Regularity estimates for nonlinear parabolic equations with measure data on nonsmooth domains

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Parabolic measure data problems

- Consider **the Cauchy-Dirichlet problem with measure data**

$$\begin{cases} u_t - \operatorname{div} \mathbf{a}(Du, x, t) = \mu & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_p \Omega_T, \end{cases} \tag{P}$$

where $Du := D_x u$, Ω is a nonsmooth bounded domain of $\mathbb{R}^n$, $n \geq 2$, and $\Omega_T := \Omega \times (0, T)$ with $\partial_p \Omega_T := (\partial\Omega \times (0,T)) \cup (\Omega \times \{0\})$.

- **μ is a signed Radon measure on Ω_T with finite mass.** Here we assume that μ is defined in $\mathbb{R}^{n+1}$ by considering the zero extension to $\mathbb{R}^{n+1} \setminus \Omega_T$; that is,

$$|\mu|(\Omega_T) = |\mu|(\mathbb{R}^{n+1}) < \infty.$$

Parabolic measure data problems with p-growth

- A typical model of (P) is **parabolic p-Laplacian**, $\mathbf{a}(\xi,x,t)=|\xi|^{p-2}\xi$;

$$u_t-\Delta_p u := u_t-\operatorname{div}(\underbrace{|Du|^{p-2}}_{\text{diffusion coefficient}} Du)=\mu$$

  - $p>2$: slow diffusion, degenerate equation
  - $p<2$: fast diffusion, singular equation
  - $p=2$: random diffusion, non-degenerate (heat) equation

- The nonlinearity $\mathbf{a}(\xi,x,t):\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ satisfies the following growth and ellipticity conditions:

$$\begin{cases} |\xi||D_\xi\mathbf{a}(\xi,x,t)|+|\mathbf{a}(\xi,x,t)|\le\Lambda_1|\xi|^{p-1}, \\ \Lambda_0|\xi|^{p-2}|\eta|^2\le\langle D_\xi\mathbf{a}(\xi,x,t)\eta,\eta\rangle\,, \end{cases} \qquad \left(p>\frac{2n}{n+1}\right)$$

for every $\eta,\xi,x\in\mathbb{R}^n$, $t\in\mathbb{R}$, and for some constants $\Lambda_0,\Lambda_1>0$.

Goal

- **Goal. Global gradient estimates for a solution to the problem** (P) as follows:

$$\int_{\Omega_T} |Du|^q \, dxdt \lesssim \int_{\Omega_T} \mathcal{M}_1(\mu)^{dq} \, dxdt + 1 \quad \text{for} \quad 0 < \forall q < \infty,$$

  where $d = d(n,p) \geq 1$.

- $\mathcal{M}_\alpha(\mu)$ is the parabolic fractional maximal function of order α for μ,

$$\mathcal{M}_\alpha(\mu)(x,t) := \sup_{r>0} \frac{r^\alpha |\mu|(Q_r(x,t))}{r^{n+2}} \quad \text{for } (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

  where $Q_r(x,t) := B_r(x) \times (t - r^2, t + r^2)$.

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where $d = d(n,p) \geq 1$.

- To derive the above estimates, we find
  1. **a notion of solution**
  2. **minimal assumption on the operator** $\mathbf{a}(\xi, x, t)$
  3. **minimal assumption on the boundary of** Ω

Motivation

- Consider
$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = \delta_0 \quad \text{in } \mathbb{R}^n \times \mathbb{R},$$
where δ_0 is the Dirac delta function charging the origin.
- The Barenblatt solution is
$$u(x,t) = \begin{cases} t^{-n\theta}\left[c(n,p) - \frac{p-2}{p}\theta^{\frac{1}{p-1}}\left(\frac{|x|}{t^\theta}\right)^{\frac{p}{p-1}}\right]_+^{\frac{p-1}{p-2}} & \text{if } p \neq 2 \ \&\ t>0, \\ (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} & \text{if } p = 2 \ \&\ t>0, \\ 0 & \text{if } t \leq 0, \end{cases}$$
where $\theta := \frac{1}{n(p-2)+p} > 0 \ \left(\iff p > \frac{2n}{n+1}\right)$.
- $u \in L^q(\mathbb{R}; W^{1,q}(\mathbb{R}^n))$ for $q < p - \frac{n}{n+1}$, i.e., $u \notin L^p\left(\mathbb{R}; W^{1,p}(\mathbb{R}^n)\right)$.
- $u \in L^1(\mathbb{R}; W^{1,1}(\mathbb{R}^n)) \iff p - \frac{n}{n+1} > 1 \iff p > 2 - \frac{1}{n+1} \left(> \frac{2n}{n+1}\right)$.
- $u \notin L^1(\mathbb{R}; W^{1,1}(\mathbb{R}^n)) \iff \frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1}$.

p-parabolic capacity

- Let $p > 1$ and $Q \subset \Omega_T$. The p-**(parabolic) capacity** of Q is defined by

$$\mathrm{cap}_p(Q) := \inf \left\{ \|u\|_W : u \in W, u \geq \chi_Q \text{ a.e. in } \Omega_T \right\},$$

where $W := \left\{ u \in L^p(0,T;V) : u_t \in L^{p'}(0,T;V') \right\}$ endowed with

$$\|u\|_W := \|u\|_{L^p(0,T;V)} + \|u_t\|_{L^{p'}(0,T;V')}.$$

Here $V := W_0^{1,p}(\Omega) \cap L^2(\Omega)$ and V' is the dual space of V.

