Existence and regularity resutls of porous medium type equations with drift term

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Monday's Nonstandard Seminar
May 31, 2021.

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Porous medium equation with drifts

Consider a nonegative weak solution of

$$\underbrace{\rho_t}_{\text{Time evolution}} = \underbrace{\Delta\rho^m}_{\text{Diffusion}} + \underbrace{\nabla\cdot(V\rho)}_{\text{Drift}} \quad \text{in } \Omega_T \tag{1}$$

- Diffusion: $\Delta\rho^m \sim \nabla\cdot\left(\rho^{m-1}\nabla\rho\right)$
- Heat equation: $m = 1$
- Degenerate (Porous media equations): $m > 1$, traffic jam
- Singular (Fast diffusion equations): $0 < m < 1$,
  scattering or gathering a flock of birds

Fluid dynamics

- Drift-diffusion equations (Silvestre-Vicol-Zlatos, ARMA 2013)

$$\partial_t \theta + u \cdot \nabla \theta + (-\Delta)^s \theta = 0, \quad \text{with} \quad \theta(0,\cdot) = \theta_0.$$

  where $s \in (0,1]$, and u is a given divergence-free vector field.
- Continuity results and counterexamples under various conditions on s and u
- divergence-free drifts: importance due to incompressibility
- surface quasi-deostrophic (SQG) model
- the drift plays an important role in the well-posedness of the problem

Keller-Segel Model *Bacillus Subtilis*

Keller-Segel model for the motion of swimming bacteria (Chung-H.-Kang-Kim, JDE 2017)

Consider the following system

$$\begin{cases} \partial_t n - \Delta n^{1+\alpha} + u \cdot \nabla n = -\nabla \cdot (\chi(c) n^q \nabla c), \\ \partial_t c - \Delta c + u \cdot \nabla c = -k(c) n, \\ \partial_t u - \Delta u + \nabla p = -n \nabla \phi, \\ \nabla \cdot u = 0 \end{cases}$$

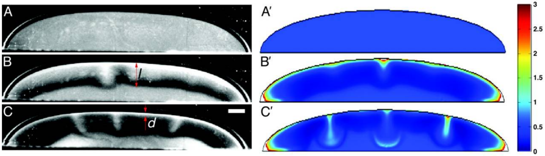
where $\alpha > 0$, $q \geq 1$, and

n : the density of bacteria,

c : the oxygen concentration,

u, p : the velocity vector of the fluid and associated pressure.

the motion of swimming bacteria, *Bacillus Subtilis*



- Let $q = 1$, and concentrate on the porous medium type equation

$$\partial_t n - \Delta n^{1+\alpha} + u \cdot \nabla n = -\nabla \cdot (\chi(c) n \nabla c)$$

where

$$\begin{aligned} \Delta n^{1+\alpha} &: \text{ diffusion, anticongestion effect,} \\ u \cdot \nabla n &: \text{ motion affected by the fluid,} \\ \nabla \cdot (\chi(c) n \nabla c) &: \text{ motion of bacteria pursuing high oxygen.} \end{aligned}$$

Degenerate diffusion-drift equations

$$\rho_t = \Delta \rho^m + \nabla \cdot (V(x,t)\,\rho)$$

- If $m = 1$, classical diffusion-drift equation.
  - $V \in L_t^p L_x^q$: Ladyzhenskaya-Ural'ceva 1961, Lieberman 1996, Nazarov-Ural'ceva 2009
  - Osada 1987 $V \in L_t^\infty (L_x^\infty)^{-1}$
  - Zhang 2004, Friedlander-Vicol 2011 $V \in L_t^\infty \mathrm{BMO}_x^{-1}$
  - Seregin-Silvestre-Sverak-Zlatos 2012
    - divergence-free drift (incompressible flows)
  - Silvestre-Vicol-Zlatos 2013
    - Fractional diffusion $((-\Delta)^s u)$ with divergence-free drift in 2-d
- If $V = 0$, classical porous medium equation.
  - Vazquez 2007, DiBenedetto-Gianazza-Vespri 2012
  - With initially integrable data, unique nonnegative weak solutions exist.
  - Immediately becomes Hölder continuous for positive times.

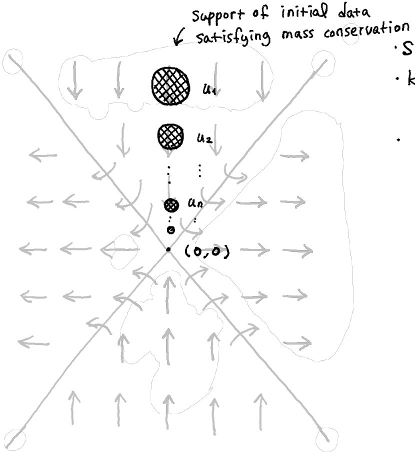
Regularity results:

What are the critical conditions on $V \in L^p_x L^q_t$ regarding the continuity of a solution?

