Lipschitz continuity of nonnegative minimizers of functionals of Bernoulli type with nonstandard growth

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Monday's nonstandard seminar
June 7, 2021

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I will comment on regularity results for nonnegative local minimizers of functionals

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Here $0 \leq \lambda(x) \in L^\infty$ and F is of $p(x)$ growth in the gradient variable.
The idea we had in mind was to see how far could we generalize the case

$$F(x, s, \eta) = \frac{|\eta|^{p(x)}}{p(x)} + b(x)s$$

$b \in L^\infty$ that we had studied previously.

Previous results

In that simpler case, if $0 < \lambda_1 \leq \lambda(x)$, $1 < p_{min} \leq p(x) \leq p_{max} < \infty$ and p Hölder continuous we proved,

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- If moreover p is Lipschtiz continuous, $\partial\{u > 0\}$ is a $C^{1,\alpha}$ surface but for a set of null $N - 1-$dimensional Hausdorff measure.
- $|\nabla u(x)| = \left(\frac{p(x)}{p(x)-1} \lambda(x)\right)^{\frac{1}{p(x)}}$ on the regular part of $\partial\{u > 0\}$.

This last result shows that Lipschitz continuity is the optimal regularity one can expect.

First assumptions

In order to get existence of minimizers with given boundary data $\varphi \in W^{1,p(x)}(\Omega)$ we assume that $F \in C_s \cap C^1_\eta$ and

$$-c_1^{-1}(1+|s|^q)+\lambda_0|\eta|^{p(x)} \leq F(x,s,\eta) \leq c_1(1+|s|^{\tau(x)})+\Lambda_0|\eta|^{p(x)}$$

with $1 < q < \min_\Omega \tau(x)$ and positive constants c_1, λ_0 and Λ_0. Here

$$\tau(x)=p^*(x)=\frac{Np(x)}{N-p(x)} \quad \textit{if} \quad p_{max} < N,$$

$$\tau(x) \in L^\infty,\ \tau(x) \geq p(x) \quad \textit{if} \quad p_{min} > N,$$

$$\tau(x)=p(x) \quad \textit{if} \quad p_{min} \leq N \leq p_{max}.$$

Existence is proved for any $0 \leq \lambda(x) \in L^\infty$.

First assumptions

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with $1 < r(x) \leq \tau(x) - 2\delta$ in $\Omega' \subset \Omega$ with $\delta > 0$ such that $\max_{\Omega'} \tau - \min_{\Omega'} \tau < \delta$, and positive constants c_1, λ_0 and Λ_0.

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Observe that this hypothesis always holds if $r(x) < \tau(x)$ in Ω, $r(x)$ and $\tau(x)$ are continuous and the diameter of Ω' is small enough.
Given $u \in W^{1,p(x)}(\Omega)$, these assumptions allow to get uniform $u + W_0^{1,p(x)}(\Omega')$ estimates for any minimizing sequence.
And we prove that there exists a minimizer $v \in u + W_0^{1,p(x)}(\Omega')$ for any $0 \leq \lambda(x) \in L^\infty$.

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As usual, under some regularity assumptions of F, if $\lambda(x) \equiv 0$, any minimizer is a solution to

$$\begin{aligned} \operatorname{div} A\big(x, v(x), \nabla v(x)\big) &= B\big(x, v(x), \nabla v(x)\big) \quad && \text{in} \quad \Omega', \\ v &= u && \text{on} \quad \partial\Omega', \end{aligned}$$

where $A(x, s, \eta) = \nabla_\eta F(x, s, \eta)$, $B(x, s, \eta) = F_s(x, s, \eta)$.

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Our next assumptions are those set by Fan (JDE, 2007) for the local $C^{1,\alpha}$ regularity of bounded weak solutions.

$$A(x,s,0)=0,$$

$$\sum_{i,j}\frac{\partial A_i}{\partial \eta_j}(x,s,\eta)\xi_i\xi_j \geq \lambda_0|\eta|^{p(x)-2}|\xi|^2,$$

$$\sum_{i,j}\left|\frac{\partial A_i}{\partial \eta_j}(x,s,\eta)\right| \leq \Lambda_0|\eta|^{p(x)-2},$$

$$\left|A(x_1,s,\eta)-A(x_2,s,\eta)\right| \leq \Lambda_0|x_1-x_2|^{\beta}\left(|\eta|^{p(x_1)-1}+|\eta|^{p(x_2)-1}\right)\left(1+\left|\log|\eta|\right|\right),$$

$$\left|A(x,s_1,\eta)-A(x,s_2,\eta)\right| \leq \Lambda_0|s_1-s_2||\eta|^{p(x)-1}.$$

$$\left|B(x,s,\eta)\right| \leq \Lambda_0\left(1+|\eta|^{p(x)}+|s|^{\tau(x)}\right),$$

