Monday's Nonstandard Seminar 2020/21

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Università di Modena e Reggio Emilia

REGULARITY FOR OBSTACLE PROBLEMS WITHOUT STRUCTURE CONDITIONS

Monday March 15th, 2021

Outline of the seminar

The aim of this seminar is to deal with the possible occurance of the Lavrentiev phenomenon on a variational obstacle problem with $p, q-$growth.

The main tool used here is a Lemma which let us move from the variational obstacle problem to the one with the relaxed functional, in order to find the solutions' regularity we want. We assume the Sobolev regularity both for the gradient of the obstacle and for the coefficients.

M. Eleuteri, P. Marcellini, E. Mascolo - Adv. Calc. Var. (2020)

Joint project with Dr. G. Bertazzoni [1]

[1] "Regularity for obstacle problems without structure conditions", preprint arXiv:2102.12906

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- **Motivation**
- Statement of the problem and main results
- A priori estimate
- Approximation in case of occurrence of Lavrentiev Phenomenon

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MOTIVATION

Motivation

This talk is focused on studying the Lipschitz continuity of the solutions to variational obstacle problems of the form

$$\min \left\{ \int_\Omega f(x, Dw) : w \in \mathcal{K}_\psi(\Omega) \right\}$$

in the case of $p, q-$growth condition, where Lavrentiev phenomenon may occur.

The relationship between the ellipticity and the growth exponent we impose is the one considered in a series of papers started with

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Lipschitz continuity of solutions

The local boundedness of the gradient Dw is a fundamental property, in fact, thanks to that, the behavior of $|Dw|$ at infinity becomes irrelevant for further regularity.

G. Mingione - Applications of Mathematics (2006)

P. Marcellini - Arch. Ration. Mech. Anal. (1989)
P. Marcellini - J. Differential Equations (1991)

P. Marcellini, G. Papi - J. Diff. Eq. (2006)
C. De Filippis, G. Mingione - preprint (2020)
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$p, q-$growth

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The Lavrentiev phenomenon is a surprising result first demonstrated in 1926 by M. Lavrentiev that, in our case, it may occurs due to the nonstandard growth conditions required on the lagrangian.

Under our assumptions, this phenomenon can be reformulated in these terms:

$$\inf_{w \in (W^{1,p} \cap \{w \geq \psi\})} \int_\Omega f(x, Dw)\, dx < \inf_{w \in (W^{1,q} \cap \{w \geq \psi\})} \int_\Omega f(x, Dw)\, dx$$

This is an obstruction to regularity, since it prevents minimizers to belong to $W^{1,q}$. The basic strategy to get regularity results is to exclude the occurrence of Lavrentiev phenomenon by imposing that the Lavrentiev gap vanishes on solutions.

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Lavrentiev phenomenon

G. Buttazzo, V. J. Mizel - J. Functional Anal. (1992)

G. Buttazzo, M. Belloni - Mathematical Applications (1995)

V.V. Zhikov - Russian J. Math. Phys. (1995)

L. Esposito, F. Leonetti, G. Mingione - J. Differential Equations (2004)

A. Esposito, F. Leonetti, P. V. Petricca - Adv. Nonlinear Anal. (2019)

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M. Eleuteri, P. Marcellini, E. Mascolo - Adv. Calc. Var. (2020)

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Sobolev dependence

We consider Sobolev dependence on the obstacle and the partial map $x \mapsto D_\xi f(x,\xi)$.

J. Kristensen, G. Mingione - Ar. Rat. Mech. Anal. (2006)
A. Passarelli di Napoli - Adv. Calc. Var. (2011)

F. Giannetti, A. Passarelli di Napoli - Mathematische Zeitschrift (2015)
R. Giova - J. Differential Equations (2015)
F. Giannetti, A. Passarelli di Napoli, C. Scheven - J. Lon. Math. Soc. (2016)
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Model functional

Double-phase functional

$$w \mapsto \int_\Omega \left[|Dw|^p + a(x)\,(1+|Dw|^2)^{\frac{q}{2}} \right] dx$$

with $q > p > 1$ and $a(\cdot)$ a bounded Sobolev coefficient

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STATEMENT OF THE PROBLEM AND MAIN RESULTS

