Sharp inequalities for the Ornstein-Uhlenbeck operator

Vit Musil

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Joint work with

Andrea Cianchi

Luboš Pick

Overview

I
- Gauss space
- Ornstein-Uhlenbeck vs. Laplace operator
- Solution & reduction principle & optimal spaces

II
- Moser-type inequalities
- Existence of extremals

- Moser inequalities in Gauss space

  Math. Ann. 2020

- Extremals in Gaussian Moser type inequalities

  Int. Math. Res. Notices 2020

- Sharp exponential inequalities for the Ornstein-Uhlenbeck operator, Submitted

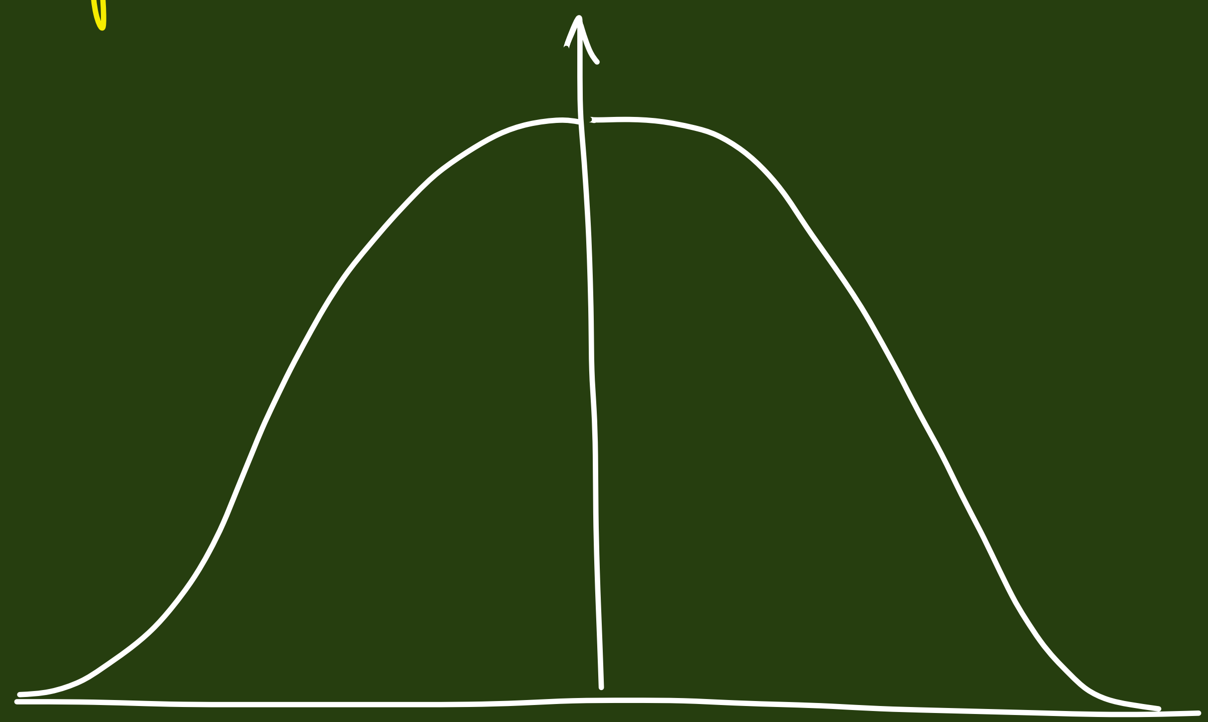
- Optimal spaces for the Ornstein-Uhlenbeck operator

  Preprint

Gauss space

= probability space $(\mathbb{R}^n, \gamma_n)$

$$d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx$$



- related to normal distribution in statistics
- can be generalised to infinite dimensional spaces

Ornstein-Uhlenbeck operator

- appropriate substitute for the Laplacian in Gauss space

$$\Delta = \operatorname{div} \circ \nabla \quad \text{in Euclidean space}$$

$$-\operatorname{div} = \text{adjoint of } \nabla \quad \text{w.r.t. } L^1(\mathbb{R}^n, \lambda_n) \text{ pairing}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$-\delta = \text{adjoint of } \nabla \quad \text{w.r.t. } L^1(\mathbb{R}^n, \gamma_n) \text{ pairing}$$

$$\delta F = \operatorname{div} F - x \cdot F$$

$$\boxed{\mathcal{L}u = \delta \circ \nabla} = \Delta - x \cdot \nabla$$

Euclidean $-\Delta u = f$

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla \phi \, dx = \int_{\mathbb{R}^n} f \phi \, dx \qquad \forall \phi$$