From the assumptions on A it is easy to see that

$$|A(x,s,\eta)| \leq \bar{\alpha}(p_{\min})N\Lambda_0|\eta|^{p(x)-1}.$$

$$A(x,s,\eta)\cdot\eta \geq \alpha(p_{\max})\lambda_0|\eta|^{p(x)}.$$

Under the assumptions above, Fan (JDE 2007) proved that bounded solutions to the equation are locally $C^{1,\alpha}$.

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Moreover, if the growth of B is

$$\left|B(x,s,\eta)\right| \leq \Lambda_0(1+|\eta|^{p(x)-1}+|s|^{p(x)-1})$$

there holds that weak solutions to the equation are locally bounded and, if the domain Ω' is smooth and the boundary datum $u \in L^\infty(\Omega')$ Fan and Zhao (Nonlinear Anal. 1999) proved that the weak solution $v \in L^\infty(\Omega')$ with norm bounded in terms of the universal constants, $\|v\|_{W^{1,p(x)}(\Omega')}$ and $\|u\|_{L^\infty(\Omega')}$.

Observe that, $F(x,s,\eta) = a(x,s)\frac{|\eta|^{p(x)}}{p(x)} + f(x,s)$ implies that

$$A(x,s,\eta) = a(x,s)|\eta|^{p(x)-2}\eta$$

and

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And, if $F(x,s,\eta) = G(x,\eta) + f(x,s)$,

$$A(x,s,\eta) = \nabla_\eta G(x,\eta)$$

and

$$B(x,s,\eta) = f_s(x,s).$$

A very important inequality for minimization problems in $W^{1,p}$ with p constant is the following. Let $u \in W^{1,p}(\Omega')$ and $v \in u + W_0^{1,p}(\Omega')$ the solution to $\Delta_p v = 0$ in Ω'. Then,

$$\int_{\Omega'} |\nabla u|^p - |\nabla v|^p \, dx \geq c_0 \begin{cases} \int_{\Omega'} |\nabla u - \nabla v|^p \, dx & \text{if } p \geq 2, \\ \int_{\Omega'} |\nabla u - \nabla v|^2 (|\nabla u| + |\nabla v|)^{p-2} \, dx & \text{if } p < 2. \end{cases}$$

These inequalities can be used to prove that convergente sequences $\{v_n\}$ of solutions with boundary data $\{u_n\}$ that are minimizers with $\lambda_n \to 0$, are such that their limits coincide.

These inequalities have been generalized to the case $F(x,s,\eta) = a(x)\frac{|\eta|^{p(x)}}{p(x)} + b(x)s$ in our previous paper. There holds,

$$\int_{\Omega'} a(x)\Big(\frac{|\nabla u|^{p(x)}}{p(x)} - \frac{|\nabla v|^{p(x)}}{p(x)}\Big) + b(x)\big(u(x) - v(x)\big)\,dx$$
$$\geq c_0\Big[\int_{\{p(x)\geq 2\}} |\nabla u - \nabla v|^{p(x)}\,dx + \int_{\{p(x)<2\}} |\nabla u - \nabla v|^2(|\nabla u| + |\nabla v|)^{p(x)-2}\,dx\Big].$$

The corresponding inequality does not hold for a general F without further assumptions.

Our assumption is

$$2|A_s(x,s,\eta)\cdot\xi w| \le \frac{1}{2}\sum_{i,j}\frac{\partial A_i}{\partial \eta_j}(x,s,\eta)\xi_i\xi_j + B_s(x,s,\eta)w^2, \qquad \text{(H)}$$

for every $\eta, \xi \in \mathbb{R}^N$, $s, w \in \mathbb{R}$, $x \in \Omega$.