Assumptions

Variational integral

$$\mathcal{F}(u) := \int_\Omega f(x, Du)\, dx$$

Obstacle problem

$$\min \{\mathcal{F}(u) : u \in \mathcal{K}_\psi(\Omega)\}$$

- Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 2$
- $f : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is a Carathéodory function, convex and of class $\mathcal{C}^2$ with respect to the second variable
- $\psi : \Omega \to [-\infty, +\infty)$, called obstacle, belongs to the Sobolev space $W^{1,p}(\Omega)$
- $\mathcal{K}_\psi(\Omega) := \left\{ w \in u_0 + W_0^{1,p}(\Omega) : w \geq \psi \text{ a.e. in } \Omega \right\}$
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Hypothesis

Exponents condition

$$1 \leq \frac{q}{p} < 1 + \frac{r-n}{r\,n} = 1 + \frac{1}{n} - \frac{1}{r}$$

where we consider $q > p \geq 2$ *and where* $r > n$

M. Eleuteri, P. Marcellini, E. Mascolo - Ann. Mat. Pura Appl. (2016)

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Hypothesis

We suppose that there exist:

- $\nu > 0$ and $L > 0$
- $h : \Omega \to [0, +\infty)$ such as $h(x) \in L^r_{\mathrm{loc}}(\Omega)$

Hypothesis on functional

$$\nu\,(1+|\xi|^2)^{\frac{p}{2}} \leq f(x,\xi) \leq L\,(1+|\xi|^2)^{\frac{q}{2}}$$

$$\nu\,(1+|\xi|^2)^{\frac{p-2}{2}}\,|\lambda|^2 \leq \sum_{i,j} f_{\xi_i\xi_j}(x,\xi)\,\lambda_i\,\lambda_j \leq L\,(1+|\xi|^2)^{\frac{q-2}{2}}\,|\lambda|^2$$

$$|f_{x\xi}(x,\xi)| \leq h(x)\,(1+|\xi|^2)^{\frac{q-1}{2}}$$

for all $\lambda, \xi \in \mathbb{R}^n$, $\lambda = \lambda_i$, $\xi = \xi_i$, $i = 1, 2, \ldots, n$ *a.e. in* Ω

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First main result

Theorem (A priori estimate)

Let $u \in \mathcal{K}_\psi(\Omega)$ be a smooth solution to the obstacle problem under the assumptions of growth and ellipticity stated before. If $\psi \in W^{2,r}_{\mathrm{loc}}(\Omega)$, then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and the following estimate

$$\|Du\|_{L^\infty(B_\rho)} \leq C \left\{ \int_{B_R} [1 + f(x, Du)]\, dx \right\}^\beta$$

holds for every $0 < \rho < R$ and with positive constants C and β depending on $n, r, p, q, \nu, L, R, \rho$ and on the local bounds for $\|D\psi\|_{W^{1,r}}$ and $\|h\|_{L^r}$.

Relaxation

Now we want to present a meaningful definition of relaxation for the variational obstacle problem we are focusing about.

Class of solutions

$$\mathcal{K}^*_\psi(\Omega) := \{w \in u_0 + W^{1,q}_0(\Omega) : w \geq \psi \text{ a.e. in } \Omega\}$$

C. De Filippis - J. Math. Anal. Appl., to appear

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Relaxation

Relaxed functional

$$\overline{\mathcal{F}}(u) := \inf_{\mathcal{C}(u)} \{ \liminf_{j \to +\infty} \mathcal{F}(u_j) \}$$

$$\mathcal{C}(u) := \left\{ \{u_j\}_{j \in \mathbb{N}} \subset \mathcal{K}^*_\psi(\Omega) : u_j \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega) \right\}$$

P. Marcellini - Ann. IHP Anal. Non Lin. (1986)

E. Acerbi, G. Bouchitté, I. Fonseca - Ann. IHP Anal. Non Lin. (2003)

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E. Acerbi, G. Bouchitté, I. Fonseca - Ann. IHP Anal. Non Lin. (2003)

Relaxation

Relaxed functional

$$\overline{\mathcal{F}}(u) := \inf_{\mathcal{C}(u)} \{\liminf_{j\to+\infty} \mathcal{F}(u_j)\}$$

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P. Marcellini - Ann. IHP Anal. Non Lin. (1986)

E. Acerbi, G. Bouchitté, I. Fonseca - Ann. IHP Anal. Non Lin. (2003)

Second main result

Theorem

Assume that f satisfies the hypothesis we stated before.
The Dirichlet problem

$$\min \left\{ \overline{\mathcal{F}}(u) : u \in \mathcal{K}_\psi(\Omega) \right\}$$

with $\overline{\mathcal{F}}$ defined above and $u_0 \in W^{1,q}(\Omega)$, has at least one locally Lipschitz continuous solution.

