Maximal regularity for local minimizers of non-autonomous functionals

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This is a joint work with Peter Hästö.

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Non-autonomous functionals

$$W^{1,1}(\Omega) \ni v \quad \mapsto \quad \mathcal{F}(v,\Omega) = \int_\Omega f(x,Dv)\,dx. \tag{1}$$

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- If the integrand f is independent of x, i.e., $\mathcal{F}(v,\Omega) = \int_\Omega f(Dv)\,dx$, the functional $\mathcal{F}$ is said to be autonomous.
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- In particular, if $z \mapsto f(x,z)$ depends only on $|z|$, i.e.,

$$\mathcal{F}(v,\Omega) = \int_\Omega \varphi(x, |Dv|)\, dx \quad \big(f(x,z) \equiv \varphi(x,|z|)\big),$$

  then we say that $\mathcal{F}$ has so-called *Uhlenbeck's structure*.

Functional with p-growth

$$\begin{cases} z \mapsto f(x,z) \text{ is } C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}), \\ 1 < p < \infty, \\ \nu |z|^p \le f(x,z) \le L|z|^p, \\ \nu |z|^{p-2}|\lambda|^2 \le \partial^2 f(x,z)\lambda \cdot \lambda \le L|z|^{p-2}|\lambda|^2. \end{cases}$$

$$W^{1,p}(\Omega) \ni v \quad \mapsto \quad \int_\Omega f(x, Dv)\, dx.$$

E-L eq: $\operatorname{div}(\partial f(x, Du)) = 0.$

- Model case

$$f(x,z) \equiv \varphi(x,|z|) = a(x)|z|^p, \quad 0 < \nu \le a(\cdot) \le L.$$

E-L eq: $\operatorname{div}\left(a(x)|Du|^{p-2}Du\right) = 0.$

Functional with p-growth

$$\omega(r) := \sup_{z\in\mathbb{R}^n\setminus\{0\}} \sup_{x,y\in B_r,\, B_r\subset\Omega} \frac{|f(x,z)-f(y,z)|}{|z|^p} \le 2L.$$

$$\Big(\text{model case: } \omega(r) := \sup_{x,y\in B_r,\, B_r\subset\Omega} |a(x)-a(y)|\Big)$$

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DeGiorgi theory

- (No additional condition) $\implies$ $u \in C^\alpha$ for some $\alpha \in (0,1)$.

Continuous for x

- $\lim_{r\to 0^+} \omega(r) = 0 \implies u \in C^\alpha$ for any $\alpha \in (0,1)$.
- $\omega(r) \lesssim r^\beta \implies u \in C^{1,\alpha}$ for some $\alpha \in (0,1)$. (Maximal regularity)

- (model case) It is well known that if $a(\cdot)$ is VMO then $u \in W^{1,q}$ for any $q > p$ hence $u \in C^\alpha$ for any $\alpha \in (0,1)$.
  Moreover, if $a(\cdot)$ is VMO only for $x_1, \dots, x_{n-1}$ $(x = (x_1, \dots, x_n))$, then $u \in W^{1,q}$ for any $q > p$. (Byun-Kim(16), Kim(18))

Non-autonomous functionals with (p, q)-growth

Marcellini introduced a general class of functionals.

(p, q)-growth condition (Marcellini, 1989)

$$\begin{cases} z \mapsto f(x,z) \text{ is } C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}), \\ 1 < p \leq q, \\ \nu |z|^p \leq f(x,z) \leq L(1 + |z|^q), \\ \nu |z|^{p-2} |\lambda|^2 \leq \partial^2 f(x,z) \lambda \cdot \lambda \leq L(|z|^{p-2} + |z|^{q-2}) |\lambda|^2. \end{cases}$$

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- (p, q)**-growth functionals with Uhlenbeck's structure**

$$f(x, z) \equiv \varphi(x, |z|)$$

$$W^{1,\varphi}(\Omega) \ni v \quad \mapsto \quad \int_\Omega \varphi(x, |Dv|)\, dx. \tag{2}$$

In this case, the previous (p, q)-growth condition can be simplified as

$$\begin{cases} t \mapsto \varphi(x,t) \text{ is in } C^1([0,\infty)) \cap C^2((0,\infty)), \\ 1 < p \le q, \\ p - 1 \le \frac{t\varphi''(x,t)}{\varphi'(x,t)} \le q - 1 \ (\Leftrightarrow t\varphi''(x,t) \approx \varphi'(x,t)). \end{cases} \tag{3}$$