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for every $\eta, \xi \in \mathbb{R}^N$, $s, w \in \mathbb{R}$, $x \in \Omega$.
Under assumption (H), for every $u \in W^{1,p(\cdot)}(\Omega)$ and $v \in W^{1,p(\cdot)}(\Omega')$ such that

$$\begin{cases} \operatorname{div} A(x,v,\nabla v) = B(x,v,\nabla v) & \text{in } \Omega', \\ v = u & \text{on } \partial\Omega', \end{cases}$$

there holds that,

$$\int_{\Omega'} \big(F(x,u,\nabla u) - F(x,v,\nabla v)\big)\,dx \geq \frac{1}{2}\alpha\lambda_0\Big(\int_{\Omega'\cap\{p(x)\geq 2\}} |\nabla u - \nabla v|^{p(x)}\,dx + \int_{\Omega'\cap\{p(x)<2\}} \Big(|\nabla u| + |\nabla v|\Big)^{p(x)-2}|\nabla u - \nabla v|^2\,dx\Big),$$

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- Hence, if $B(x, 0, 0) \equiv 0$ and the boundary datum u is bounded, there holds that the solution v satisfies that $\|v\|_{L^\infty(\Omega')} \leq \|u\|_{L^\infty(\Omega')}$.
- With the growth assumption of Fan and Zhao, the solution w with boundary datum $M = \|u\|_{L^\infty(\Omega')}$ is a bounded in terms of M and $\|w\|_{W^{1,p(x)}(\Omega')}$. But, with constant boundary data we can see from the proof of the minimization argument that this last norm can be bounded in terms of M and universal constants. So that, there holds that $\|v\|_{L^\infty(\Omega')} \leq \|w\|_{L^\infty(\Omega')} \leq C\big(\|u\|_{L^\infty(\Omega')}\big)$.

The problem with $0 \leq \lambda(x) \in L^\infty$, $0 \not\equiv \lambda(x)$

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If moreover, $-M_2 \leq \varphi \leq M_1$ with $M_1 > 0,\ M_2 \geq 0$ and for instance,

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$$f(x,\cdot) \text{ nonincreasing in } (-\infty, 0] \text{ and nondecreasing in } [0,+\infty),$$

there holds that $-M_2 \leq u \leq M_1$ in Ω.

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In fact, both $w_1 = u - (u - M_1)^+$ and $w_2 = u + (u + M_2)^-$ are admissible functions. Moreover,

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So that on the one hand,

$$0 \leq \int_\Omega F(x, w_1, \nabla w_1) - F(x, u, \nabla u) = \int_{u>M_1} F(x, M_1, 0) - F(x, u, \nabla u)$$
$$= \int_{u>M_1} f(x, M_1) - f(x, u) - \int_{u>M_1} G(x, u, \nabla u)$$
$$\leq - \int_{u>M_1} G(x, u, \nabla u) \leq 0.$$

Hence, $G(x, u, \nabla u) = 0$ in $\{u > M_1\}$. So that, $\nabla(u - M_1)^+ = 0$ in Ω. As $(u - M_1)^+ = 0$ on $\partial\Omega$, we deduce that $u \leq M_1$ in Ω.
And, proceeding in a similar way we find that $u \geq -M_2$.

Regularity of nonnegative, bounded minimizers

One first observation is that a local minimizer is a subsolution to the equation

$$\operatorname{div} A(x, u(x), \nabla u(x)) \geq B(x, u(x)\nabla u(x)) \quad \text{in} \quad \Omega.$$

The reason is that for every $0 \leq \phi \in C_0^\infty(\Omega)$, $\{u - \varepsilon\phi > 0\} \subset \{u > 0\}$.

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$$0 \geq \int_\Omega F(x, u(x), \nabla u(x)) - F(x, u(x) - \varepsilon\phi(x), \nabla u(x) - \varepsilon\nabla\phi(x))\, dx.$$

Proceeding as usual we get that for every $0 \leq \phi \in C_0^\infty(\Omega)$,

$$0 \geq \int_\Omega A(x, u(x), \nabla u(x))\nabla\phi(x) + B(x, u(x), \nabla u(x))\phi(x)\, dx.$$

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So that for every $0 \le \phi \in C_0^\infty(\{u > 0\}^\circ)$,

$$0 \le \int_\Omega A(x, u(x), \nabla u(x))\nabla\phi(x) + B(x, u(x), \nabla u(x))\phi(x)\, dx.$$

Regularity of nonnegative, bounded minimizers

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- A first conclusion is that such a minimizer is a solution to the equation in its positivity set.
- I will give some idea of the proof. In particular, in order to show one use of the main inequality.