A PRIORI ESTIMATE

The linearization procedure

F. Duzaar - Dissertation Thesis (1985)
M. Fuchs - Analysis (1985)

F. Duzaar - J. Reine Angew. Math. (1987)
M. Fuchs - Ann. Mat. Pura Appl. (1990)
M. Fuchs - Advanced Lectures in Mathematics (1994)
M. Fuchs, L. Gongbao - Abstr. Appl. Anal. (1998)
M. Fuchs, G. Mingione - Manuscripta Math. (2000)

C. Benassi, M. Caselli - Rendiconti Lincei (2020)

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C. Benassi, M. Caselli - Rendiconti Lincei (2020)

Variational inequality

$$\int_\Omega D_\xi f(x, Du) \cdot D(\varphi - u)\, dx \geq 0$$

that holds true for all $\varphi \in W^{1,q}_{\mathrm{loc}}(\Omega)$, $\varphi \geq \psi$

$$g := -div(D_\xi f(x, Du))\chi_{[u=\psi]}$$

Higher differentiability

$$D\psi \in W^{1,r}_{\mathrm{loc}}(\Omega) \quad \Longrightarrow \quad \left(1 + |Du|^2\right)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega)$$

C. Gavioli - J. Elliptic Parabol. Equ. (2019)

The linearization procedure

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C. Gavioli - J. Elliptic Parabol. Equ. (2019)

The linearization procedure

Starting point

$$\int_\Omega D_\xi f(x, Du) \cdot D\eta \, dx = \int_\Omega g \, \eta \, dx \qquad \forall \, \eta \in C_0^1(\Omega)$$

Estimate on g

$$|g| \leq h(x) \left(1 + |D\psi|^2\right)^{\frac{q-1}{2}} + L \left(1 + |D\psi|^2\right)^{\frac{q-2}{2}} |D^2 \psi|$$

The linearization procedure

Starting point

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"Second variation" system

$$\int_\Omega \left(\sum_{i,j=1}^{n} f_{\xi_i \xi_j}(x, Du)\, u_{x_j x_s}\, D_{x_i}\varphi + \sum_{i=1}^{n} f_{\xi_i x_s}(x, Du)\, D_{x_i}\varphi \right) dx = \int_\Omega g\, D_{x_s}\varphi\, dx$$

$$\textit{for all } s = 1, \ldots, n \textit{ and for all } \varphi \in W_0^{1,2}(\Omega).$$

- $0 < \rho < R$ with B_R compactly contained in Ω
- $\eta \in C_0^1(\Omega)$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on B_ρ, $\eta \equiv 0$ outside B_R, $|D\eta| \le \frac{C}{R-\rho}$
- $\gamma \ge 0$

$$\varphi = \eta^2 \left(1 + |Du|^2\right)^\gamma u_{x_s}$$

A priori estimate

"Second variation" system

$$\int_\Omega \left(\sum_{i,j=1}^{n} f_{\xi_i\xi_j}(x, Du)\, u_{x_j x_s}\, D_{x_i}\varphi + \sum_{i=1}^{n} f_{\xi_i x_s}(x, Du)\, D_{x_i}\varphi \right) dx = \int_\Omega g\, D_{x_s}\varphi\, dx$$

for all $s = 1, \dots, n$ *and for all* $\varphi \in W_0^{1,2}(\Omega)$.

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A priori estimate

Summing up the 9 integrals and using the hypothesis we obtain

$$
\begin{aligned}
&\int_\Omega \eta^2 \left(1+|Du|^2\right)^{\frac{p-2}{2}+\gamma} |D^2 u|^2 \, dx \\
\leq \quad & C\,\Theta\,(1+\gamma^2) \left[\int_\Omega (\eta^{2m} + |D\eta|^{2m}) \left(1+|Du|^2\right)^{\left(q-\frac{p}{2}+\gamma\right)m} dx \right]^{\frac{1}{m}}
\end{aligned}
$$

where the constant C depends on ν, L, n, p, q but it is independent of γ

$$\Theta = 1 + \|g\|^2_{L^r(\Omega)} + \|h\|^2_{L^r(\Omega)}$$

$$m = \frac{r}{r-2}$$

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The iteration process

C. De Filippis, G. Mingione - J. Geom. Anal. (2020)

C. De Filippis - J. Math. Anal. Appl., to appear

Final result

$$\|Du\|_{L^\infty(B_\rho)} \leq C \left\{ \int_{B_R} [1 + f(x, Du)]\, dx \right\}^\beta$$

Holds for every $0 < \rho < R$ *and with positive constants* C *and* β *depending on* $n, r, p, q, \nu, L, R, \rho$ *and on the local bounds for* $\|D\psi\|_{W^{1,r}}$ *and* $\|h\|_{L^r}$.