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- **Bounded or Hölder continuous minimizers**

- **functionals and PDEs with generalized Orlicz growth**

$$W^{1,\varphi}(\Omega) \ni v \quad \mapsto \quad \int_\Omega f(x, Dv)\, dx.$$

$$\begin{cases} z \mapsto f(x,z) \in \mathbb{R} \text{ is } C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}), \\ \nu\varphi(x,|z|) \le f(x,z) \le L\varphi(x,|z|), \\ \nu\frac{\varphi'(x,|z|)}{|z|}|\lambda|^2 \le \partial^2 f(x,z)\lambda\cdot\lambda \le L\frac{\varphi'(x,|z|)}{|z|}|\lambda|^2. \end{cases}$$

$$\operatorname{div} A(x, Du) = 0.$$

$$\begin{cases} z \mapsto A(x,z) \in \mathbb{R}^n \text{ is } C^1(\mathbb{R}^n \setminus \{0\}), \\ |A(x,|z|)| + |z||\partial A(x,z)| \le L\varphi'(x,|z|), \\ \nu\frac{\varphi'(x,|z|)}{|z|}|\lambda|^2 \le \partial A(x,z)\lambda\cdot\lambda. \end{cases}$$

Generalized Orlicz function

- We say $f : U \subset \mathbb{R} \to \mathbb{R}$ is almost increasing (or almost decreasing) if $f(t) \le Lf(s)$ (or $f(s) \le Lf(t)$) for any $t < s$ for some $L \ge 1$.

Let $\varphi = \varphi(x,t) : \Omega \times [0,\infty) \to [0,\infty)$ and $\gamma > 0$.
(aInc)$_\gamma$: $t \mapsto \frac{\varphi(x,t)}{t^\gamma}$ is almost increasing unlformly x with $L \ge 1$.
(aDec)$_\gamma$: $t \mapsto \frac{\varphi(x,t)}{t^\gamma}$ is almost decreasing uniformly x with $L \ge 1$.

- When $L = 1$, (Inc)$_\gamma$ = (aInc)$_\gamma$ and (Dec)$_\gamma$ = (aDec)$_\gamma$.
- $p - 1 \le \frac{t\varphi''(x,t)}{\varphi'(x,t)} \le q - 1 \quad \overset{\varphi \in C^2}{\Longleftrightarrow} \quad \varphi'$ satisfies (Inc)$_{p-1}$ and (Dec)$_{q-1}$.
  $\Longrightarrow \quad \varphi$ satisfies (Inc)$_p$ and (Dec)$_q$.

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- $p - 1 \le \frac{t\varphi''(x,t)}{\varphi'(x,t)} \le q - 1 \overset{\varphi \in C^2}{\iff} \varphi'$ satisfies (Inc)$_{p-1}$ and (Dec)$_{q-1}$.
  $\implies \varphi$ satisfies (Inc)$_p$ and (Dec)$_q$.

(A0) $\varphi(\cdot,1) \approx 1$ (i.e., $\exists L \ge 1$ s.t. $L^{-1} \le \varphi(x,1) \le L \quad {}^\forall x \in \Omega$).

From now on, $\varphi \in \Phi_{\mathrm{w}}(\Omega)$ and satisfies (A0), (aInc)$_p$ and (aDec)$_q$.

Perturbed Orlicz(general growth)

$$\varphi(x,t) = a(x)\varphi_0(t),$$

where $\begin{cases} 0 < \nu \le a(\cdot) \le L & (\Longleftrightarrow \varphi \text{ is (A0) and } \varphi \approx \varphi_0), \\ t\varphi_0''(t) \approx \varphi_0'(t) & (\Longrightarrow \varphi \text{ is } (\mathrm{Inc})_p \text{ and } (\mathrm{Dec})_q,\ 1 < p \le q). \end{cases}$

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$$W^{1,\varphi_0}(\Omega) \ni v \quad \mapsto \quad \int_\Omega a(x)\varphi_0(|Dv|)\,dx.$$

Define

$$\omega(r) := \sup_{x,y\in B_r,\ B_r\subset\Omega} |a(x) - a(y)| \le 2L.$$

Regularity (Lieberman(91), ...)

- (No additional condition) $\Longrightarrow$ $u \in C^\alpha$ for some $\alpha \in (0,1)$.
- $\lim_{r\to 0^+} \omega(r) = 0$ $(a(\cdot) \in C^0)$ $\Longrightarrow$ $u \in C^\alpha$ for any $\alpha \in (0,1)$.
- $\omega(r) \lesssim r^\beta$ $(a(\cdot) \in C^\beta)$ $\Longrightarrow$ $u \in C^{1,\alpha}$ for some $\alpha \in (0,1)$.

