Unbounded supersolutions with generalized Orlicz growth

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We say that $\varphi : \Omega \times [0, \infty) \to [0, \infty]$ is a *weak Φ-function*, and write $\varphi \in \Phi_{\mathrm{w}}(\Omega)$, if the following conditions hold:

- For every measurable function $f : \Omega \to \mathbb{R}$ the function $x \mapsto \varphi(x, f(x))$ is measurable and for every $x \in \Omega$ the function $t \mapsto \varphi(x, t)$ is non-decreasing.
- $\varphi(x, 0) = \lim_{t \to 0^+} \varphi(x, t) = 0$ and $\lim_{t \to \infty} \varphi(x, t) = \infty$ for every $x \in \Omega$.
- The function $t \mapsto \frac{\varphi(x,t)}{t}$ is L-almost increasing on $(0, \infty)$ with L independent of x.

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Some special cases of Φ-functions:

- $\varphi(x,t) = t^p$ the classical Lebesgue space
- $\varphi(x,t) = \varphi(t)$ the Orlicz space
- $\varphi(x,t) = t^{p(x)}a(x)$ the variable exponent Lebesgue space
- $\varphi(x,t) = t^{p(x)}\log(e+t)$
- $\varphi(x,t) = t^p + a(x)t^q$ the double phase case

We assume that $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following φ-growth conditions:

$$\nu\varphi(x, |\xi|) \leqslant f(x, \xi) \cdot \xi \quad \text{and} \quad |f(x, \xi)|\,|\xi| \leqslant \Lambda\varphi(x, |\xi|)$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, and fixed but arbitrary constants $0 < \nu \leqslant \Lambda$. We are interested in local (weak) supersolutions:

Definition 1

A function $u \in W^{1,\varphi}_{\mathrm{loc}}(\Omega)$ is a supersolution if

$$\int_\Omega f(x, \nabla u) \cdot \nabla h \, dx \geqslant 0,$$

for all non-negative $h \in W^{1,\varphi}(\Omega)$ with compact support in Ω.

If φ is differentiable wrt second variable, then our assumptions covers also the equation

$$\int_\Omega \frac{\varphi'(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla h \geqslant 0,$$

for all non-negative $h \in W_0^{1,\varphi}(\Omega)$.

Instead of supersolutions, you can think local superminimizers: Every open set $D \Subset \Omega$ and for every non-negative $v \in W^{1,\varphi}(\Omega)$ with a compact support in D, we have

$$\int_D F(x, |\nabla u|)\, dx \leqslant \int_D F(x, |\nabla (u+v)|)\, dx.$$

Here $F(x,t) \approx \varphi(x,t)$.

Special case: $\varphi(x,t) = t^p$.

The standard p-Laplace equation $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$, $1 < p < \infty$. The non-negative weak supersolutions satisfies the weak Harnack inequality

$$\left(\fint_{2B} u^s \, dx \right)^{\frac{1}{s}} \lesssim \operatorname*{ess\,inf}_B u,$$

where

- the constant is independent of u,
- $0 < s < \frac{n}{n-p}(p-1)$ when $p < n$, and $s \in (0,\infty)$ when $p \geqslant n$.

Trudinger (1967)

Special case: Orlicz $\varphi(x,t) = \varphi(t)$.

Theorem 2 (Arriagada–Huentutripay (2018))

Assume that $1 < p \leqslant \frac{t\psi(t)}{\varphi(t)} \leqslant q < \infty$ *and* $\varphi(t) = \int_0^t \psi(t)\,dt$. *Let* $u \geqslant 0$ *be bounded supersolution. Then*

$$\left(\fint_B u^s\,dx \right)^{\frac{1}{s}} \lesssim \operatorname*{ess\,inf}_B u + \operatorname{diam}(B).$$

- Bounded solutions, Lieberman (1987, 1991)

There have to be some results for corresponding minimizers.

Special case: variable exponent $\varphi(x,t) = t^{p(x)}$.

Theorem 3 (Lukkari (2010))

Assume that p is log-Hölder continous and $1 < p^- \leqslant p^+ < \infty$. Let $t > 0$, $0 < s < \frac{n}{n-1}(p^- - 1)$, and let $u \geqslant 0$ be supersolution. Then

$$\left(\fint_{2B} u^s \, dx \right)^{\frac{1}{s}} \lesssim \operatorname*{ess\,inf}_B u + \operatorname{diam}(B),$$

where the constant depends on $L^t(4B)$-norm of u.