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APPROXIMATION IN CASE OF OCCURRENCE OF LAVRENTIEV PHENOMENON

L. Boccardo, P. Marcellini - Ann. Mat. Pura Appl. (1976)

Lemma

For each $u \in \mathcal{K}_\psi(\Omega)$, there exists a sequence $u_k \in \mathcal{K}^*_\psi(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and

$$\overline{\mathcal{F}}(u) = \lim_{k \to +\infty} \mathcal{F}(u_k)$$

This Lemma's proof is based on a diagonal argument with sequences of elements in the class of solutions $\mathcal{K}^*_\psi(\Omega)$.

Convergence Lemma

L. Boccardo, P. Marcellini - Ann. Mat. Pura Appl. (1976)

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For each $u \in \mathcal{K}_\psi(\Omega)$*, there exists a sequence* $u_k \in \mathcal{K}^*_\psi(\Omega)$ *such that* $u_k \rightharpoonup u$ *weakly in* $W^{1,p}(\Omega)$ *and*

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This Lemma's proof is based on a diagonal argument with sequences of elements in the class of solutions $\mathcal{K}^*_\psi(\Omega)$.

Theorem

Let f be satisfying the growth conditions and strictly convex at infinity and $f_{\xi\xi}$ and $f_{\xi x}$ be two Carathéodory functions, satisfying ellipticity and growing conditions. Then there exists a sequence of $\mathcal{C}^2-$functions

$$f^{lk} : \Omega \times \mathbb{R}^n \to [0, +\infty)$$

with f^{lk} convex in the last variable and strictly convex at infinity, such that f^{lk} converges to f as $l \to \infty$ and $k \to \infty$ for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^n$ and uniformly in $\Omega_0 \times K$, where $\Omega_0 \Subset \Omega$ and K being a compact set of $\mathbb{R}^n$.
Moreover the functions f^{lk} satisfy the hypothesis with constants which are independent on k and satisfy the additional hypothesis necessaries to conclude our proof with constants which are dependent only on k.

I. Fonseca, N. Fusco, P. Marcellini - ESAIM Control Optim. Calc. Var. (2002)

Regularization of f

Theorem

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I. Fonseca, N. Fusco, P. Marcellini -
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The existence Theorem

Theorem

Assume that f satisfies the hypothesis we stated before. The Dirichlet problem

$$\min \left\{ \overline{\mathcal{F}}(u) : u \in \mathcal{K}_{\psi}(\Omega) \right\}$$

with $\overline{\mathcal{F}}$ defined above and $u_0 \in W^{1,q}(\Omega)$, has at least one locally Lipschitz continuous solution.

Variational problems

$$\inf\left\{\int_\Omega f^{lk}(x,Du)\,dx : u\in\mathcal{K}^*_\psi(\Omega)\right\}$$

with $f^{lk}(x,\xi) \;=\; f^{l}(x,\xi)+\frac{1}{k}\,(1+|\xi|^2)^{\frac{q}{2}}$

There exists a solution $u^{lk}\in\mathcal{K}^*_\psi(\Omega)$ with $u_0\in W^{1,q}(\Omega)$.

Moreover, we can consider $u_0\in\mathcal{K}^*_\psi(\Omega)$.

Remark

Let us notice that, by replacing u_0 *by* $\tilde{u}_0=\max\{u_0,\psi\}$*, we may assume that the boundary value function* u_0 *satisfies* $u_0\geq\psi$ *in* Ω*. Moreover assumptions* $f(x,Du)\in L^1_{\mathrm{loc}}(\Omega)$ *and* $f(x,Du_0)\in L^1_{\mathrm{loc}}(\Omega)$ *imply* $f(x,D\tilde{u}_0)\in L^1_{\mathrm{loc}}(\Omega)$.

