in S parametrised by a locally Noetherian $\Bbbk$-scheme U. Let J be a fixed homogeneous ideal in S. Then the set*

$$\{u \in U \mid \mathcal{I}_u \text{ is saturated with respect to } J\}$$

is an open subset of U.

The most relevant case of theorem is when U is the multigraded Hilbert scheme.

Corollary 1.2. *Suppose that S is a positively graded polynomial ring and that $J \subset S$ is a homogeneous ideal. Let Hilb^h_S be the multigraded Hilbert scheme of S for some Hilbert function $h\colon A \to \mathbb{Z}_{\geqslant 0}$. Consider the subset $\mathrm{Hilb}^{h,\mathrm{sat}\,J}_S \subset \mathrm{Hilb}^h_S$ consisting of the points representing ideals saturated with respect to J. Then $\mathrm{Hilb}^{h,\mathrm{sat}\,J}_S$ is Zariski open.*

Now suppose X is a smoothcomplete toric variety over $\Bbbk$ and let $S = S[X] = \bigoplus_{D\in \mathrm{Pic}(X)} H^0(X, D)$ is its Cox ring graded by $\mathrm{Pic}(X)$. Suppose $\Bbbk \subset \mathbb{K}$ is a field extension and $Z \subset X \times_{\Bbbk} \mathbb{K}$ is a subscheme. Denote by $\mathrm{I}(Z) \subset S[X] \otimes_{\Bbbk} \mathbb{K}$ the homogeneous ideal of Z, that is the ideal generated by sections of line bundles on $X \times_{\Bbbk} \mathbb{K}$ that vanish on Z. The *Hilbert function* of Z is $h_Z\colon \mathrm{Pic}(X) \to \mathbb{Z}_{\geqslant 0}$ defined by

$\mathbb{Q}$-Cartier should be enough

or $\mathrm{Cl}(X)$

$$h_Z(D) = \dim_{\mathbb{K}} \left((S[X] \otimes_{\Bbbk} \mathbb{K}) \,/\, \mathrm{I}(Z) \right)_D .$$

Fix a Noetherian $\Bbbk$-scheme U as a base scheme and suppose $\mathcal{Z} \subset U \times X$ is a closed subscheme flat over U (that is, $\mathcal{Z}$ is a flat family of subschemes of X). Then there are only finitely many possible Hilbert functions of fibres $\mathcal{Z}_u$ and this determines a finite stratification of U by locally closed subsets.

Proposition 1.3. *Let X, U and S be as above. For a flat family $\mathcal{Z} \to U$ of subschemes of X consider a map from the set of points of U to the set of integer valued functions from $\mathrm{Pic}(X)$ to $\mathbb{Z}_{\geqslant 0}$: $u \mapsto h_{\mathcal{Z}_u}$. This map takes only finitely many values $h_1, \ldots, h_k$ and the preimage of any function is a locally closed subset U_i of U. In particular, if U is irreducible with the generic point $\eta \in U$, then there is a Zariski open dense subset $U_i \subset U$ such that every fibre $\mathcal{Z}_u$ for $u \in U_i$ has the same Hilbert function as the generic fibre $h_{\mathcal{Z}_\eta}$.*

Another application concerns configurations of points on a smooth complete toric variety X. Fix an integer $r > 0$, and by $h_{r,X}\colon \mathrm{Pic}\, X \to \mathbb{N}$ denote the *generic Hilbert function of r points on X*, that is $h_{r,X}(D) = \min\left(r, \dim_{\Bbbk} H^0(X, D)\right)$. In [BB19, Lem. 3.9] we show that (over complex numbers) a very general configuration of r points in X has the Hilbert function $h_{r,X}$. Here we show that the same is actually true for a general configuration. Moreover, we extend the result to an arbitrary algebraically closed base field.