Decomposition of measure

$$\mathfrak{M}_b(\Omega_T) := \{\text{all signed Radon measures on } \Omega_T \text{ with finite mass}\}.$$
$$\mathfrak{M}_a(\Omega_T) := \{\mu \in \mathfrak{M}_b(\Omega_T) : \mu \text{ is absolutely continuous w.r.t } p\text{-capacity}\}.$$
$$\mathfrak{M}_s(\Omega_T) := \{\mu \in \mathfrak{M}_b(\Omega_T) : \mu \text{ has support on a set of zero } p\text{-capacity}\}.$$

- For $\mu \in \mathfrak{M}_b(\Omega_T)$,

$$\mu = \mu_a + \mu_s, \quad \mu_a \in \mathfrak{M}_a(\Omega_T), \quad \mu_s \in \mathfrak{M}_s(\Omega_T).$$

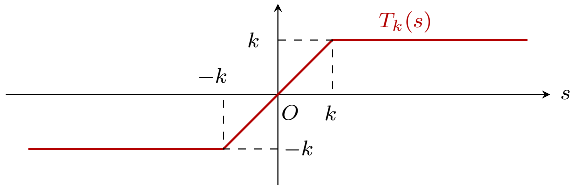
- $$\mu_a \in \mathfrak{M}_a(\Omega_T) \iff \mu_a = f + g_t + \operatorname{div} G,$$

  where $f \in L^1(\Omega_T)$, $g \in L^p(0,T;V)$ and $G \in L^{p'}(\Omega_T)$.

A notion of weak gradient

- Let us define the truncation operator

$$T_k(s) := \max\{-k, \min\{k, s\}\} \quad \text{for any } k > 0 \text{ and } s \in \mathbb{R}.$$



- If u is a measurable function defined in Ω_T, finite a.e., such that $T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega))$ for any $k > 0$, then there exists a unique measurable function U such that $DT_k(u) = U\chi_{\{|u|<k\}}$ a.e. in Ω_T for all $k > 0$.
- In this case, $Du := U$. If $u \in L^1(0, T; W_0^{1,1}(\Omega))$, then it coincides with the usual weak gradient.

Renormalized solution

Definition (Petitta & Porretta '15)

Let $p > 1$ and $\mu = \mu_a + \mu_s$. A function $u \in L^1(\Omega_T)$ is a **renormalized solution** of the problem (P) if $T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega))$ for every $k > 0$ and the following property holds: for any $k > 0$ there exist sequences of nonnegative measures $\nu_k^+, \nu_k^- \in \mathfrak{M}_a(\Omega_T)$ such that

$$\nu_k^+ \to \mu_s^+, \ \nu_k^- \to \mu_s^- \quad \text{tightly as } k \to \infty$$

and

$$-\int_{\Omega_T} T_k(u)\varphi_t \ dxdt + \int_{\Omega_T} \langle \mathbf{a}(DT_k(u), x, t), D\varphi \rangle \ dxdt = \int_{\Omega_T} \varphi \ d\mu_k$$

for every $\varphi \in W \cap L^\infty(\Omega_T)$ with $\varphi(\cdot, T) = 0$, where $\mu_k := \mu_a + \nu_k^+ - \nu_k^- \in \mathfrak{M}_a(\Omega_T)$.

- $\mu_s = \mu_s^+ - \mu_s^-$, where μ_s^+ and μ_s^- are the positive and negative parts, respectively.
- We say that a sequence $\{\nu_k\} \subset \mathfrak{M}_b(\Omega_T)$ converges *tightly* to $\nu \in \mathfrak{M}_b(\Omega_T)$ if

$$\lim_{k\to\infty} \int_{\Omega_T} \varphi \ d\nu_k = \int_{\Omega_T} \varphi \ d\nu \qquad \forall \varphi \in C_b(\Omega_T).$$

Various notions of solutions

- Very weak solution, Distributional solution, Duality solution, $\cdots$
- **Renormalized solution**
  - DiPerna & Lions '89
  - Dal Maso & Murat & Orsina & Prignet '99
  - Petitta '08
  - Petitta & Ponce & Porretta '11
  - Petitta & Porretta '15, $\cdots$
- **Entropy solution**
  - Bénilan & Boccardo & Gallouët & Gariepy & Pierre & Vázquez '95
  - Boccardo & Gallouët & Orsina '96
  - Prignet '97, $\cdots$
- **SOLA (Solution Obtained by Limits of Approximations)**
  - Boccardo & Gallouët '89, '92
  - Dall'Aglio '96
  - Boccardo & Dall'Aglio & Gallouët & Orsina '97, $\cdots$

Small BMO assumption

- Let $R > 0$ and $\delta \in (0, \frac{1}{8})$. The vector field $\mathbf{a}$ satisfies

$$\sup_{t_1,t_2\in\mathbb{R}} \sup_{0<r\le R} \sup_{y\in\mathbb{R}^n} \fint_{t_1}^{t_2} \fint_{B_r(y)} \Theta\left(\mathbf{a}, B_r(y)\right)(x,t)\, dxdt \le \delta,$$

  where

$$\Theta\left(\mathbf{a}, B_r(y)\right)(x,t) := \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \left| \frac{\mathbf{a}(\xi,x,t)}{|\xi|^{p-1}} - \fint_{B_r(y)} \frac{\mathbf{a}(\xi,s,t)}{|\xi|^{p-1}}\, ds \right|.$$

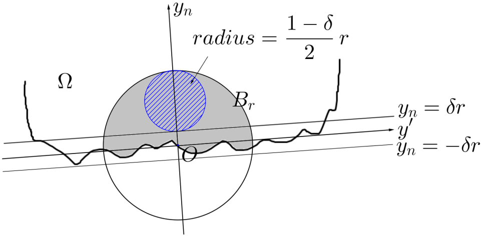
- The map $x \mapsto \frac{\mathbf{a}(\xi,x,t)}{|\xi|^{p-1}}$ is of **Bounded Mean Oscillation (BMO)** such that its BMO seminorm is less than or equal to δ, uniformly in ξ and t.

- $L^\infty \subset$ BMO $\subset L^q$ for any $q < \infty$.