- Kim-Zhang (SIMA, 2018)
- Chung-H.-Kang-Kim (JDE, 2017)
- H.-Zhang (to appear at NLA)

joint work with Kyungkeun Kang (Yonsei Univ.) and Yuming Paul Zhang (UCSD).

Diffusion Vs. Drift



- Silvestre-Vicol-Zlatoš 2013 (linear)
- Kim-Zhang 2018 (degenerate)

- $u_t = \Delta u^m + \nabla(u \cdot V)$, $m > 1$

  Δu^m: $\mathrm{div}(u^{m-1} \nabla u)$

  - When u hits 0, then the drift is dominating.
  - Construct $\{u_n\}$ for given initial data in a shrinking compactly supported domain
  - Possible to build that u_∞, in fact, breaks any modulus of continuity at $(0,0)$.
  - $0 < m < 1$ (Singular), the drift is dominating when $u \to \infty$.

Counterexamples

Consider $$u_t = \Delta u^m + \nabla \cdot (uV), \quad \text{with} \quad u(\cdot, 0) = u_0. \qquad (2)$$

Loss of regularity (Kim-Zhang, SIMA 2018)

(a) Let $d \geq 2$ and $1 \leq p \leq d$. Then there exists a sequence of vector fields $\{\nabla \Phi_A(x)\}_{A \in \mathbb{N}}$ which are uniformly bounded in $L^p(\mathbb{R}^d)$ such that the following holds. Let u_A solve (2) with $V = \nabla \Phi_A$ and with a smooth, compactly supported initial data u_0. Then

$$\sup_{x \in \mathbb{R}^d, t>0} u_A(x,t) \to \infty \quad \text{as} \quad A \to \infty.$$

(b) There exist a family of potentials Φ_A such that $\nabla \Phi_A \in L^d(\mathbb{R}^d)$ and a family of initial data u_0^A which are uniformly bounded in $L^1 \cap L^\infty \cap C^\infty$ such that the following holds: The solutions u^A with $V = \nabla \Phi_A$ with initial data u_0^A stays uniformly bounded but lacks any uniform modulus of continuity as $A \to \infty$.

Scaling invariant classes of $V(x)$ in L^∞-sense

- Consider
$$\rho_t = \Delta\rho^m + \nabla\cdot(V\rho) \quad \text{for} \quad m \geq 1.$$
- For given $\lambda > 0$ $r > 0$, let
$$\rho_{\lambda,r}(x,t) := \lambda\,\rho(rx, \lambda^{m-1}r^2 t) \;\text{ and }\; V_{\lambda,r} := \lambda^{m-1} r V(rx),$$
which solves
$$\partial_t\rho_{\lambda,r} = \Delta\rho^m_{\lambda,r} + \nabla\cdot(V_{\lambda,r}\rho_{\lambda,r}).$$

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which solves
$$\partial_t \rho_{\lambda,r} = \Delta\rho^m_{\lambda,r} + \nabla\cdot(V_{\lambda,r}\rho_{\lambda,r}).$$
- Note that
$$\|V_{\lambda,r}(\cdot)\|_{L^p(\mathbb{R}^d)} = \lambda^{m-1} r^{1-\frac{d}{p}} \|V(\cdot)\|_{L^p(\mathbb{R}^d)}.$$
- Critical case: $p = d$
- If $p < d$, then $1 - d/p < 0$. Boundeness breaks when $r \to 0$.

Scaling invariant classes of $V(x,t)$ in L^∞-sense

- Let
$$V_{\lambda,r}(x,t) = \lambda^{1-m} r V(rx, \lambda^{1-m} r^2 t).$$
- Then we make an observation that
$$\|V_{\lambda,r}\|_{L_t^{2\hat{q}_1} L_x^{2\hat{q}_2}} = \left(\int_{-\lambda^{1-m} r^2}^{0} \left[\int_{B_r} V_{\lambda,r}^{2\hat{q}_2}(y,s)\, dy \right]^{\frac{2\hat{q}_1}{2\hat{q}_2}} ds \right)^{\frac{1}{2\hat{q}_2}}$$
$$= \lambda^{(1-m)\left(1-\frac{1}{2\hat{q}_1}\right)} r^{1-\frac{1}{2}\left(\frac{2}{\hat{q}_1}+\frac{d}{\hat{q}_2}\right)} \|V\|_{L_t^{2\hat{q}_1} L_x^{2\hat{q}_2}}$$
- Equivalently, we observe that
$$V \in L_x^{2\hat{q}_2} L_t^{2\hat{q}_1}, \quad \text{where} \quad \frac{d}{\hat{q}_2} + \frac{2}{\hat{q}_1} = 2 - d\kappa$$
for some $\kappa \in [0, 2/d)$.