Regularity of nonnegative, bounded minimizers

The idea is to prove that, given $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ such that the diameter of Ω'' is small enough, there exists $\rho_0 > 0$ such that, if $\rho \le \rho_0$, $x_0 \in \Omega'$,

$$\left(\frac{1}{\rho^N}\int_{B_\rho(x_0)} |\nabla u|^{p_-}\,dx\right)^{1/p_-} \le C\rho^{\alpha-1}$$

for some $0 < \alpha < 1$ and some positive constant C.Then, $u \in C^\alpha(\Omega')$.
Here $p_- = \min_{\Omega''} p$.

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In order to get this inequality we use as comparison function the solution $v \in u + W_0^{1,p(x)}(B_r(x_0))$ to the equation $A - B$.

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We take $r \leq r_0$ with r_0 small enough so that $B_{r_0}(x_0) \subset \Omega''$ and the diameter of Ω'' small so that r_0 is small so that this solution exists.

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We take $r \le r_0$ with r_0 small enough so that $B_{r_0}(x_0) \subset \Omega''$ and the diameter of Ω'' small so that r_0 is small so that this solution exists. Here we are assuming that

$$-c_1^{-1}(1 + |s|^{r(x)}) + \lambda_0|\eta|^{p(x)} \le F(x,s,\eta) \le c_1(1 + |s|^{\tau(x)}) + \Lambda_0|\eta|^{p(x)}$$

with $1 < r(x) < \tau(x)$ in Ω and $r \in C(\Omega)$.

From the main inequality and the fact that u is a minimizer of J we get

$$\int_{B_r(x_0)\cap\{p\geq 2\}} |\nabla u - \nabla v|^{p(x)}\, dx \leq Cr^N,$$

$$\int_{B_r(x_0)\cap\{p<2\}} |\nabla u - \nabla v|^2(|\nabla u| + |\nabla v|)^{p(x)-2}\, dx \leq Cr^N.$$

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Then, we take $\varepsilon > 0$ to be chosen and r_0 small such that $r_0^\varepsilon \leq 1/2$ and let $\rho = r^{1+\varepsilon}$. Then, applying Young's inequality to the integrand we get

$$\int_{\{p<2\}\cap B_\rho(x_0)} |\nabla u - \nabla v|^{p(x)}\,dx \leq C_\theta r^N + C\theta \int_{B_\rho(x_0)\cap\{p<2\}} (|\nabla u| + |\nabla v|)^{p(x)}$$

So that,

$$\int_{B_\rho(x_0)} |\nabla u - \nabla v|^{p(x)}\,dx \leq C_\theta r^N + C\theta \int_{B_\rho(x_0)\cap\{p<2\}} (|\nabla u| + |\nabla v|)^{p(x)}\,dx,$$

and by choosing θ small enough we conclude that,

$$\int_{B_\rho(x_0)} |\nabla u|^{p(x)}\,dx \leq Cr^N + C\int_{B_\rho(x_0)} |\nabla v|^{p(x)}\,dx,$$

The aim is to prove that

$$\sup_{B_{r/2}(x_0)} |\nabla v| \leq \frac{CM}{r},$$

where $M = \|v\|_{L^\infty(B_r(x_0))} \leq C\big(\|u\|_{L^\infty(B_r(x_0))}\big)$.

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If this inequality holds, we get

$$\int_{B_\rho(x_0)} |\nabla u|^{p(x)}\, dx \le Cr^N + C\rho^N r^{-p_+}.$$

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Here $p_+ = \max_{\Omega''} p$.
Hence, if we take $\varepsilon \leq \frac{p_{\min}}{N}$, we have that

$$\frac{1}{\rho^N}\int_{B_\rho(x_0)} |\nabla u|^{p_-}\, dx \leq C_N + \frac{1}{\rho^N}\int_{B_\rho(x_0)} |\nabla u|^{p(x)}\, dx$$
$$\leq C_N + C\Big(\frac{r}{\rho}\Big)^N + Cr^{-p_+} \leq C_N + Cr^{-\varepsilon N} + Cr^{-p_+} \leq Cr^{-p_+} = C\rho^{-\frac{p_+}{(1+\varepsilon)}}$$

Regularity of nonnegative, bounded minimizers

We conclude that,

$$\Big(\frac{1}{\rho^N}\int_{B_\rho(x_0)}|\nabla u|^{p_-}\,dx\Big)^{1/p_-}\leq C\rho^{-\frac{p_+}{p_-}\frac{1}{1+\varepsilon}}.$$