Gaussian $-\mathcal{L} u = f$

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla \phi \, d\gamma_n = \int_{\mathbb{R}^n} f \phi \, d\gamma_n \qquad \forall \phi$$

Definition of the solution

For $f \in L^2(\mathbb{R}^n, \gamma_n)$: $u \in W^{1,2}(\mathbb{R}^n, \gamma_n)$ s.t.

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla \phi \, d\gamma_n = \int_{\mathbb{R}^n} f \phi \, d\gamma_n \qquad \forall \phi \in W^{1,2}(\mathbb{R}^n, \gamma_n)$$

For $f \in L^1(\mathbb{R}^n, \gamma_n)$: $u \in L^1(\mathbb{R}^n, \gamma_n)$ s.t.

$\{f_k\} \subseteq L^2$ and $\{u_k\} \subseteq W^{1,2}$ exist s.t.

- $-\mathcal{L} u_k = f_k$
- $f_k \to f$ in L^1
- $u_k \to u$ a.e.

Existence of the solution

<u>Theorem</u> For every $f \in L^1(\mathbb{R}^n, \gamma_n)$ satisfying $\mathrm{mv}(f) = 0$, there exists a unique solution u (up to additive constants) to the Ornstein-Uhlenbeck equation. Moreover

$$\|u\|_{L^{1,\infty}\log L(\mathbb{R}^n,\gamma_n)} \leq c\,\|f\|_{L^1(\mathbb{R}^n,\gamma_n)}.$$

Rearrangement estimates

Theorem Let $f \in L^1(\mathbb{R}^n, \gamma_n)$. Then

$$(u - \mathrm{med}(u))^*_\pm(s) \le \Theta(s) \int_0^s f^*_\pm + \int_s^{1/2} f^*_\pm(r)\,\Theta(r)\,dr$$

for $s \in (0, 1/2)$, where

$$\Theta(s) = \int_s^{1/2} \frac{dr}{I(r)^2} \approx \frac{1}{2s\,\ell(s)} \quad \text{as} \quad s \to 0_+$$

and I is the Gaussian isoperimetric profile. $\ell(s) = 1 + \log\frac{1}{s}$.

Main objectives

- Transfer of integrability from datum f to the solution u $\longrightarrow$ Sobolev-type embeddings

- Attention to its optimality

- Competing spaces = rearrangement invariant spaces (r.i.)

  $\supseteq$ Lebesgue, Lorentz, Orlicz, ...

r.i. norms measure "how big and spread out" functions are

Reduction principle

Theorem Let X and Y be r.i. spaces. Then

$$\|u - \operatorname{med}(u)\|_{Y(\mathbb{R}^n, \gamma_n)} \le c_1 \|\mathcal{L}u\|_{X(\mathbb{R}^n, \gamma_n)}$$

for every u if and only if

$$\left\| \frac{1}{s\ell(s)} \int_0^s g + \int_s^1 \frac{g(r)}{r\ell(r)}\, dr \right\|_{Y(0,1)} \le c_2 \|g\|_{X(0,1)}$$

for every g. Moreover $c_1 \approx c_2$.

Notes

- Full-dimensional problem reduces to one-dimensional
- Differential operator from RHS transfers to integral operator on LHS
- Reduced inequality is independent of dimension n
- Optimal spaces can be characterised

Comparison to Laplacian

Embedding

$$\|u\|_{Y(\Omega)} \leq c_1 \|\Delta u\|_{X(\Omega)}$$

is equivalent to

$$\left\| s^{\frac{2}{n}-1} \int_0^s g + \int_s^1 g(r)\, r^{\frac{2}{n}-1} dr \right\|_{Y(0,1)} \leq c_2 \|g\|_{X(0,1)} .$$

Compare the weights

$$\frac{1}{r\ell(r)} \qquad \text{vs.} \qquad r^{\frac{2}{n}-1}$$

Examples (Laplace)