In Kim-Zhang (SIMA, 2018), there are two results.

- time-independent $V(x) \in L^p_x$
  - Hölder regularity if
$$p > d + \frac{4}{d+2}$$
  - counterexamples when $p = d$
    Constructing a limit solution that breaks any modulus of continuity.
- time-dependent $V(x,t)$
  - Hölder continuity if
$$V \in L^p_x L^\infty_t \quad \text{for} \quad p > d + \frac{4}{d+2}$$

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  $$V \in L^p_x L^\infty_t \quad \text{for} \quad p > d + \frac{4}{d+2}$$
- Question marks on the range $d < p \leq d + \frac{4}{d+2}$

Hölder regularity

Local Hölder continuity Result (CHKK, 2017)

Let ρ be a nonnegative bounded weak solution under

$$V \in L^{2\hat{q}_1, 2\hat{q}_2}_{x,t}, \quad \nabla V \in L^{\hat{q}_1,\hat{q}_2}_{x,t} \text{ where } \frac{d}{\hat{q}_1} + \frac{2}{\hat{q}_2} = 2 - d\kappa$$

for some $\kappa \in (0, 2/d)$. Then u is locally Hölder continuous. Moreover, there exist positive constants $\beta \in (0,1)$ and γ depending on data (that is, $m, d, \Omega_T, \Omega_T', \|V\|_{2\hat{q}_1, 2\hat{q}_2}$) such that

$$|\rho(x_1, t_1) - \rho(x_2, t_2)| \le \gamma \|\rho\|_\infty \left(\frac{|x_1 - x_2| + \|\rho\|_\infty^{\frac{m-1}{2}} |t_1 - t_2|^{1/2}}{\text{dist}_p(\Omega_T'; \partial_p \Omega_T)} \right)^\beta .$$

- p-Laplace equation with lower order terms (DiBenedetto, 1983)
- $G-$Laplacian (H.-Lieberman, 2015)

- Recall
$$V \in L_x^{2\hat{q}_2} L_t^{2\hat{q}_1}, \quad \frac{d}{\hat{q}_2} + \frac{2}{\hat{q}_1} = 2 - d\kappa$$
for some $\kappa \in [0, 2/d)$.
- Subcritical Regime: in case $\kappa \in (0, 2/d)$
  - Hölder continuity for a bounded weak solution
- Critical Regime: in case $\kappa = 0$ where $\hat{q}_1 \in (1, \infty]$ and $\hat{q}_2 \in [d, \infty)$
  - counterexample failing any modulus of continuity
  - Uniform continuity under the divergence-free condition on V
  - Not sure when $\hat{q}_2 = \infty$ due to the regularity in terms of t
- Supercritical Regime: in case $\kappa < 0$.
  - counterexample of $(-\Delta)^s u$, $s \in (0, 1]$, Silvestre-Vicol-Zlatos (ARMA 2013)
  - open question for porous medium type equation
  - unknown whether or not there exist discontinuous solutions for $m > 1$

Subcritical regime: Hölder continuity

Theorem H.-Zhang (NLA, to appear)

Let u be a non-negative bounded weak solution with $m \geq 1$ in Q_1 under B is in subcritical regime. Then u is uniformly Hölder continuous in $Q_{\frac{1}{2}}$.

- Let $v = \rho^m$. Then we rewrite the equation to

$$\partial_t v^{1/m} = \Delta v + \nabla \cdot (B\, v^{1/m}).$$

- DiBenedetto 1993 : $\quad |b(x,t,u,Du)| \leq c|Du|^p + \varphi(x,t), \quad \varphi \in L_x^{\hat{q}} L_t^{\hat{r}}$
- Energy estimate: $A_{k,\rho}^{+} := Q_\rho \cap \{(v - \mu_+ + k)_+ > 0\}$

$$\begin{aligned}
&\mu_+^{-\beta} \sup_{t_0 \leq t \leq t_1} \int_{K_\rho \times \{t\}} v_+^2 \zeta^2 \, dx + \iint_{Q_\rho} \left|\nabla (v_+ \zeta)\right|^2 \, dx\, dt \\
&\leq C(\mu_+ - k)^{-\beta} \iint_{Q_\rho} v_+^2 |\zeta \zeta_t| \, dx\, dt + C \iint_{Q_\rho} v_+^2 |\nabla \zeta|^2 \, dx\, dt \\
&+ C\mu_+^{2/m} \|B\|^2_{L_t^{2\hat{q}_1} L_x^{2\hat{q}_2}(Q_\rho)} \left[\int_{t_0}^{t_1} \left[A_{k,\rho}^{+}(t)\right]^{\frac{q_1}{q_2}} \, dt\right]^{\frac{2(1+\kappa)}{q_1}}.
\end{aligned} \tag{3}$$