Reifenberg flat domain

- Let $R > 0$ and $\delta \in (0, \frac{1}{8})$. The domain Ω is (δ, R)-**Reifenberg flat**, that is, for each $x_0 \in \partial\Omega$ and each $r \in (0, R)$, there exists a coordinate system $\{y_1, \cdots, y_n\}$ such that in this new coordinate system, the origin is x_0 and

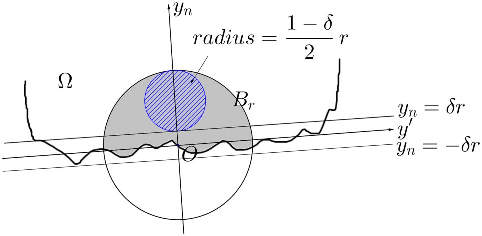
$$B_r \cap \{y_n > \delta r\} \subset B_r \cap \Omega \subset B_r \cap \{y_n > -\delta r\}.$$



Reifenberg flat domain (cont.)

- This domain includes C^1 domain, Lipschitz domain with a small Lipschitz constant, and so on.
- This domain also satisfies the **measure density condition**:

$$\sup_{0<r\leq R}\sup_{y\in\Omega}\frac{|B_r(y)|}{|\Omega\cap B_r(y)|}\leq\left(\frac{2}{1-\delta}\right)^n.$$



Global Calderón-Zygmund type estimates 1

$$(P)\begin{cases} u_t - \operatorname{div}\mathbf{a}(Du,x,t) = \mu & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_p\Omega_T, \end{cases} \qquad \left(p > \frac{2n}{n+1}\right).$$

- Assume that $(\mathbf{a},\Omega)$ is (δ,R)-vanishing.
  ($\iff$ $x \mapsto \frac{\mathbf{a}(\xi,x,t)}{|\xi|^{p-1}}$ is δ-small BMO and Ω is (δ,R)-Reifenberg flat)

Theorem (Byun & P. & Shin '21, P. & Shin submitted)

Let u be a renormalized solution of (P). For $p \geq 2$, we have

$$\int_{\Omega_T} |Du|^q \, dxdt \lesssim \left(\int_{\Omega_T} [\mathcal{M}_1(\mu)]^q \, dxdt\right)^{\frac{(n+2)(p-1)}{(n+1)p-n}} + 1 \quad \textit{for } \; 0 < \forall q < \infty.$$

For $\frac{2n}{n+1} < p \leq 2$, we have

$$\int_{\Omega_T} |Du|^q \, dxdt \lesssim \int_{\Omega_T} [\mathcal{M}_1(\mu)]^{\frac{2q}{(n+1)p-2n}} \, dxdt + 1 \quad \textit{for } \; 0 < \forall q < \infty.$$

- For elliptic problems, see Mingione '11, Phuc '14, Nguyen & Phuc '19.

Global Calderón-Zygmund type estimates 2

- Assume that the following decomposition holds:

$$\mu = \mu_0 \otimes f,$$

where μ_0 is a finite signed Radon measure on Ω and $f \in L^{\frac{q}{p-1}}(0,T)$.

- Assume that $(\mathbf{a}, \Omega)$ is (δ, R)-vanishing.

Theorem (Byun & P. & Shin '21, P. & Shin submitted)

Let u be a renormalized solution of (P). For $p \geq 2$, we have

$$\int_{\Omega_T} |Du|^q \, dxdt \lesssim \left(\int_{\Omega_T} [\mathcal{M}_1(\mu_0) f]^{\frac{q}{p-1}} \, dxdt \right)^{\frac{(n+2)(p-1)^2}{(n+1)p-n}} + 1 \ \text{for } p-1 < \forall q < \infty.$$

For $\frac{2n}{n+1} < p \leq 2$, we have

$$\int_{\Omega_T} |Du|^q \, dxdt \lesssim \int_{\Omega_T} [\mathcal{M}_1(\mu_0) f]^{\frac{q}{p-1}} \, dxdt + 1 \quad \text{for } \ p-1 < \forall q < \infty.$$

Known results for parabolic measure data problems

- **Calderón-Zygmund type estimates**
  - Nguyen '15: linear problems
  - Byun & P. '18: $p = 2$
  - Byun & P. & Shin, '21: $p > 2 - \frac{1}{n+1}$
  - P. & Shin, submitted: $\frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1}$
- **Pointwise potential estimates**
  - Duzaar & Mingione '11: $p = 2$
  - Kuusi & Mingione '13, '14: $p > 2 - \frac{1}{n+1}$
  - P. & Shin, in preparation: $\frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1}$
- **Marcinkiewicz estimates**
  - Baroni '14, Bui & Duong '18: $p \geq 2$
  - Baroni '17: $2 - \frac{1}{n+1} < p < 2$
  - P., in preparation: $\frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1}$
- **In this talk, we only focus on when $\frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1}$.**

Idea of proof

- **Our proof consists of comparison estimates via approximation arguments and covering argument.**

- **We obtain comparison estimates under intrinsic cylinders.**

- **Covering argument is based on maximal function technique, introduced by Caffarelli & Peral '98, developed in Byun & Palagachev & Shin '20.**

  $*$ Maximal function free technique (Acerbi & Mingione '07).

Intrinsic scaling

- **Dimensional analysis**
  We denote by $[\omega]$ the dimension of the quantity ω (e.g. $[\tau]$ is time, $[\rho]$ is a length). Consider $v_t = \mathrm{div}(|Dv|^{p-2}Du)$. Then we have dimension

$$\frac{[v]}{[\tau]} = \frac{[v]^{p-1}}{[\rho]^p} \;\Rightarrow\; [\tau] = [v]^{2-p}[\rho]^p = \left(\frac{[v]}{[\rho]}\right)^{2-p}[\rho]^2 = \lambda^{2-p}[\rho]^2,$$

  provided that $\lambda := [v]/[\rho](= [Dv])$.
- Intrinsic parabolic cylinders: for $\lambda > 0$,

$$Q_\rho^\lambda(x_0,t_0) := B_\rho(x_0) \times \underbrace{\left(t_0 - \lambda^{2-p}\rho^2, t_0 + \lambda^{2-p}\rho^2\right)}_{=:I_\rho^\lambda(t_0)}.$$

  If $p = 2$ or $\lambda = 1$, then $Q_\rho^\lambda(x_0,t_0) = Q_\rho(x_0,t_0)$.
- **The concept of intrinsic scaling means that the size of cylinders depends on the solution itself (introduced by DiBenedetto).**