Lebesgue

Δu	u	
L^1	L^q	$q < \frac{n}{n-2}$
L^p	$L^{\frac{np}{n-2p}}$	$1 < p < \frac{n}{2}$
$L^{\frac{n}{2}}$	L^q	$q < \infty$
L^p	L^∞	$p > \frac{n}{2}$

Lorentz

Δu	u	
L^1	$L^{\frac{n}{n-2},\infty}$	
L^p	$L^{\frac{np}{n-2p},p}$	$1 < p < \frac{n}{2}$
$L^{\frac{n}{2},1}$	L^∞	

Notes

- always a gain
- dependence of n
- $\Delta \neq$ second order ∇^2

Orlicz

Δu	u
L^1	no optimal one
$L^p(\log L)^\alpha$	$L^{\frac{np}{n-2p}}(\log L)^{\frac{n\alpha}{n-2p}}$
$L^{\frac{n}{2}}$	$\exp L^{\frac{n}{n-2}}$

Examples (Ornstein-Uhlenbeck)

$\mathcal{L}u$	u		
$L(\log\log L)^{\alpha+1}$	$L(\log\log L)^{\alpha}$	$\alpha > 0$	loss
$L(\log L)^{\alpha}$	$L(\log L)^{\alpha}$	$\alpha > 0$	equalizer
$L^p(\log L)^{\alpha}$	$L^p(\log L)^{\alpha+1}$	$1<p<\infty$	gain
$\exp L^{\beta}$	$\exp L^{\beta}$	$\beta > 0$	equalizer
$\exp\exp L^{\beta+1}$	$\exp\exp L^{\beta}$	$\beta > 0$	loss
L^{∞}	$\exp\exp L$		

- Gain is not guaranteed
- no dependence on n

Expected behaviour

First order embedding on Gauss space

∇u	u		
L^p	$L^p (\log L)^{\frac{p}{2}}$	$1 \le p < \infty$	gain
$\exp\left(\frac{1}{2} \log L\right)^2$	$\exp\left(\frac{1}{2} \log L\right)^2$		equalizer
$\exp L^\beta$	$\exp L^{\frac{2\beta}{2+\beta}}$	$\beta > 0$	loss
L^∞	$\exp L^2$		

Ciandhi, Pick, J. Funct. Anal., 09

Summary

- Existence of solution to Ornstein-Uhlenbeck equation
- Reduction principle
- Optimal transfer of integrability $f \mapsto u$

II Moser-type inequalities

sharp forms of exponential Sobolev-type ineq.

$$\|u\|_{\exp L^{\frac{n}{n-1}}(\Omega)} \leq c \, \|\nabla u\|_{L^n(\Omega)} \qquad \text{(Moser)}$$

$$\|u\|_{\exp L^{\frac{n}{n-2}}(\Omega)} \leq c \, \|\Delta u\|_{L^{\frac{n}{2}}(\Omega)} \qquad \text{(Adams)}$$

Moser, Indiana Univ. Math J. 70/71

There is a certain α_n s.t. for all $\alpha \le \alpha_n$:

(s) $$\sup_u \int_\Omega \exp^{\frac{n}{n-1}}(\alpha\,|u|)\, dx < \infty$$

where the supremum is extended over all u s.t. $\overline{\operatorname{supp} u} \subset \Omega$

$$\int_\Omega |\nabla u|^n \, dx \le 1$$

and (s) fails to hold if $\alpha > \alpha_n$.

Supremum (s) is attained.

Adams, Ann. of Math., 88

There is β_n s.t. for every $\beta \leq \beta_n$ s.t.

$$(S) \qquad \sup_u \int_\Omega \exp^{\frac{n}{n-2}}(\beta |u|)\, dx < \infty$$

where the sup is over all u s.t. $\overline{\operatorname{supp} u} \subset \Omega$

$$\int_\Omega |\Delta u|^{\frac{n}{2}} dx \leq 1$$

and (S) fails if $\beta > \beta_n$.

Supremum (S) is attained $(n=4)$.