Critical regime

Let

$$\varrho_B(r) := \sup_{(x_0,t_0)} \|B\|_{L_t^{2\hat{q}_1} L_x^{2\hat{q}_2}(((x_0,t_0)+Q_r)\cap Q_1)} \quad \text{with} \quad Q_r := K_r \times (-r^2, 0].$$

Theorem H.-Zhang (NLA)

Suppose that B is divergence-free. Let ρ be a non-negative bounded weak solution with $m \geq 1$ in Q_1. Then ρ is uniformly continuous in $Q_{\frac{1}{2}}$ depending on $m, d, \hat{q}_1, \hat{q}_2$ and $\varrho_B(r)$.

- From $\nabla \cdot B = 0$, we are able to obtain local energy estimate.
- Roughly speaking, $\int \nabla(\rho B)\rho^m \sim \int B \cdot \nabla\rho\, \rho^m \sim \int B \cdot \nabla \rho^m \rho \sim \epsilon \int |\nabla \rho^m|^2 + c(\epsilon) \int |B|^2 \rho^2$

$$\begin{aligned}
&\mu_+^{-\beta} \sup_{t_0 \leq t \leq t_1} \int_{K_\rho \times \{t\}} v_+^2 \zeta^2 \, dx + \iint |\nabla(v_+ \zeta)|^2 \, dxdt \\
&\leq C \frac{k^2}{(\mu_+ - k)^\beta} \iint_{v_+ > 0} |\zeta \zeta_t| \, dx + Ck^2 \iint_{v_+ > 0} |\nabla \zeta|^2 \, dxdt \qquad (4) \\
&\quad + C(\mu_+ - k)^{-2\beta} \|B\|^2_{L_t^{2\hat{q}_1} L_x^{2\hat{q}_2}(Q_\rho)} \|v_+ \zeta\|^2_{L_t^{q_1} L_x^{q_2}(Q_\rho)}.
\end{aligned}$$

Theorem H.-Zhang (NLA)

There exist sequences of vector fields $\{B_n\}_n$, which are uniformly bounded in $L^d(\mathbb{R})$ and

$$\varrho_{B_n}(r) := \sup_{(x_0,t_0)} \|B_n\|_{L^d((x_0,t_0)+Q_r)} \leq \omega(r) \quad \text{for some modulus of continuity } \omega,$$

along with sequences of uniformly bounded functions $\{u_n\}$ in K_1 which are stationary solutions with $B = B_n$ such that, they do not share any common mode of continuity.

- The divergence free condition is essential, otherwise it fails uniform continuity.
- sharp condition even for linear diffusion cases

Existence Results

- with Kyungkeun Kang (Yonsei Univ.) and Hwakil Kim (Hannam Univ.)

Let us recall the condition

$$V \in L_x^{q_1} L_t^{q_2} \quad \text{where} \quad \frac{d}{q_1} + \frac{2}{q_2} \leq 1$$

- appears also for linear equations.
- so called Serrin conditions
  - Considering Navier-Stokes equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \Delta u = -\nabla \pi, \\ \nabla \cdot u = 0. \end{cases}$$

    If $u \in L^p(0, T; L^q(\Omega))$ for $\frac{2}{p} + \frac{3}{q} \leq 1$, $p \in [2, \infty)$, then u is regular.
  - by Prodi (1959) and Serrin (1962), independently.

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  - by Prodi (1959) and Serrin (1962), independently.
- Anyway to find out connection of nonlinear factor and drift ?

- Let, for $q \geq 1$ and $\lambda > 0$,

$$\rho_\lambda(x,t) = \lambda^{d/q}\rho(\lambda x, \lambda^\beta t) \quad \text{and} \quad V_\lambda(x,t) = \lambda^\alpha V(\lambda x, \lambda^\beta t),$$

where positive constants α, β to be determined later.

- The factor $\lambda^{d/q}$ is to preserve L^q_x norm.

- Let, for $q \geq 1$ and $\lambda > 0$,

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where positive constants α, β to be determined later.