Theorem 1.4. *Suppose the base field $\Bbbk$ is algebraically closed, X is a smooth projective toric variety over $\Bbbk$, and let $S = S[X]$ be the Cox ring of X. Suppose that $Z := \{x_1, \ldots, x_r\}$ is a general configuration of points on X (that is, $(x_1, \ldots, x_r) \in X^{\times r}$ is a general point). Then the Hilbert function h_Z is equal to $h_{r,X}$. Moreover, the set of such r-tuples of points is open in $X^{\times r}$.*

2 Saturation is an open property

The proofs in this section are largely suggested by Joachim Jelisiejew (private communication).

Lemma 2.1. *Let $f \in S$ be a fixed homogeneous element of nonzero degree. Let U be a locally Noetherian $\Bbbk$-scheme and $\mathcal{I}$ be a flat family of homogeneous ideals in S. Then the set*

$$\{u \in U \mid f \text{ is not a zero divisor in } (\kappa(u) \otimes_{\Bbbk} S)/\mathcal{I}_u\} \tag{2.2}$$

is an open subset of U.

Proof. This argument resembles one of the forms of the local criterion for flatness [Stac17, Tag 00ME], but in the graded setting.

Everything is local on U, so we assume $U = \mathrm{Spec}(R)$ for a Noetherian $\Bbbk$-algebra R, $\mathcal{I}$ is a homogeneous ideal in $R \otimes_{\Bbbk} S$ and let $B = R \otimes_{\Bbbk} S/\mathcal{I}$ be the quotient algebra. Pick a point $u \in U$ and let $\mathfrak{m} \subset R$ be the maximal ideal corresponding to u and $\kappa(u) = R/\mathfrak{m}$. Suppose that f is not a zero-divisor in $B/\mathfrak{m}B$. We will prove that f is not a zero-divisor in $B_{\mathfrak{m}}$. First, we show that f is not a zero-divisor in $B_{\mathfrak{m}}/\mathfrak{m}^d B_{\mathfrak{m}}$ for all positive d. We do this by induction. The case $d = 1$ is the assumption that f is not a zero-divisor in $B/\mathfrak{m}B$. For the induction step, consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_{\mathfrak{m}} \otimes_R \mathfrak{m}^d/\mathfrak{m}^{d+1} & \longrightarrow & B_{\mathfrak{m}}/\mathfrak{m}^{d+1}B_{\mathfrak{m}} & \longrightarrow & B_{\mathfrak{m}}/\mathfrak{m}^d B_{\mathfrak{m}} & \longrightarrow & 0 \\
 & & \downarrow{\cdot f} & & \downarrow{\cdot f} & & \downarrow{\cdot f} & & \\
0 & \longrightarrow & B_{\mathfrak{m}} \otimes_R \mathfrak{m}^d/\mathfrak{m}^{d+1} & \longrightarrow & B_{\mathfrak{m}}/\mathfrak{m}^{d+1}B_{\mathfrak{m}} & \longrightarrow & B_{\mathfrak{m}}/\mathfrak{m}^d B_{\mathfrak{m}} & \longrightarrow & 0
\end{array}$$

whose rows come from tensoring $0 \to \mathfrak{m}^d/\mathfrak{m}^{d+1} \to R_{\mathfrak{m}}/\mathfrak{m}^{d+1} \to R_{\mathfrak{m}}/\mathfrak{m}^d \to 0$ with the flat $R_{\mathfrak{m}}$-module $B_{\mathfrak{m}}$. We have

$$B_{\mathfrak{m}} \otimes_A \mathfrak{m}^d/\mathfrak{m}^{d+1} = B_{\mathfrak{m}}/\mathfrak{m}B_{\mathfrak{m}} \otimes_{\kappa(u)} \mathfrak{m}^d/\mathfrak{m}^{d+1} \simeq (B/\mathfrak{m}B)^{\dim_{\kappa(u)} \mathfrak{m}^d/\mathfrak{m}^{d+1}}$$

as B-modules, so the multiplication by f is injective on this B-module. The multiplication by f on $B_{\mathfrak{m}}/\mathfrak{m}^d B_{\mathfrak{m}}$ is injective by the inductive assumption. Snake Lemma implies that also the multiplication by f on $B_{\mathfrak{m}}/\mathfrak{m}^{d+1}B_{\mathfrak{m}}$ is injective, which concludes the proof of induction step and of the induction claim.