Moser's question in Gauss space

First order

$$\|u\|_{\exp L^{\frac{2\beta}{2+\beta}}(\mathbb{R}^n,\gamma_n)} \le c\, \|\nabla u\|_{\exp L^{\beta}(\mathbb{R}^n,\gamma_n)}$$

$$\left[\|u\|_{\exp L^{2}(\mathbb{R}^n,\gamma_n)} \le c\, \|\nabla u\|_{L^{\infty}(\mathbb{R}^n,\gamma_n)}\right]$$

Second order - Ornstein-Uhlenbeck

$$\|u\|_{\exp L^{\beta}(\mathbb{R}^n,\gamma_n)} \le c\, \|\mathcal{L} u\|_{\exp L^{\beta}(\mathbb{R}^n,\gamma_n)}$$

$$\left[\|u\|_{\exp\exp L(\mathbb{R}^n,\gamma_n)} \le c\, \|\mathcal{L} u\|_{L^{\infty}(\mathbb{R}^n,\gamma_n)}\right]$$

Moser's question in Gauss space (Orustein-Uhlenbeck)

let $\beta > 0$. Find largest possible constant θ s.t.

$$(S) \qquad \sup_{u} \int_{\mathbb{R}^n} \exp^{\beta}(\theta |u|)\, d\gamma_n < \infty$$

sup over u satisfying

$$\int_{\mathbb{R}^n} \mathrm{Exp}^{\beta}(|\mathcal{L}u|)\, d\gamma_n \leq M \qquad \text{for some } M > 1$$

and normalized by $\mathrm{med}(u) = 0$.

Questions

- Is there a threshold ?
- Is it attained ?
- Do extremals exist ?

The threshold

for $\beta > 0$, we set

$$\boxed{\theta_\beta = \frac{2}{\beta}}$$

New type of threshold – unseen in Euclidean case

$$\beta \in (0,1] \qquad \text{and} \qquad \beta \in (1,\infty).$$

Result: case $\beta \in [0,1]$

Theorem

(i) If $\theta \leq \theta_\beta$, then (s) holds for any $M > 1$.

(ii) If $\theta > \theta_\beta$, then (s) fails for any $M > 1$, witnessed by a single function.

Result: case $\beta \in (1, \infty)$

Theorem

(i) If $\theta < \theta_\beta$, then (S) holds for any $M > 1$.

(ii) If $\theta = \theta_\beta$, then

- there exists $M > 1$ s.t. (S) holds
- there exists $M > 1$ s.t. (S) fails

(iii) If $\theta > \theta_\beta$, then (S) fails for any $M > 1$, witnessed by a single function

Differences between Gaussian and Euclidean

- Threshold value $\beta = 1$ ... phenomenon only in Gauss

- The value M in the constraint is irrelevant when $\beta \in (0,1]$ but affects the conclusion if $\beta \in (1, \infty)$.

  By contrast, the value 1 in the Euclidean case

$$\int_\Omega |\Delta u|^{\frac{n}{2}} \, dx \le 1$$

is crucial.

- When $\theta > \theta_\beta$, there is a single function u s.t.

$$\int_{\mathbb{R}^n} \exp^\beta(\theta |u|)\, d\mu = \infty .$$

Instead, in Euclidean case

$$\int_\Omega \exp^{\frac{n}{n-2}}(c|u|)\, dx < \infty$$

for every $c > 0$ and every eligible u.

Existence of extremals

<u>**Theorem**</u> Let $\beta \in (0,1]$

Supremum (s) is attained for $\theta = \theta_\beta$ and every $M > 1$.

Main issue = lack of compactness

Moser: $W_0^{1,n}(\Omega) \to \exp L^{\frac{n}{n-1}}(\Omega)$ is not compact

Direct variational methods fail - the convergence of maximizing sequence is not guaranteed.

Moreover

$$\sup_u \int_\Omega \exp^{\frac{n}{n-1}}(\alpha_n |u|)\, \varphi(|u|)\, dx = \infty$$

for every unbounded increasing φ.

This is not the case in Gauss space.

- Embedding is also non-compact
- However:

Theorem Let $\beta \in (0,1]$. Then

$$\sup_u \int_{\mathbb{R}^n} \exp^\beta(\theta_\beta |u|)\, \varphi(|u|)\, d\mu < \infty$$

for every $M > 0$, provided that $\varphi \nearrow \infty$ sufficiently slowly.

Take-home

- Moser type inequalities in Gauss setting for the first and second order (∇, $\mathcal{L}$)
- Thresholds known & extremals exist
- New features foreign to Euclidean setup

That's all Folks!