- The factor $\lambda^{d/q}$ is to preserve L^q_x norm.
- Then the PDE provides

$$\lambda^{d/q+\beta}\rho_t = \lambda^{\frac{md}{q}+2}\Delta\rho^m + \lambda^{d/q+\alpha+1}\nabla\cdot(\rho V).$$

Then the equality $\frac{d}{q} + \beta = \frac{md}{q} + 2 = \frac{d}{q} + \alpha + 1$ determines

$$\alpha = q_{m,d} + 1, \quad \beta = q_{m,d} + 2, \quad \text{for} \quad q_{m,d} := \frac{d(m-1)}{q}.$$

- Let, for $q \geq 1$ and $\lambda > 0$,

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$$\alpha = q_{m,d} + 1, \quad \beta = q_{m,d} + 2, \quad \text{for} \quad q_{m,d} := \frac{d(m-1)}{q}.$$

- With such α and β, we observe the following, for $q_1, q_2 > 0$,

$$\|V_\lambda(x,t)\|_{L^{q_1,q_2}_{x,t}} = \lambda^{\alpha - \frac{d}{q_1} - \frac{\beta}{q_2}} \|V(x,t)\|_{L^{q_1,q_2}_{x,t}},$$

and then see the following relation

$$\alpha - \frac{d}{q_1} - \frac{\beta}{q_2} = 0 \quad \Longleftrightarrow \quad \frac{d}{q_1} + \frac{2+q_{m,d}}{q_2} = 1 + q_{m,d}.$$

Scaling invariant classes in L^q-sense

- For $m \geq 1$ and $q \geq 1$, let
$$q_{m,d} := \frac{d(m-1)}{q}.$$
- Scaling invariant classes
$$\mathcal{S}_{m,q}^{(q_1,q_2)} := \left\{ V : \|V\|_{L_{x,t}^{q_1,q_2}} < \infty \text{ where } \frac{d}{q_1} + \frac{2+q_{m,d}}{q_2} = 1 + q_{m,d} \right\}.$$
- Sacling invariant norm
$$\|V\|_{\mathcal{S}_{m,q}^{(q_1,q_2)}} := \|V\|_{L_{x,t}^{q_1,q_2}}$$
- Note that
$$q_{m,d} \to 0 \quad \text{as either} \quad m = 1, \quad \text{or} \quad q = \infty.$$

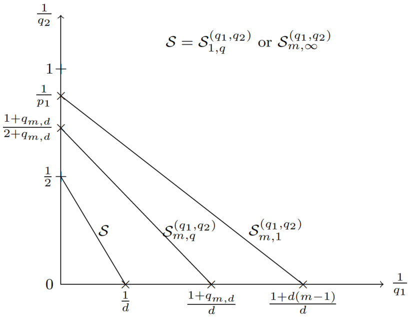


Figure 1. pairs $(\frac{1}{q_1}, \frac{1}{q_2})$ in $\mathcal{S}_{m,q}^{(q_1,q_2)}$ in (1.4)

Existence of a linear type equation

$$\partial_t \rho = \nabla \cdot (\nabla \rho - v\rho) \quad \text{with} \quad \rho(\cdot, 0) = \rho_0.$$

Theorem (Kang-Kim, SIMA 19)

For $\alpha > d$, suppose that

$$\rho_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d), \quad \text{for } p \geq \frac{\alpha}{\alpha - 2}.$$

And assume further that

$$v \in L^\beta(0, T; L^\alpha(\mathbb{R}^d)) \quad \text{for } \frac{d}{\alpha} + \frac{2}{\beta} \leq 1.$$

Then, there exists an absolutely continuous curve $\rho \in AC(0, T; \mathcal{P}_2(\mathbb{R}^d))$ such that ρ solves

$$\partial_t \rho = \Delta \rho - \nabla \cdot (v\rho) \quad \rho(0) = \rho_0,$$

in the sense of distributions.

Wasserstein space - $\mathcal{P}_2(\mathbb{R}^d)$

- $\mathcal{P}_2(\mathbb{R}^d) := \{\mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 d\mu(x) < \infty\}$,
  where $\mathcal{P}(\mathbb{R}^d)$ is the set of all Borel probability measures on $\mathbb{R}^d$.
- Given $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, we define *Wasserstein distance* $W_2(\mu, \nu)$ by

$$W_2^2(\mu,\nu) := \min_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\gamma(x,y),$$

  where $\gamma \in \Gamma(\mu,\nu)$ means $\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$\gamma(A \times \mathbb{R}^d) = \mu(A), \quad \gamma(\mathbb{R}^d \times A) = \nu(A),$$

  for any Borel subsets $A \subset \mathbb{R}^d$.
- $(\mathcal{P}_2(\mathbb{R}^d), W_2)$: a complete separable metric space.
- Villani (2009), Ambrogio-Gigli-Savare (2008)