Comparison estimates

$$\begin{cases} u_t - \operatorname{div} \mathbf{a}(Du, x, t) = \mu & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_p \Omega_T. \end{cases} \tag{P}$$

Let w be the weak solution of the homogeneous problem

$$\begin{cases} w_t - \operatorname{div} \mathbf{a}(Dw, x, t) = 0 & \text{in } K_{8r}^{\lambda}(x_0, t_0), \\ w = u & \text{on } \partial_p K_{8r}^{\lambda}(x_0, t_0). \end{cases} \tag{P_w}$$

- $K_{8r}^{\lambda}(x_0, t_0) := Q_{8r}^{\lambda}(x_0, t_0) \cap \Omega_{\mathfrak{T}}$, where $\Omega_{\mathfrak{T}} := \Omega \times (-\infty, T)$.
- Suppose that $(\mathbf{a}, \Omega)$ is (δ, R)-vanishing for some $R > 0$, where $\delta \in \left(0, \frac{1}{8}\right)$ is to be determined later. Fix any $\lambda > 0$, $(x_0, t_0) \in \Omega_{\mathfrak{T}}$ and $0 < r \leq \frac{R}{8}$ satisfying

$$B_{8r}(x_0) \cap \{x_n > 0\} \subset B_{8r}(x_0) \cap \Omega \subset B_{8r}(x_0) \cap \{x_n > -16\delta r\}.$$

- From now on, for simplicity, we omit the center (x_0, t_0) of $K_{8r}^{\lambda}(x_0, t_0)$.

Higher integrability

Lemma (Bögelein & Parviainen '10)

Let $\frac{2n}{n+2} < p \leq 2$. *There exist constants* $\sigma = \sigma(n, \Lambda_0, \Lambda_1, p, \theta) \in (0,1)$ *and* $c = c(n, \Lambda_0, \Lambda_1, p, \theta) \geq 1$ *such that*

$$\fint_{K_{4r}^{\lambda}} |Dw|^{p(1+\sigma)}\, dxdt \leq c\lambda^{p(1+\sigma)} \left[\left(\lambda^{-\theta} \fint_{K_{8r}^{\lambda}} |Dw|^{\theta}\, dxdt \right)^{\frac{p(1+\mathfrak{d}\sigma)}{p-p\mathfrak{d}+\mathfrak{d}\theta}} + 1 \right]$$

for every $\frac{(2-p)n}{2} < \theta \leq p$, *where* $\mathfrak{d} := \frac{2p}{(n+2)p-2n}$.

- In an interior case, see Kinnunen & Lewis '00.
- The lower bound of θ comes from $p - p\mathfrak{d} + \mathfrak{d}\theta > 0$ (**scaling deficit!**).
- For elliptic equations with p-growth $(p > 1)$, there has no scaling deficit (see Giusti's book).

Comparison estimates below L^1 spaces

Lemma (Comparison estimate between (P) and (P_w))

Let $\frac{3n+2}{2n+2} < p \leq 2 - \frac{1}{n+1}$. *There exists a constant* $c = c(n, \Lambda_0, p, \theta) \geq 1$ *such that*

$$\left(\fint_{K_{8r}^{\lambda}} |Du - Dw|^{\theta} \, dxdt \right)^{\frac{1}{\theta}} \leq c \left[\frac{|\mu|(K_{8r}^{\lambda})}{|K_{8r}^{\lambda}|^{\frac{n+1}{n+2}}} \right]^{\frac{n+2}{(n+1)p-n}} + c \left[\frac{|\mu|(K_{8r}^{\lambda})}{|K_{8r}^{\lambda}|^{\frac{n+1}{n+2}}} \right] \left(\fint_{K_{8r}^{\lambda}} |Du|^{\theta} \, dxdt \right)^{\frac{(2-p)(n+1)}{\theta(n+2)}}$$

for any constant θ *such that* $\frac{n+2}{2(n+1)} < \theta < p - \frac{n}{n+1} \leq 1$.

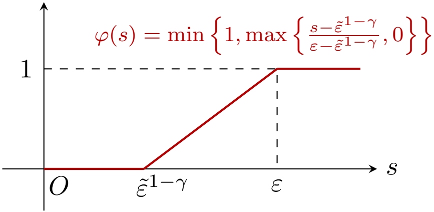
- For $p > 2 - \frac{1}{n+1}$, see Kuusi & Mingione '13, '14.
- For elliptic comparison estimate (with $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$), see Nguyen & Phuc '19.

Outline of proof (1/5)

- For any fixed $\varepsilon, \tilde{\varepsilon}$ with $\varepsilon > \tilde{\varepsilon}^{1-\gamma} > 0$, choose a test function

$$\varphi_1 = \pm \min\left\{1, \max\left\{\frac{(u-w)_\pm^{1-\gamma} - \tilde{\varepsilon}^{1-\gamma}}{\varepsilon - \tilde{\varepsilon}^{1-\gamma}}, 0\right\}\right\}\zeta(t), \quad (0 \leq \gamma < 1)$$

where $\zeta \in C^\infty(\mathbb{R})$ is nonincreasing with $0 \leq \zeta \leq 1$ and $\zeta(t) = 0$ for $t \geq \tau$.



$$D\varphi_1 = \frac{1-\gamma}{\varepsilon - \tilde{\varepsilon}^{1-\gamma}}(u-w)_\pm^{-\gamma}(Du - Dw)\chi_{\{\tilde{\varepsilon}^{1-\gamma} < (u-w)_\pm^{1-\gamma} < \varepsilon\}}\zeta.$$

- $(u-w)_\pm^{-\gamma} \leq \tilde{\varepsilon}^{-\gamma}$ in the set $\left\{\tilde{\varepsilon} < (u-w)_\pm < \varepsilon^{\frac{1}{1-\gamma}}\right\}$.