- Bounded supersolutions, Alkhutov (1997).
- Bounded supersolutions and $0 < s < \frac{n}{n-1}(p_0 - 1)$, Alkhutov–Krasheninnikova (2004).
- Unbounded supersolutions, H–Kinnunen–Lukkari (2007)
- Bounded superminimizers, Fan–Zhao (1999, 2000)
- Unbounded superminimizers, H–Kuusi–Lukkari–Marola–Parviainen (2008)

Special case: variable exponent $\varphi(x, t) = t^{p(x)}$.

- "$+\operatorname{diam}(B)$" is not needed if $p \in C^1$, Julin (2015)
- It is not know is "$+\operatorname{diam}(B)$" necessary or not.
- In the Harnack's inequality the constant cannot be independent of u, example in H–Kinnunen–Lukkari (2007)

Special case: double phase $\varphi(x,t) = t^p + a(x)t^q$.

Theorem 4 (Baroni–Colombo–Mingione (2015))

Let $a \in C^{0,\alpha}$, $\alpha \geqslant \frac{n}{p}(q-p)$. Let $u \geqslant 0$ be bounded supersolution. Then there exists $s > 0$ such that

$$\left(\fint_B u^s \, dx \right)^{\frac{1}{s}} \lesssim \operatorname*{ess\,inf}_B u.$$

Here the constant depends on $\|u\|_\infty$.

Other related results:

- $\varphi(x,t) = t^{p(x)}$ and general structural conditions, Latvala–Toivanen (2017)
- $\varphi(x,t) = t^{p(x)}$ and p makes a jump at a hyperplane, Alkhutov–Surnachev (2019)
- $\varphi(x,t) = t^{p(x)}$ and p is piecewise constant, Alkhutov–Surnachev (2019, 2020)
- $\varphi(x,t) = t^{p(x)} \log(e+t)$, Ok (2018)
- generalized double phase functional, Byen–Oh (2020)

Let $p, q, s > 0$ and let $\omega : \Omega \times [0, \infty) \to [0, \infty)$ be almost increasing. We say that $\varphi : \Omega \times [0, \infty) \to [0, \infty)$ satisfies

- (A0) if there exists $\beta \in (0, 1]$ such that $\beta \leqslant \varphi^{-1}(x, 1) \leqslant \frac{1}{\beta}$ for a.e. $x \in \Omega$,
- (A1-ω) if there exists $\beta \in (0, 1]$ such that, for every ball B and a.e. $x, y \in B \cap \Omega$,

$$\varphi(x, \beta t) \leqslant \varphi(y, t) \quad \text{when} \quad \omega_B^-(t) \in \left[1, \frac{1}{|B|}\right];$$

- (A1-s) if it satisfies (A1-ω) for $\omega(x, t) := t^s$;
- (A1) if it satisfies (A1-φ);
- (aInc)$_p$ if $t \mapsto \frac{\varphi(x,t)}{t^p}$ is L_p-almost increasing in $(0, \infty)$ for some $L_p \geqslant 1$ and a.e. $x \in \Omega$;
- (aDec)$_q$ if $t \mapsto \frac{\varphi(x,t)}{t^q}$ is L_q-almost decreasing in $(0, \infty)$ for some $L_q \geqslant 1$ and a.e. $x \in \Omega$.

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$\varphi(x,t) :=$	(A0)	(A1)	(A1-s)	(aInc)	(aDec)
$\varphi(t)$	true	true	true	∇_2	Δ_2
$t^{p(x)}a(x)$	$a \approx 1$	$p \in C^{\log}$	$p \in C^{\log}$	$p^- > 1$	$p^+ < \infty$
$t^{p(x)}\log(e+t)$	true	$p \in C^{\log}$	$p \in C^{\log}$	$p^- > 1$	$p^+ < \infty$
$t^p + a(x)t^q$	$a \in L^\infty$	$a \in C^{0,\frac{n}{p}(q-p)}$	$a \in C^{0,\frac{n}{s}(q-p)}$	$p > 1$	$q < \infty$

Table: Assumptions in some special cases

Theorem 5 (Benyaiche-H-Hästö-Karppinen (accepted))

Suppose φ *satisfies* (A0), $(\mathrm{aInc})_p$ and $(\mathrm{aDec})_q$, $1 < p \leqslant q < \infty$. Let $u \geqslant 0$ be a supersolution. Assume one of the following:

1. φ satisfies (A1-s_*) and $\|u\|_{L^s(B_{2R})} \leqslant d$, where $s_* := \frac{ns}{n+s}$ and $s \in [q-p, \infty]$.
2. φ satisfies (A1) and $\|u\|_{W^{1,\varphi}(B_{2R})} \leqslant d$.