Lie-Trotter formula and splitting method for ODE

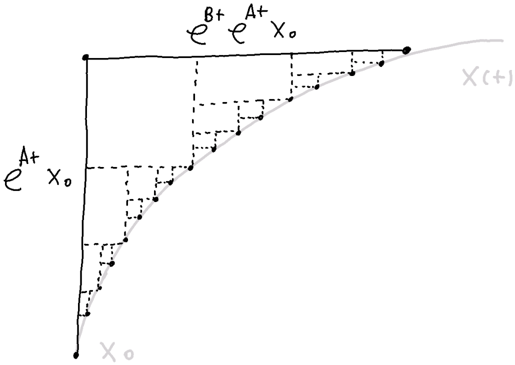
- Splitting method for the ODE

$$x'(t) = (A+B)x(t), \quad x(0) = x_0$$

we have

$$x(t) = e^{(A+B)t}x_0 = \lim_{n\to\infty}\left(e^{\frac{At}{n}}e^{\frac{Bt}{n}}\right)^n x_0.$$

Lie-Trotter formula and splitting method for ODE



Splitting method for linear type equation

$$\rho_t = \Delta\rho + \nabla \cdot (v\rho) \Longrightarrow \rho_t = \nabla \cdot \left(\left\{ \frac{\nabla \rho}{\rho} + v \right\} \rho \right)$$

First, we define two flows Φ_F and Φ_v on $\mathcal{P}_2(\mathbb{R}^d)$ as follows.

- Let $\Phi_F : [0,\infty) \times \mathcal{P}_{2,ac}(\mathbb{R}^d) \mapsto \mathcal{P}_{2,ac}(\mathbb{R}^d)$ be the gradient flow of the entropy function F .i.e.

$$\begin{cases} \partial_t \Phi_F(t,\rho) = \Delta \Phi_F(t,\rho) \\ \Phi_F(0,\rho) = \rho. \end{cases}$$

- For given vector field v, let $\psi_v : [0,\infty) \times [0,\infty) \times \mathbb{R}^d \mapsto \mathbb{R}^d$ be the flow map generated by the vector field. That is,

$$\begin{cases} \frac{d}{dt}\psi_v(s;t,x) = v(t,\psi_v(s;t,x)) \\ \psi_v(s;s,x) = x. \end{cases}$$

We define $\Phi_v : [0,\infty) \times [0,\infty) \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathcal{P}_2(\mathbb{R}^d)$ be by

$$\Phi_v(s;t,\rho) := \psi_v(s;t,\cdot)_{\#}\rho.$$

Splitting method

Linear type equation

$$\rho_t = \Delta\rho + \nabla\cdot(v\rho) \Longrightarrow \rho_t = \nabla\cdot\left(\left\{\frac{\nabla\rho}{\rho} + v\right\}\rho\right)$$

Porous medium type equation

$$\rho_t = \Delta\rho^m + \nabla\cdot(V\rho) \Longrightarrow \rho_t = \nabla\cdot\left(\left\{\frac{\nabla\rho^m}{\rho} + V\right\}\rho\right)$$

- a priori estimates, p-moment and speed estimates
- convergence of ρ^m
- Aubin-Lions theorem, uniform Hölder continuity upto $t = 0$

Summary of existence results

- For $m > 1$, $q \geq 1$, let us denote

$$\left\{\rho_0 \in \mathcal{P}(\mathbb{R}^d) : \ \rho_0 \geq 0 \text{ and } \|\rho_0\|_{L^1_x(\mathbb{R}^d)} = 1\right\},$$

- Further, let

$$1 < p \leq p_q := \min\left\{2, 1 + \frac{d(q-1)+q}{d(m-1)+q}\right\}$$

and, for $\langle x\rangle^p = \left(1+|x|^2\right)^{\frac{p}{2}}$,

$$\left\{\rho_0 \in \mathcal{P}_p(\mathbb{R}^d) : \ \rho_0 \in \mathcal{P}(\mathbb{R}^d) \text{ and } \int_{\mathbb{R}^d} \rho_0 \langle x\rangle^p \, dx < \infty\right\}.$$

Summary of existence results

- For $m > 1$, $q \geq 1$, let us denote

$$\left\{\rho_0 \in \mathcal{P}(\mathbb{R}^d) : \ \rho_0 \geq 0 \text{ and } \|\rho_0\|_{L^1_x(\mathbb{R}^d)} = 1\right\},$$

- Further, let

$$1 < p \leq p_q := \min\left\{2, 1 + \frac{d(q-1)+q}{d(m-1)+q}\right\}$$

and, for $\langle x \rangle^p = \left(1+|x|^2\right)^{\frac{p}{2}}$,

$$\left\{\rho_0 \in \mathcal{P}_p(\mathbb{R}^d) : \ \rho_0 \in \mathcal{P}(\mathbb{R}^d) \text{ and } \int_{\mathbb{R}^d} \rho_0 \langle x \rangle^p \, dx < \infty\right\}.$$

- Three types of initial data
  - I. $\rho_0 \in \mathcal{P}_p(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} \rho_0 \log \rho_0 \, dx < \infty$, for $1 < p \leq p_1$.
  - II. $\rho_0 \in \mathcal{P}(\mathbb{R}^d)$ and $L^q_x(\mathbb{R}^d)$, for $q > 1$.
  - III. $\rho_0 \in \mathcal{P}_p(\mathbb{R}^d)$ and $L^q_x(\mathbb{R}^d)$, for $q > 1$ and $1 < p \leq p_q$.