Let $K \subset B$ be the kernel of the multiplication by f on B. Then $K_{\mathfrak{m}} \subset B_{\mathfrak{m}}$ is the kernel of the multiplication by f on $B_{\mathfrak{m}}$, so $K_{\mathfrak{m}} \subset \mathfrak{m}^d B_{\mathfrak{m}}$ for all d, hence $K_{\mathfrak{m}}$ is a subset of $\bigcap_d \mathfrak{m}^d B_{\mathfrak{m}}$. We claim that the latter module is zero. By assumption, the ring B is $\mathbb{N}$-graded and its degree r part is an image of S_r, so it is a finite R-module. The image of $R \setminus \mathfrak{m}$ in B consists of elements of degree zero, hence $B_{\mathfrak{m}}$ is $\mathbb{N}$-graded and its degree r part is a finite $R_{\mathfrak{m}}$-module. Similarly, $\bigcap_d \mathfrak{m}^d B_{\mathfrak{m}}$ is $\mathbb{N}$-graded with degree r part equal to $\bigcap_d \mathfrak{m}^d (B_{\mathfrak{m}})_r$. But $(B_{\mathfrak{m}})_r$ is a finite module over the Noetherian local ring $R_{\mathfrak{m}}$, so by Krull Intersection Theorem [Eise95, Cor. 5.4], we have $\bigcap_d \mathfrak{m}^d (B_{\mathfrak{m}})_r = 0$. It follows that $\bigcap_d \mathfrak{m}^d B_{\mathfrak{m}} = 0$, so $K_{\mathfrak{m}} = 0$.

As B is Noetherian, the ideal $K \subset B$ is finitely generated, say by elements $k_1, \dots, k_m$. From $K_{\mathfrak{m}} = 0$ we see that there exist $s_1, \dots, s_m \in R \setminus \mathfrak{m}$ such that $s_i k_i = 0$ for all i. Let $s = \prod_{i=1}^m s_i$. Then $sk_i = 0$ for all i, so $sK = 0$ so $K_s = 0$. The ideal $K_s \subset B_s$ is the kernel of the multiplication by f on B_s. As it is zero, the element f is not a zero-divisor in B_s. It follows that the open neighbourhood ($s \neq 0$) of u is contained in the set (2.2). This completes the proof. $\square$

Lemma 2.3. *Assume that S is $\mathbb{Z}$-graded, and that the degrees of all variables α_i have positive degree. Let $\Bbbk \subset \mathbb{K}$ be a field extension, let $S_{\mathbb{K}} = S \otimes_{\Bbbk} \mathbb{K}$ and let $B = S_{\mathbb{K}}/I$ for a homogeneous ideal $I \subset S_{\mathbb{K}}$. Let $J \subset S$ be a homogeneneous ideal. Then I is saturated with respect to $J \cdot S_{\mathbb{K}}$ if and only if there exists a homogeneous element $f \in J$ which is not a zero-divisor in B.*

We stress that the homogeneous element f in the conclusion of the lemma should exist in J, not only in $J \cdot S_{\mathbb{K}}$.

Proof. If $J = S$, then there is nothing to prove. Thus assume that J is generated by homogeneous elements of positive degrees.

One implication is formal: if $f \in J$ is not a zero divisor in B, then consider $g \in (I\colon J \cdot S_{\mathbb{K}})$ and set $\bar{g} \in B$ to be the class of g modulo I. Thus $g \cdot J \subset I$ and $\bar{g} \cdot J = 0$. In particular, $\bar{g} \cdot f = 0$ and since f is not a zero divisor, $\bar{g} = 0$, that is $g \in I$, and $(I\colon J \cdot S_{\mathbb{K}}) = I$ as claimed.