Finally, if the diameter of Ω'' is small enough there holds that

$$\frac{p_+}{p_-}\leq 1+\frac{\varepsilon}{2},$$

so that,

$$\Big(\frac{1}{\rho^N}\int_{B_\rho(x_0)}|\nabla u|^{p_-}\,dx\Big)^{1/p_-}\leq C\rho^{-\frac{(1+\frac{\varepsilon}{2})}{(1+\varepsilon)}}=C\rho^{-(1-\alpha)}.$$

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For that purpose we consider the rescaled function $w(x) = \frac{v(x_0+rx)}{M}$ and prove that $|\nabla w| \leq C$ in $B_{1/2}$. So that,

$$\frac{r}{M}|\nabla v| \leq C \quad \text{in} \quad B_{r/2}(x_0).$$

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There holds that w is the solution of a rescaled equation

$$\mathrm{div}\bar{A}(x, w, \nabla w) = \bar{B}(x, w, \nabla w) \quad \text{in} \quad B_1$$

where

$$\bar{A}(x, s, \eta) = A(x_0 + rx, Ms, \tfrac{M}{r}\eta), \qquad \bar{B}(x, s, \eta) = rB(x_0 + rx, Ms, \tfrac{M}{r}\eta).$$

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So, we let

$$\widetilde{A}(x,s,\eta) = \left(\frac{r}{M}\right)^{p_- -1} \bar{A}(x,s,\eta), \qquad \widetilde{B}(x,s,\eta) = \left(\frac{r}{M}\right)^{p_- -1} \bar{B}(x,s,\eta),$$

and observe that $w \in W^{1,\bar{p}(\cdot)}(B_1) \cap L^\infty(B_1)$ satisfies

$$\mathrm{div}\widetilde{A}(x,w,\nabla w) = \widetilde{B}(x,w,\nabla w) \quad \text{in} \quad B_1,$$

where $\bar{p}(x) = p(x_0 + rx)$, and this equation is under the hypotheses of the paper by Fan on the $C^{1,\alpha}$ regularity. And, we are done.

Regularity of nonnegative, bounded minimizers

The proof of the Lipschitz continuity is much more involved. It is performed through a contradiction argument. So we have to deal with sequences of solutions. Moreover, within the proof we have to perform 2 rescalings.

I will not talk about this proof. If someone is interested, the paper has been published in Mathematics in Engineering (October, 2020) (volume in honor of Sandro Salsa).

Thank you for your attention

Examples

Some examples of application of our results are:

$$F(x, s, \eta) = G(x, s, \eta) + f(x, s)$$

with

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For example,

$$f(x, s) = b(x)(1 + s^2)^{\frac{\tau(x)}{2}}$$

with $0 \leq b \in L^\infty(\Omega)$

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  If the minimizer u lies between 0 and M, condition (4) only needs to hold for $s \in [0, M]$.

  For instance, if the boundary datum $\varphi \in [0, M]$ and condition (4) above holds for $s \in [0, M]$ and the others hold for s in a neighborhood of $[0, M]$, there holds that the minimizer u moves in that range and it is locally Lipschitz continuous in Ω.

Examples

A possible example would be

$$a(x,s) = \begin{cases} (1+s)^{-q(x)} & \text{if } -1/2 \le s \le M+1, \\ 2^{q(x)} & \text{if } s \le -1/2, \\ (2+M)^{-q(x)} & \text{if } s \ge M+1, \end{cases}$$

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with $q \in L^{\infty}(\Omega)$ a Hölder continuous function such that $0 < q(x) < \frac{1}{\gamma(x)-1}$.

With this choice, a minimizer always exists, it lies between 0 and M and it is locally Lipschitz continuous.

Examples

- $G(x,s,\eta) = a(x)\widetilde{G}(|\eta|^{p(x)})$ with $\widetilde{G} \in C^2([0,\infty))$,

$$c_0 \le \widetilde{G}'(t) \le C_0,$$
$$0 \le \widetilde{G}''(t) \le \frac{C_0}{1+t} \qquad c_0, C_0 \text{ positive constants.}$$

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  Also

- $G(x,s,\eta) = \big(A(x)\eta \cdot \eta\big)|\eta|^{p(x)-2}$ with A uniformly positive definite and bounded matrix.