Let us consider

the following equation for a non-negative function $\rho : \mathbb{R}^d \times [0, T] \mapsto \mathbb{R}$, $d \geq 2$, which is given as

$$\begin{cases} \partial_t \rho = \nabla \cdot (\nabla \rho^m - V\rho) \\ \rho(\cdot, 0) = \rho_0 \end{cases} \quad \text{in} \quad Q_T := \mathbb{R}^d \times [0, T], \tag{5}$$

where V is a vector field.

Main Theorem I-(i)

Let $1 < m \leq 2$ and $1 < p \leq p_1 := 1 + \frac{1}{d(m-1)+1}$. Suppose that

$$\rho_0 \in \mathcal{P}_p(\mathbb{R}^d), \quad \text{and} \quad \int_{\mathbb{R}^d} \rho_0 \log \rho_0 \, dx < \infty. \tag{6}$$

Moreover, assume that

$$V \in \mathcal{S}_{m,1}^{(q_1,q_2)} \quad \text{for} \quad \begin{cases} 2 \leq q_2 \leq \frac{m}{m-1}, & \text{if } d > 2, \\ 2 \leq q_2 < \frac{m}{m-1}, & \text{if } d = 2. \end{cases} \tag{7}$$

Then, there exists an absolutely continuous curve $\rho \in AC(0,T;\mathcal{P}_p(\mathbb{R}^d))$ such that $\rho(\cdot,0) = \rho_0$ which solves (5) in the sense of distribution, namely, for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times [0,T))$, satisfying

$$\int_0^T \int_{\mathbb{R}^d} \left\{ [\partial_t \varphi + \nabla\varphi \cdot V]\,\rho + \Delta\varphi \rho^m \right\} dx\,dt = -\int_{\mathbb{R}^d} \varphi(\cdot,0)\rho_0 \, dx. \tag{8}$$

Furthermore, ρ satisfies

$$\begin{aligned} &\sup_{0\leq t\leq T} \int_{\mathbb{R}^d \times \{t\}} \rho \left(|\log\rho| + \langle x \rangle^p \right) dx + \iint_{Q_T} \left| \nabla \rho^{\frac{m}{2}} \right|^2 dx\,dt \\ &\quad + \iint_{Q_T} \left(\left| \frac{\nabla \rho^m}{\rho} \right|^p + |V|^p \right) \rho \, dx\,dt \leq C \end{aligned} \tag{9}$$

and

$$W_p(\rho(t),\rho(s)) \leq C(t-s)^{\frac{p-1}{p}}, \qquad \forall \ 0 \leq s \leq t \leq T, \tag{10}$$

where the constant $C = C(\|V\|_{\mathcal{S}_{m,1}^{(q_1,q_2)}}, \int_{\mathbb{R}^d} (\rho_0 \log \rho_0 + \rho_0 \langle x \rangle^p) \, dx)$.

Main Theorem I-(ii), divergence free case

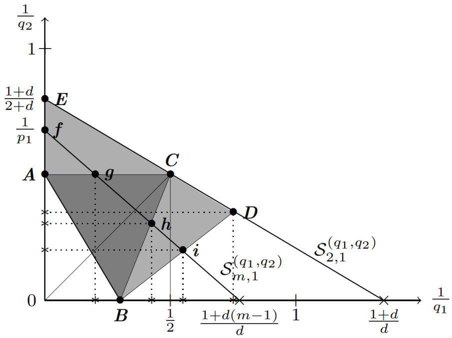
Let $m > 1$. Assume

$$\rho_0 \in \mathcal{P}_p(\mathbb{R}^d), \quad \text{and} \quad \int_{\mathbb{R}^d} \rho_0 \log \rho_0 \, dx < \infty, \tag{11}$$

and the divergence-free V (i.e., $\nabla \cdot V = 0$) satisfying

$$V \in \mathcal{S}_{m,1}^{(q_1,q_2)} \quad \text{for} \quad \begin{cases} p_1 \leq q_2 \leq \frac{p_1 m}{m-1}, & \text{if } d > 2, \\ p_1 \leq q_2 < \frac{p_1 m}{m-1}, & \text{if } d = 2. \end{cases} \tag{12}$$

Then the same conclusion holds as in above theorem.



$$\overline{\boldsymbol{AB}} = \mathcal{S}^{(q_1,q_2)}_{1,1}$$

$\mathcal{R}(\boldsymbol{ABC})$: $(\frac{1}{q_1}, \frac{1}{q_2})$ *satisfying* (1.14).