Now consider the oposite implication: suppose by contradiction that all homogeneous elements of J are zero divisors on B. Primary decomposition implies that the set of zero divisors of B is a set-theoretic union of prime ideals $\overline{\mathfrak{p}}_1, \ldots, \overline{\mathfrak{p}}_r$ where $\overline{\mathfrak{p}}_i = (0 : \bar{g}_i)$ for some nonzero $\bar{g}_i \in B$, see [AM69, Prop. 4.7]. Moreover, all the ideals $\overline{\mathfrak{p}}_i$ are homogeneous by [Eise95, Prop. 3.12]. By the same proposition, $\bar{g}_i$ can be chosen homogeneous. Let $\mathfrak{p}_i \subset S_{\mathbb{K}}$ be the preimage of $\overline{\mathfrak{p}}_i$ and $g_i \in S_{\mathbb{K}}$ be a homogeneous preimage of $\bar{g}_i \in B$. All homogeneous elements of J are zero divisors on B, so are contained in $\bigcup_{i=1}^{r} \mathfrak{p}_i$. As they lie in S, they are even contained in $\bigcup_{i=1}^{r}(\mathfrak{p}_i \cap S)$. But J and $\mathfrak{p}_i \cap S$ are homogeneous ideals of S, with $\mathfrak{p}_i \cap S$ prime, so $J \subset \mathfrak{p}_i \cap S$ for some index i by homogeneous prime avoidance [Eise95, Lem. 3.3]. It follows that $J\bar{g}_i = 0$ and so $g_i \in (I : J) \setminus I$, so I is not saturated with respect to J. $\square$

Lemma 2.4. *Suppose S is A-graded and the grading is positive. Then there exists a homomorphism $\varphi\colon A \to \mathbb{Z}$, such that the induced $\mathbb{Z}$-grading on S has degrees of all variables positive.*

Proof. Consider $A_{\mathbb{R}} = A \otimes_{\mathbb{Z}} \mathbb{R}$, the real vector space of A. The set of classes of $\deg(\alpha_i)$ spans a rational convex polyhedral cone in $A_{\mathbb{R}}$. This cone is strictly convex: indeed, if there is a line contained in the cone, then there are two nontrivial monomials m_1 and m_2 of opposite degree (up to torsion). Then $\deg\big((m_1 m_2)^d\big) = 0$ for some d, contradciting the positivity of the grading.

Thus there exists an integral hyperplane $\phi = 0$ supporting the vertex 0 in the cone. Then the integral form ϕ defining the hyperplane determines the desired homomorphism $A \to \mathbb{Z}$. $\square$

Proof of Theorem 1.1. By Lemma 2.4 we may assume that the grading group A is equal to $\mathbb{Z}$ and that all the degrees of variables are positive. Both ideals $\mathcal{I}$ and J are still homogeneous with respect to the new grading.

Let $u \in U$ be a point such that $\mathcal{I}_u \subset S \otimes \kappa(u)$ is saturated with respect to J. By Lemma 2.3, there exists a homogeneous $f \in J$ which is not a zero divisor on $(S \otimes \kappa(u))/\mathcal{I}_u$. By Lemma 2.1, the same element f is not a zero divisor on an open set $U' \subset U$ of fibers which contains u. By Lemma 2.3 every fiber in U' corresponds to saturated ideal. This concludes the proof. □

3 Families of schemes in toric varieties

Here we prove Proposition 1.3.

Let $\mathrm{Irrel}_X \subset S[X]$ be the irrelevant ideal of X. Throughout this subsection we fix Irrel_X as the denominator of the saturation, that is, we whenever we say an ideal is saturated, we mean saturated with respect to Irrel_X.

Suppose $\mathcal{Z} \subset U \times X$ is a flat family of subschemes of X. The statement of the proposition is about points of U only, it does not take into account the scheme structure of U. Thus we may assume U is reduced. Moreover, if U is reducible, then we argue for each irreducible component separately. Thus we may assume U is integral.