Outline of proof (2/5)

- Weak formulation

$$\underbrace{\int_{K_{8r}^{\lambda}} (u-w)_t \varphi_1 \, dxdt}_{:=I_1} + \underbrace{\int_{K_{8r}^{\lambda}} \langle \mathbf{a}(Du,x,t) - \mathbf{a}(Dw,x,t), D\varphi_1 \rangle \, dxdt}_{\geq 0 \quad \text{by monotonicity}} = \underbrace{\int_{K_{8r}^{\lambda}} \varphi_1 \, d\mu}_{\leq |\mu|(K_{8r}^{\lambda})} .$$

- The integration by parts gives

$$I_1 = \int_{K_{8r}^{\lambda}} \left[\int_{\tilde{\varepsilon}}^{(u-w)_{\pm}} \min\left\{ 1, \frac{s^{1-\gamma} - \tilde{\varepsilon}^{1-\gamma}}{\varepsilon - \tilde{\varepsilon}^{1-\gamma}} \right\} \, ds \right] (-\zeta_t) \, \, dxdt \geq 0$$

- Letting $\tilde{\varepsilon} \to 0$, we have

$$\int_{K_{8r}^{\lambda}} \left[\int_{0}^{(u-w)_{\pm}} \min\left\{ 1, \frac{s^{1-\gamma}}{\varepsilon} \right\} \, ds \right] (-\zeta_t) \, \, dxdt \leq |\mu|(K_{8r}^{\lambda})$$

- Letting $\varepsilon \to 0$ and letting ζ approximate the characteristic function $\chi_{(-\infty,\tau)}$, we derive

$$\sup_{\tau \in I_{8r}^{\lambda}} \int_{\Omega_{8r} \times \{\tau\}} |u-w| \, dx \leq |\mu|(K_{8r}^{\lambda}). \tag{1}$$

Outline of proof (3/5)

- Choose another test function

$$\varphi_2 = \frac{\varphi_1}{\left(\alpha^{1-\gamma} + (u-w)_{\pm}^{1-\gamma}\right)^{\xi-1}},$$

where $0 \le \gamma < 1$, $\alpha > 0$ and $\xi > 1$ are to be determined later.

- Weak formulation

$$\underbrace{\int_{K_{8r}^{\lambda}} (u-w)_t \varphi_2 \, dxdt}_{\le \alpha^{(1-\gamma)(1-\xi)}|\mu|(K_{8r}^{\lambda})} + \underbrace{\int_{K_{8r}^{\lambda}} \langle \mathbf{a}(Du,x,t) - \mathbf{a}(Dw,x,t), D\varphi_2 \rangle \, dxdt}_{:=I_2} = \underbrace{\int_{K_{8r}^{\lambda}} \varphi_2 \, d\mu}_{\le \alpha^{(1-\gamma)(1-\xi)}|\mu|(K_{8r}^{\lambda})}.$$

- From calculations of I_2, we derive

$$\int_{K_{8r}^{\lambda}} \frac{|u-w|^{-\gamma}\left(|Du|^2 + |Dw|^2\right)^{\frac{p-2}{2}} |Du - Dw|^2}{\left(\alpha^{1-\gamma} + |u-w|^{1-\gamma}\right)^{\xi}} \, dxdt \le c \frac{\alpha^{(1-\gamma)(1-\xi)}}{(1-\gamma)(\xi-1)} |\mu|(K_{8r}^{\lambda}). \tag{2}$$

Outline of proof (4/5)

- **Parabolic embedding theorem** $(q \geq 1,\ m > 0)$

$$\int_{K_{8r}^{\lambda}} |f|^{q\frac{n+m}{n}}\, dxdt \leq c \left(\int_{K_{8r}^{\lambda}} |Df|^{q}\, dxdt \right) \left(\sup_{t \in I_{8r}^{\lambda}} \int_{\Omega_{8r}\times\{t\}} |f|^{m}\, dx \right)^{\frac{q}{n}}.$$

- Take $f = |u-w|^{\frac{p-\beta}{p}}$, $q = 1$ and $m = \frac{p}{p-\beta}$ $\quad (0 \leq \beta < p)$.

$$\begin{aligned}\fint_{K_{8r}^{\lambda}} |u-w|^{\frac{(n+1)p-n\beta}{np}}\, dxdt &\leq c \fint_{K_{8r}^{\lambda}} \left| D|u-w|^{\frac{p-\beta}{p}} \right| dxdt \\ &\qquad \times \left(\sup_{t \in I_{8r}^{\lambda}} \int_{\Omega_{8r}\times\{t\}} |u-w|\, dx \right)^{\frac{1}{n}} \\ &\leq c \left[|\mu|(K_{8r}^{\lambda}) \right]^{\frac{1}{n}} \fint_{K_{8r}^{\lambda}} \left| D|u-w|^{\frac{p-\beta}{p}} \right| dxdt \quad \text{(by (1))}.\end{aligned}$$

Outline of proof (5/5)

- $\fint_{K^\lambda_{8r}} \left| D|u-w|^{\frac{p-\beta}{p}} \right| \, dxdt$ can be estimated by (2).
(Here, choose $0 \le \gamma < 1$, $\alpha > 0$ and $\xi > 1 \Longrightarrow$ **the range of p, β is determined**.)

- Since $\left| D|u-w|^{\frac{p-\beta}{p}} \right| = \dfrac{p-\beta}{p} |u-w|^{-\frac{\beta}{p}} |Du-Dw|$, we compute

$$
\begin{aligned}
&\fint_{K^\lambda_{8r}} |Du-Dw|^{\theta} \, dxdt \\
&\quad = \fint_{K^\lambda_{8r}} \left(|u-w|^{-\frac{\beta}{p}} |Du-Dw| \right)^{\theta} |u-w|^{\frac{\beta\theta}{p}} \, dxdt \\
&\quad \le c \left(\fint_{K^\lambda_{8r}} \left| D|u-w|^{\frac{p-\beta}{p}} \right| \, dxdt \right)^{\theta} \left(\fint_{K^\lambda_{8r}} |u-w|^{\frac{\beta\theta}{p(1-\theta)}} \, dxdt \right)^{1-\theta} \\
&\quad \le c \left[|\mu|(K^\lambda_{8r}) \right]^{\frac{1-\theta}{n}} \fint_{K^\lambda_{8r}} \left| D|u-w|^{\frac{p-\beta}{p}} \right| \, dxdt,
\end{aligned}
$$

by taking $\frac{\beta\theta}{p(1-\theta)} = \frac{(n+1)p-n\beta}{np}$ ($\Longrightarrow$ **the range of θ is determined**).