Then there exist positive constants ℓ_0 and C such that

$$\left(\fint_{B_{2R}} (u+R)^{\ell_0}\,dx \right)^{\frac{1}{\ell_0}} \leqslant C(\operatorname*{ess\,inf}_{B_R} u + R).$$

If (1) holds with $s > \max\{\frac{n}{p}, 1\}(q-p)$ or if (2) holds with $p^* > q$, then the weak Harnack inequality holds for any $\ell_0 < \ell(p)$, where $\ell(p) = \frac{n}{n-p}(p-1)$ if $p < n$, and $\ell(p) = \infty$ if $p \geqslant n$.

Other results on generalized Orlicz spaces:

- Bounded supersolutions, Benyaiche–Khlifi (2020).
- Bounded supersolutions, Shan–Skrypnik–Voitovych (preprint)
- Bounded superminimizers, H–Hästö–Toivanen (2017).
- Bounded superminimizers, H–Hästö–Lee (to appear).

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Proposition 6 (Benyaiche–H–Hästö–Karppinen (accepted))

The (A1-s_*) *assumption in the prevous theorem is sharp, since for any* $s' < s_*$ *if, instead of (1),* φ *satisfies* (A1-s') *and* $\|u\|_{L^s(B_{2R})} \leqslant d$, *then the weak Harnack inequality need not hold.*

Let $\varphi \in \Phi_{\mathrm{w}}(\mathbb{R})$ be defined by $\varphi(x,0) := 0$ and

$$\varphi'(x,t) := \max\{t^{p-1}, a(x)t^{q-1}\},$$

so that $\varphi(x,t) \approx \max\{t^p, a(x)t^q\} \approx t^p + a(x)t^q$.
Let u be a solution of $\left(\varphi'(x,|u'|)\frac{u'}{|u'|}\right)' = 0$ on the interval (a,b).
We assume that $\lim_{x\to a^+} u(x) < \lim_{x\to b^-} u(x)$, so u is increasing and $\frac{u'}{|u'|} = 1$. Then the differential equation reduces to $\varphi'(x,u') \equiv c$, i.e.

$$u'(x) = \begin{cases} c^{\frac{1}{p-1}}, & \text{when } c^{-\frac{q-p}{p-1}} \geqslant a(x), \\ (c/a(x))^{\frac{1}{q-1}}, & \text{otherwise.} \end{cases}$$

We further assume that $a(x) := \max\{-x, 0\}^{\alpha}$. Since a is decreasing, we obtain that

$$u'(x) = \begin{cases} c^{\frac{1}{p-1}}, & \text{when } x \geqslant -x_0, \\ (c|x|^{-\alpha})^{\frac{1}{q-1}}, & \text{when } x < -x_0, \end{cases} \quad \text{for } x_0 := c^{-\frac{1}{\alpha}\frac{q-p}{p-1}}.$$

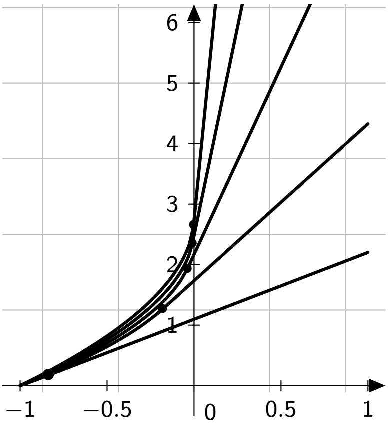

Figure: Solution for $c = 1.01, 1.1, 1.2, 1.3, 1.4$ in $[-1, 1]$. The parameters are $p = 1.1$, $q = 2$ and $\alpha = 0.5$. The right boundary values have been partly cut away but they are in the range $[2, 32]$. The point indicates x_0.