Let $\eta \in U$ be the generic point. Consider the scheme theoretic closure $\overline{\mathcal{Z}_\eta}$. Since $\mathcal{Z}_\eta \subset \mathcal{Z}$ and $\mathcal{Z}$ is closed, we have $\overline{\mathcal{Z}_\eta} \subset \mathcal{Z}$. By flatness of $\mathcal{Z} \to U$ also its restrictions to subschemes of U are flat [Hart77, Prop. III.9.2(b)]. Thus by the flatness criterion over the base of dimension 1 [Hart77, Prop. III.9.8], we must also have $\mathcal{Z} \subset \overline{\mathcal{Z}_\eta}$, that is $\mathcal{Z} = \overline{\mathcal{Z}_\eta}$. Therefore the saturated ideal $\mathrm{I}(\mathcal{Z}_\eta)$ determines the homogeneous ideal sheaf $\mathcal{I}_{\mathcal{Z}} \subset \mathcal{O}_U \otimes_{\Bbbk} S[X]$. [find better, non-projective reference]

By generic flatness there is an open dense subset $U' \subset U$ such that $\mathcal{I}_{\mathcal{Z}}$ is a flat family of homogeneous ideals on U'. [ref] By Theorem 1.1 there is another open subset $U_1 \subset U'$ such that for each $u \in U_1$ the ideal $\mathcal{I}_{\mathcal{Z},u}$ is saturated. That is $\mathcal{I}_{\mathcal{Z},u} = \mathrm{I}(\mathcal{Z}_u)$ for $u \in U_1$ and $\dim_{\kappa(u)} \mathrm{I}(\mathcal{Z}_u)_D = \dim_{\kappa(\eta)} \mathrm{I}(\mathcal{Z}_\eta)_D$ for all $D \in \mathrm{Pic}(X)$, and thus on U_1 the Hilbert function is constant. [ref]

Now consider the complement of U_1, and replace U with this complement and proceed by induction. Since U is Noetherian, the induction will stop after finitely many steps.

By semicontinuity of Hilbert functions in a family, each stratum must be open its closure, which concludes the proof of Proposition 1.3.

4 Set of ideals of points with generic Hilbert function

Here we prove Theorem 1.4.

Let $X^{r,\circ} \subset X^{\times r}$ denote the difference of the r-fold product of X and the big diagonal, that is, the set of tuples in which at least two of the coordinates coincide. In other words, $X^{r,\circ}$ is the *ordered configuration space* of X. Then $X^{r,\circ}$ naturally determines a flat family of subschemes of X, that is, $\mathcal{Z} \subset X^{r,\circ} \times X$, with $\mathcal{Z}_{(x_1,\dots,x_r)} = \{x_1, \dots, x_r\} \subset X$.

By Proposition 1.3 the Hilbert function of a general configuration is equal to the Hilbert function of the generic configuration $\mathcal{Z}_\eta$. Here $\eta \in X^{r,\circ}$ and $\overline{\{\eta\}} = X^{r,\circ}$. Moreover, the set of those $u \in X^{r,\circ}$ which have the same Hilbert function as $\mathcal{Z}_\eta$ is open in $X^{r,\circ}$, and thus also in $X^{\times r}$.

For each $D \in \mathrm{Pic}(X)$ consider the set:

$$(X^{r,\circ})_D = \left\{ u \in X^{r,\circ} \mid \dim_{\kappa(u)} \left(\left(H^0(X,D) \otimes_{\Bbbk} \kappa(u) \right) / I(\mathcal{Z}_u)_D \right) < h_{r,X}(D) \right\}.$$

This is a closed subset of $X^{r,\circ}$ and since $\Bbbk$ is algebraiclaly closed it is straightforward to show that $(X^{r,\circ})_D \neq X^{r,\circ}$. Thus $\eta \in X^{r,\circ} \setminus \bigcup_{D \in \mathrm{Pic}(X)} (X^{r,\circ})_D$, that is, $\mathcal{Z}_\eta$ has the generic Hilbert function as claimed.

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