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Proof of the existence Theorem

Variational problems

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Proof of the existence Theorem

By the growth conditions, the minimality of u^{lk} and the previous remark

$$\int_\Omega |Du^{lk}|^p\,dx \le \int_\Omega f^l(x, Du_0)\,dx + \frac{1}{k}\int_\Omega (1+|Du_0|^2)^{\frac{q}{2}}\,dx$$

$$\lim_{l\to+\infty}\int_\Omega |Du^{lk}|^p\,dx \le \int_\Omega f(x, Du_0)\,dx + \frac{1}{k}\int_\Omega (1+|Du_0|^2)^{\frac{q}{2}}\,dx$$

By the previous Theorem, the functions f^{lk} satisfy the hypothesis, so we can apply the a-priori estimate on u^{lk} and obtain for all $B \Subset \Omega$ that

$$\|Du^{lk}\|_{L^\infty(B)} \le C\left\{\int_\Omega [1+f(x, Du_0)+\frac{1}{k}(1+|Du_0|^2)^{\frac{q}{2}}]\,dx\right\}^{\frac{\gamma}{p}}$$

where C, γ depend on all the parameters except for l, k.

Therefore there exist $u^k \in \mathcal{K}_\psi(\Omega)$, for all $k \in \mathbb{N}$, such that

$$u^{lk} \overset{l\to\infty}{\longrightarrow} u^k \text{ weakly in } W^{1,p}(\Omega)$$

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Therefore there exist $u^k \in \mathcal{K}_\psi(\Omega)$, for all $k \in \mathbb{N}$, such that

$$u^{lk} \overset{l\to\infty}{\rightharpoonup} u^k \text{ weakly in } W^{1,p}(\Omega)$$

$$u^{lk} \overset{l\to\infty}{\rightharpoonup^*} u^k \text{ weakly star in } W^{1,\infty}_{\text{loc}}(\Omega)$$

Proof of the existence Theorem

Following the previous estimates we also have

$$\|Du^k\|_{L^p(\Omega)} \leq \int_\Omega f(x, Du_0)\, dx + \int_\Omega (1 + |Du_0|^2)^{\frac{q}{2}}\, dx$$

$$\|Du^k\|_{L^\infty(B)} \leq C \left\{ \int_\Omega [1 + f(x, Du_0) + \frac{1}{k}(1 + |Du_0|^2)^{\frac{q}{2}}]\, dx \right\}^{\frac{\gamma}{p}}$$

So there exists, up to subsequences, $\overline{u} \in \mathcal{K}_\psi(\Omega)$ such that

$$u^k \overset{k\to\infty}{\longrightarrow} \overline{u} \text{ weakly in } W^{1,p}(\Omega)$$

$$u^k \overset{k\to\infty}{\longrightarrow} \overline{u} \text{ weakly star in } W^{1,\infty}_{\text{loc}}(\Omega)$$

C. Gavioli - J. Elliptic Parabol. Equ. (2019)

Strong convergence

$$u^k \to \overline{u} \text{ in } W^{1,p}_0(\Omega) + u_0, \qquad \overline{u} \in K_\psi(\Omega)$$

Proof of the existence Theorem

Following the previous estimates we also have

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Strong convergence

$$u^k \to \overline{u} \text{ in } W_0^{1,p}(\Omega) + u_0, \qquad \overline{u} \in K_\psi(\Omega)$$

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Proof of the existence Theorem

For any fixed $k \in \mathbb{N}$, using the uniform convergence of f^l to f in $\Omega_0 \times K$ (for any K compact subset of $\mathbb{R}^n$) and the minimality of u^{lk}, we get for all $w \in \mathcal{K}^*_\psi(\Omega)$

$$\int_{\Omega_0} f(x, Du^k)\, dx \leq \liminf_{l \to \infty} \int_\Omega f^l(x, Dw)\, dx + \frac{1}{k} \int_\Omega (1 + |Dw|^2)^{\frac{q}{2}}\, dx$$

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By the relaxed functional's definition, we have

$$\overline{\mathcal{F}}(\overline{u}) \leq \liminf_{k \to \infty} \int_\Omega f(x, Du^k)\, dx \leq \int_\Omega f(x, Dw)\, dx \qquad \forall\, w \in \mathcal{K}^*_\psi(\Omega)$$

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Let $v \in \mathcal{K}_\psi(\Omega)$. By the Convergence Lemma, there exists $u_k \in \mathcal{K}^*_\psi(\Omega)$ such that $u_k \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$ and

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By the last estimate on the relaxed functional

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THANK YOU FOR THE ATTENTION!