- **Comparison estimate between** (P) **and** (P_w)
  Let $\frac{3n+2}{2n+2} < p \leq 2 - \frac{1}{n+1}$. There exists a constant $c = c(n, \Lambda_0, p, \theta) \geq 1$ such that

$$\left(\fint_{K_{8r}^{\lambda}} |Du - Dw|^{\theta} \, dxdt \right)^{\frac{1}{\theta}} \leq c \left[\frac{|\mu|(K_{8r}^{\lambda})}{|K_{8r}^{\lambda}|^{\frac{n+1}{n+2}}} \right]^{\frac{n+2}{(n+1)p-n}} + c \left[\frac{|\mu|(K_{8r}^{\lambda})}{|K_{8r}^{\lambda}|^{\frac{n+1}{n+2}}} \right] \left(\fint_{K_{8r}^{\lambda}} |Du|^{\theta} \, dxdt \right)^{\frac{(2-p)(n+1)}{\theta(n+2)}}$$

  for any constant θ such that $\frac{n+2}{2(n+1)} < \theta < p - \frac{n}{n+1} \leq 1$.

- **Higher integrability** (Bögelein & Parviainen '10)
  Let $\frac{2n}{n+2} < p \leq 2$. There exist constants $\sigma = \sigma(n, \Lambda_0, \Lambda_1, p, \theta) \in (0, 1)$ and $c = c(n, \Lambda_0, \Lambda_1, p, \theta) \geq 1$ such that

$$\fint_{K_{4r}^{\lambda}} |Dw|^{p(1+\sigma)} \, dxdt \leq c \lambda^{p(1+\sigma)} \left[\left(\lambda^{-\theta} \fint_{K_{8r}^{\lambda}} |Dw|^{\theta} \, dxdt \right)^{\frac{p(1+\mathfrak{d}\sigma)}{p - p\mathfrak{d} + \mathfrak{d}\theta}} + 1 \right]$$

  for every $\frac{(2-p)n}{2} < \theta \leq p$, where $\mathfrak{d} := \frac{2p}{(n+2)p-2n}$.

Why $p > \frac{2n}{n+1}$?

- From comparison estimates and higher integrability, the valid range of p is

$$\frac{2n}{n+1} < p \leq 2 - \frac{1}{n+1},$$

not $\max\left\{\frac{3n+2}{2n+2}, \frac{2n}{n+2}\right\} < p \leq 2 - \frac{1}{n+1}$, since the constant θ exists only when

$$\max\left\{\frac{n+2}{2(n+1)}, \frac{(2-p)n}{2}\right\} < \theta < p - \frac{n}{n+1}.$$

- $\frac{2n}{n+1} \geq \max\left\{\frac{2n}{n+2}, \frac{3n+2}{2n+2}\right\}$, where the equality holds iff $n = 2$.

- $\frac{2n}{n+2} < \frac{3n+2}{2n+2}$ if $n = 2, 3, 4$; $\frac{2n}{n+2} > \frac{3n+2}{2n+2}$ if $n \geq 5$.

Idea of proof

- **Our proof consists of comparison estimates via approximation arguments and covering argument.**
- **We obtain comparison estimates under intrinsic cylinders.**
- **Covering argument is based on maximal function technique, introduced by Caffarelli & Peral '98, developed in Byun & Palagachev & Shin '20.**

  $*$ Maximal function free technique (Acerbi & Mingione '07).

Modified Vitali covering lemma

Lemma

Let $0 < \varepsilon < 1$, $\lambda > 0$ *and* $\Omega_{\mathfrak{T}} := \Omega \times (-\infty, T)$, *where* Ω *is* (δ, R)*-Reifenberg flat. Let* $\mathfrak{C} \subset \mathfrak{D} \subset \Omega_{\mathfrak{T}}$ *be two bounded measurable subsets such that*

1. $|\mathfrak{C}| < \varepsilon \left|Q^{\lambda}_{R/10}\right|$, *and*
2. *for any* $(y, s) \in \Omega_{\mathfrak{T}}$ *and any* $r \in (0, R/10]$ *with* $|\mathfrak{C} \cap Q^{\lambda}_{r}(y, s)| \geq \varepsilon |Q^{\lambda}_{r}|$, $Q^{\lambda}_{r}(y, s) \cap \Omega_{\mathfrak{T}} \subset \mathfrak{D}$.

Then we have

$$|\mathfrak{C}| \leq \left(\frac{10}{1-\delta}\right)^{n+2} \varepsilon |\mathfrak{D}| =: \varepsilon_0 |\mathfrak{D}|. \tag{3}$$

- The covering lemma above is obtained under $\mathfrak{C} \subset \mathfrak{D} \subset \Omega_{\mathfrak{T}}$, not $\mathfrak{C} \subset \mathfrak{D} \subset \Omega_{T} := \Omega \times (0, T)$. Then the relation (3) is independent of λ.