$\boldsymbol{A} = (0, \frac{1}{2})$, $\boldsymbol{B} = (\frac{1}{d}, 0)$, $\boldsymbol{C} = (\frac{1}{2}, \frac{1}{2})$

$\boldsymbol{g} = (\frac{m-1}{2}, \frac{1}{2})$, $\boldsymbol{h} = (\frac{(2-m)+d(m-1)}{md}, \frac{m-1}{m})$

$\mathcal{R}(\boldsymbol{ABDE})$: $(\frac{1}{q_1}, \frac{1}{q_2})$ *satisfying* (1.18).

$\boldsymbol{D} = (\frac{1+d}{2d}, \frac{1+d}{2(2+d)})$, $\boldsymbol{E} = (0, \frac{1+d}{2+q})$

$\boldsymbol{f} = (0, \frac{1}{p_1})$, $\boldsymbol{i} = (\frac{1+d(m-1)}{md}, \frac{m-1}{p_1 m})$

Figure 3-(i). (1.14) *and* (1.18) *in Theorem 1.3 for* $d > 2$

- Think of testing $\log \rho$:

$$\rho_t \log \rho = \partial_t (\rho \log \rho) - \rho \partial_t \log \rho = \partial_t (\rho \log \rho - \rho),$$

$$\nabla \rho^m \nabla \log \rho \sim |\nabla \rho^{m/2}|^2,$$

$$V \rho \nabla log \rho = V \rho \frac{\nabla \rho}{\rho} = V \nabla \rho \sim V \rho^{1-\frac{m}{2}} \nabla \rho^{m/2}.$$

- $1 - \frac{m}{2} \geq 0$ implies that $1 < m \leq 2$.
- the p-entropy inequality:

$$\int_{\mathbb{R}^d} \rho |\log \rho| \, dx \leq \int_{\mathbb{R}^d} \rho \log \rho \, dx + \int_{\mathbb{R}^d} \rho \langle x \rangle^p \, dx + c(d).$$

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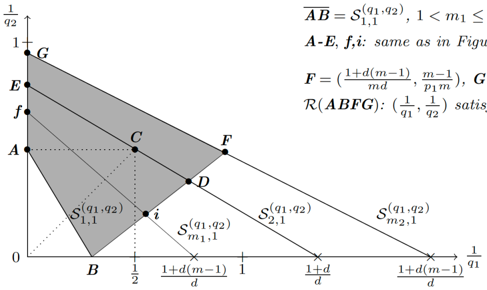
- divergence-free case

$$V \rho \nabla log \rho = V \nabla \rho \sim \nabla V \rho$$



$\overline{\boldsymbol{AB}} = \mathcal{S}_{1,1}^{(q_1,q_2)}$, $1 < m_1 \leq 2$, *and* $m_2 > 2$.

***A*-*E*, *f*,*i*:** *same as in Figure 3-(i).*

$\boldsymbol{F} = (\frac{1+d(m-1)}{md}, \frac{m-1}{p_1 m})$, $\boldsymbol{G} = (0, \frac{1+d(m-1)}{2+d(m-1)})$, *for* $m > 2$

$\mathcal{R}(\boldsymbol{ABFG})$: $(\frac{1}{q_1}, \frac{1}{q_2})$ *satisfying* (1.18)

Figure 3-(ii). (1.18) *in Theorem 1.4 for* $d > 2$

Thank you for your attention!!

- The results hold for $m = 1$. Embedding property for t-direction.
- The scaling invariant classes can be replaced by sub-scaling cases

$$\mathfrak{S}_{m,q}^{(q_1,q_2)} := \left\{ V : \ \|V\|_{L_{x,t}^{q_1,q_2}} < \infty \ \text{ where } \ \frac{d}{q_1} + \frac{2+q_{m,d}}{q_2} \leq 1 + q_{m,d} \right\}$$

- There are corresponding results for scaling invariant classes of ∇V

$$\tilde{\mathfrak{S}}_{m,q}^{(\tilde{q}_1,\tilde{q}_2)} := \left\{ V : \ \|\nabla V\|_{L_{x,t}^{\tilde{q}_1,\tilde{q}_2}} < \infty \ \text{ where } \ \frac{d}{\tilde{q}_1} + \frac{2+q_{m,d}}{\tilde{q}_2} \leq 2 + q_{m,d} \right\}$$

- Application: an repulsion type of Keller-Segel equations, which is given of the form

$$\rho_t - \Delta\rho^m = \nabla\left(\rho\nabla c\right), \qquad c_t - \Delta c = \rho. \tag{13}$$