- For any fixed $\varepsilon \in (0,1)$, we set

$$\lambda_0 := \left[\frac{|\mu|(\Omega_T)}{|\Omega_T|^{\frac{n+1}{n+2}}}\right]^{\beta\theta} \frac{|\Omega_T|}{\varepsilon\left|Q_{R/10}\right|} + \left[\frac{|\mu|(\Omega_T)}{\delta T^{\frac{n+1}{2}}}\right]^{d} + 1,$$

where $\beta := \frac{n+2}{(n+1)p-n}$, $d := \frac{2}{(n+1)p-2n}$, and θ is a constant such that

$$\max\left\{\frac{n+2}{2(n+1)}, \frac{(2-p)n}{2}, p-1\right\} < \theta < p - \frac{n}{n+1} \leq 1.$$

- For $\lambda \geq \lambda_0 \geq 1$ and $N > 1$, we write

$$\mathfrak{C} := \left\{\mathcal{M}^{\lambda}(|Du|^{\theta}) > (N\lambda)^{\theta}\right\}, \quad \mathfrak{D} := \left\{\mathcal{M}^{\lambda}(|Du|^{\theta}) > \lambda^{\theta}\right\} \cup \left\{\mathcal{M}_1^{\lambda}(\mu) > \delta\lambda\right\},$$

where

$$\mathcal{M}^{\lambda}(|Du|^{\theta})(x,t) := \sup_{r>0} \fint_{Q_r^{\lambda}(x,t)} |Du(y,s)|^{\theta}\, dyds,$$

$$\mathcal{M}_1^{\lambda}(\mu)(x,t) := \sup_{r>0} \frac{|\mu|(Q_r^{\lambda}(x,t))}{r^{n+1}}.$$

Verifying two conditions

- **Energy type estimates**
  Let $\frac{3n+2}{2n+2} < p \leq 2 - \frac{1}{n+1}$. Then there is a constant $c = c(n, \Lambda_0, p, \theta) \geq 1$ such that
  $$\left(\fint_{\Omega_T} |Du|^{\theta} \, dxdt \right)^{\frac{1}{\theta}} \leq c \left[\frac{|\mu|(\Omega_T)}{|\Omega_T|^{\frac{n+1}{n+2}}} \right]^{\frac{n+2}{(n+1)p-n}}$$
  for any constant θ such that $0 < \theta < p - \frac{n}{n+1}$.
- **(1st condition)** There exists a constant $N_1 = N_1(n, \Lambda_0, p, \theta) > 1$ such that for any fixed $N \geq N_1$ and $\lambda \geq \lambda_0$, we have $|\mathfrak{C}| < \varepsilon \left| Q^{\lambda}_{R/10} \right|$.

Pf. Since $\theta > p - 1$ and $\lambda \geq \lambda_0 \geq 1$, we note that $\lambda^{-\theta} \leq \lambda^{1-p} \leq \lambda^{2-p} \lambda_0^{-1}$.

$$\begin{aligned} |\mathfrak{C}| &\leq \frac{c}{(\lambda N)^{\theta}} \int_{\Omega_T} |Du|^{\theta} \, dxdt \leq \frac{c|\Omega_T|}{(\lambda N)^{\theta}} \left[\frac{|\mu|(\Omega_T)}{|\Omega_T|^{\frac{n+1}{n+2}}} \right]^{\beta\theta} \\ &< \frac{c\lambda^{2-p} \varepsilon \left| Q_{R/10} \right|}{N_1{}^{\theta}} \leq \varepsilon \left| Q^{\lambda}_{R/10} \right|, \end{aligned}$$

by selecting N_1 large enough.

Verifying two conditions (cont.)

- For $\lambda \geq \lambda_0 \geq 1$ and $N > 1$, we write

$$\mathfrak{C} := \left\{\mathcal{M}^{\lambda}(|Du|^{\theta}) > (N\lambda)^{\theta}\right\}, \quad \mathfrak{D} := \left\{\mathcal{M}^{\lambda}(|Du|^{\theta}) > \lambda^{\theta}\right\} \cup \left\{\mathcal{M}_1^{\lambda}(\mu) > \delta\lambda\right\}.$$

- **(1st condition)** There exists a constant $N_1 = N_1(n, \Lambda_0, p, \theta) > 1$ such that for any fixed $N \geq N_1$ and $\lambda \geq \lambda_0$, we have $|\mathfrak{C}| < \varepsilon \left|Q^{\lambda}_{R/10}\right|$.

- **From previous comparison estimates, we can obtain 2nd condition:** There exists $N_2 = N_2(n, \Lambda_0, \Lambda_1, p, \theta) > 1$ such that for any fixed $\lambda \geq \lambda_0$, $N \geq N_2$, $r \in \left(0, \frac{R}{10}\right]$ and $(y, s) \in \Omega_{\mathfrak{T}}$ with $\left|\mathfrak{C} \cap Q^{\lambda}_r(y, s)\right| \geq \varepsilon \left|Q^{\lambda}_r\right|$, we have $K^{\lambda}_r(y, s) \subset \mathfrak{D}$.

- **Take $N = \max\{N_1, N_2\}$. We apply the covering lemma to derive the decay estimates as follows:** for any $\lambda \geq \lambda_0$,

$$\left|\left\{\mathcal{M}^{\lambda}(|Du|^{\theta}) > (N\lambda)^{\theta}\right\}\right| \leq \varepsilon_0 \left|\left\{\mathcal{M}^{\lambda}(|Du|^{\theta}) > \lambda^{\theta}\right\}\right| + \varepsilon_0 \left|\left\{\mathcal{M}_1^{\lambda}(\mu) > \delta\lambda\right\}\right|.$$

Relation between classical and intrinsic fractional maximal functions

- $\mathcal{M}_1(\mu)(x,t) := \sup_{r>0} \frac{|\mu|(Q_r(x,t))}{r^{n+1}}$ & $\mathcal{M}_1^{\lambda}(\mu)(x,t) := \sup_{r>0} \frac{|\mu|(Q_r^{\lambda}(x,t))}{r^{n+1}}$.
- Let $\delta > 0$ and let $\lambda \geq 1$. If $\frac{2n}{n+1} < p \leq 2$, then we have

$$\left\{\mathcal{M}_1^{\lambda}(\mu) > \delta\lambda\right\} \subset \left\{[\mathcal{M}_1(\mu)]^{\frac{2}{(n+1)p-2n}} > \delta^{\frac{2}{(n+1)p-2n}}\lambda\right\}.$$

Pf. Set $r_0 := \lambda^{\frac{2-p}{2}} r$. Since $Q_r^{\lambda} \subset Q_{r_0}$, $\frac{|\mu|(Q_r^{\lambda}(y,s))}{r^{n+1}} \leq \frac{|\mu|\left(Q_{r_0}(y,s)\right)}{r_0^{n+1}} \lambda^{\frac{(2-p)(n+1)}{2}}$.

If $\frac{|\mu|(Q_r^{\lambda}(y,s))}{r^{n+1}} > \delta\lambda$, then $\frac{|\mu|(Q_{r_0}(y,s))}{r_0^{n+1}} > \delta\lambda^{\frac{(n+1)p-2n}{2}}$.

Relation between classical and intrinsic fractional maximal functions (cont.)

- Let $\delta > 0$ and let $\lambda \geq 1$. If $p > 1$ and $\mu = \mu_0 \otimes f$, then we have

$$\left\{\mathcal{M}_1^\lambda(\mu) > \delta\lambda\right\} \subset \left\{\left[2(\mathcal{M}_1(\mu_0))(\mathcal{M}f)\right]^{\frac{1}{p-1}} > \delta^{\frac{1}{p-1}}\lambda\right\},$$

where $\mathcal{M}f(t) := \sup_{r>0} \fint_{t-r}^{t+r} |f(s)|\, ds$ and $\mathcal{M}_1(\mu_0)(x) := \sup_{r>0} \dfrac{|\mu_0|(B_r(x))}{r^{n-1}}$.

Pf. Since $\mu = \mu_0 \otimes f$, $\dfrac{|\mu|(Q_r^\lambda(x,t))}{\lambda^{2-p} r^{n+1}} = 2\dfrac{|\mu_0|(B_r(x))}{r^{n-1}} \fint_{t-\lambda^{2-p}r^2}^{t+\lambda^{2-p}r^2} |f(s)|\ ds.$

If $\dfrac{|\mu|(Q_r^\lambda(x,t))}{r^{n+1}} > \delta\lambda$, then $2\dfrac{|\mu_0|(B_r(x))}{r^{n-1}} \fint_{t-\lambda^{2-p}r^2}^{t+\lambda^{2-p}r^2} |f(s)|\ ds > \delta\lambda^{p-1}$.

Decay estimates

- For any $\lambda \geq \lambda_0$, we have

$$\left|\left\{\mathcal{M}^\lambda(|Du|^\theta) > (N\lambda)^\theta\right\}\right| \leq \varepsilon_0 \left|\{\mathcal{M}^\lambda(|Du|^\theta) > \lambda^\theta\}\right| + \varepsilon_0 \left|\left\{[\mathcal{M}_1(\mu)]^d > \delta^d \lambda\right\}\right|,$$

where $d := \frac{2}{(n+1)p-2n} \geq 1$. Moreover, if $\mu = \mu_0 \otimes f$, then we have

$$\begin{aligned}&\left|\left\{\mathcal{M}^\lambda(|Du|^\theta) > (N\lambda)^\theta\right\}\right| \\ &\qquad \leq \varepsilon_0 \left|\{\mathcal{M}^\lambda(|Du|^\theta) > \lambda^\theta\}\right| + \varepsilon_0 \left|\left\{[2(\mathcal{M}_1(\mu_0))(\mathcal{M}f)]^{\frac{1}{p-1}} > \delta^{\frac{1}{p-1}} \lambda\right\}\right|,\end{aligned}$$

where $\varepsilon_0 := \left(\frac{10}{1-\delta}\right)^{n+2} \varepsilon$.

Decay estimates of integral type

- Weak $(1,1)$-estimate for the λ-maximal function

$$\left|\{\mathcal{M}^{\lambda} f > 2\alpha\}\right| \leq \frac{c(n)}{\alpha} \int_{\{|f|>\alpha\}} |f| \; dxdt \quad \text{for any } \alpha > 0.$$

- For any $\theta_0 \in \left(\theta, p - \frac{n}{n+1}\right)$ and any $\lambda \geq \lambda_0$, there exists a constant $c = c(n, \Lambda_0, \Lambda_1, p, \theta_0) \geq 1$ such that

$$\begin{aligned} \int_{\{|Du|>N\lambda\}} |Du|^{\theta_0} \; dxdt \leq c\varepsilon \int_{\left\{|Du|>\frac{\lambda}{2}\right\}} |Du|^{\theta_0} \; dxdt \\ + \frac{c\varepsilon}{\delta^{d\theta_0}} \int_{\left\{[\mathcal{M}_1(\mu)]^d > \delta^d \lambda\right\}} \left[\mathcal{M}_1(\mu)\right]^{d\theta_0} \; dxdt. \end{aligned}$$

- Using the above decay estimate, we can obtain our main global gradient estimate.

Summary

- Consider **the Cauchy-Dirichlet problem with measure data**

$$\begin{cases} u_t - \operatorname{div} \mathbf{a}(Du, x, t) = \mu & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_p \Omega_T, \end{cases} \tag{P}$$

  where μ **is a signed Radon measure on** Ω_T **with finite mass.**

- $(\mathbf{a}, \Omega)$ **is** (δ, R)**-vanishing**
  ($\iff$ $x \mapsto \frac{\mathbf{a}(\xi,x,t)}{|\xi|^{p-1}}$ is δ-small BMO & Ω is (δ, R)-Reifenberg flat)

- **Global gradient estimate.** Let u be a **renormalized solution** of (P).
  For $p \geq 2$, we have

$$\int_{\Omega_T} |Du|^q \, dxdt \leq c \left(\int_{\Omega_T} [\mathcal{M}_1(\mu)]^q \, dxdt \right)^{\frac{(n+2)(p-1)}{(n+1)p-n}} + c \quad \text{for } 0 < \forall q < \infty.$$

  For $\frac{2n}{n+1} < p \leq 2$, we have

$$\int_{\Omega_T} |Du|^q \, dxdt \leq c \int_{\Omega_T} [\mathcal{M}_1(\mu)]^{\frac{2q}{(n+1)p-2n}} \, dxdt + c \quad \text{for } 0 < \forall q < \infty.$$

Thank you for your attention!