Non-standard growth

Standard growth cases:

$$\varphi(x,t) = a(x)t^p \quad \big(\text{or} \quad a(x)\varphi_0(t)\big).$$

The power, or growth, of t is independent of x.

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Zhikov's examples

(*On Lavrentiev's phenomenon*, Russian J. Math. Phys. (1995))

(Anna Balci's talk on next week!)

- **Variable exponent**

$$\varphi(x,t) = t^{p(x)},$$

$$1 < p \le \inf p(\cdot) \le \sup p(\cdot) \le q \quad \text{and} \quad p(\cdot) \in C^0.$$

- **Double phase**

$$\varphi(x,t) = t^p + b(x)t^q,$$

$$0 \le b(\cdot) \le L \text{ and } \ b(\cdot) \in C^{0,\beta}, \quad \beta \in (0,1].$$

In last two decades, there have been a lot of researches on regularity theory for these problems.

General non-autonomous problems

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Harjulehto and Hästö found the following crucial conditions on φ:

(A1) There exists $L \geq 1$ such that for any $B_r \Subset \Omega$ with $|B_r| < 1$,

$$\varphi^+_{B_r}(t) \leq L\varphi^-_{B_r}(t) \quad \text{for any} \quad t > 0 \quad \text{with} \quad \varphi^-_{B_r}(t) \in [1, |B_r|^{-1}].$$

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Here, $\varphi^+_U(t) := \sup_{x\in U} \varphi(x,t)$ and $\varphi^-_U(t) := \inf_{x\in U} \varphi(x,t)$.

- (A1) means $\varphi^+_{B_r}(t)$ and $\varphi^-_{B_r}(t)$ are comparable uniformly for $t > 0$ with $\varphi^-_{B_r}(t) \in [1, |B_r|^{-1}] \;\approx\; t \in [1, (\varphi^-_{B_r})^{-1}(|B_r|^{-1})]$.

Hölder continuity for non-autonomous problems

Theorem (Harjulehto-Hästö-Lee, to appear in Ann. Sc. Norm. Super. Pisa)

If φ satisfies (A1), then $u \in C^{\alpha}_{loc}(\Omega)$ for some $\alpha = \alpha(n, p, q, L) \in (0, 1)$.
If φ satisfies (A1-n) and $u \in L^{\infty}(\Omega)$, then $u \in C^{\alpha}_{loc}(\Omega)$ for some $\alpha = \alpha(n, p, q, L) \in (0, 1)$.

- The paper considers more general functionals and quasi-minimizers and proves Harnack's inequality.

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What about C^{α}-regularity for any $\alpha \in (0,1)$ and $C^{1,\alpha}$-regularity for some $\alpha \in (0,1)$?

Note: In particular cases, the proofs of these regularities use perturbation arguments that depend on their particular structures.

Non-autonomous functional with Uhlenbeck's structure

(VA1): Vanishing (A1)

There exists a non-decreasing, bounded, continuous function $\omega : [0,\infty) \to [0,1]$ with $\omega(0)=0$ such that for any small $B_r \Subset \Omega$,

$$\varphi^+_{B_r}(t) \le (1+\omega(r))\varphi^-_{B_r}(t) \quad {}^\forall t>0 \ \text{ with } \ \varphi^-_{B_r}(t) \in [\omega(r), |B_r|^{-1}].$$

$$\text{(A1)}: \quad \varphi^+_{B_r}(t) \le L\varphi^-_{B_r}(t) \quad {}^\forall t>0 \ \text{ with } \ \varphi^-_{B_r}(t) \in [1, |B_r|^{-1}].$$

- (VA1) implies (A1).
- (VA1) implies that $x \mapsto \varphi(x,t)$ is continuous for all $t \in (0,\infty)$.

Non-autonomous functional with Uhlenbeck's structure

Theorem (Hästö-Ok, to appear in JEMS)

Let $\varphi(x,\cdot) \in C^1([0,\infty))$ for every $x \in \Omega$ with φ' satisfying (A0), (Inc)$_{p-1}$ and (Dec)$_{q-1}$ for some $1 < p \le q$.

(1) *If φ satisfies (VA1), then $u \in C^{\alpha}_{loc}(\Omega)$ for any $\alpha \in (0,1)$.*

(2) *If φ satisfies (VA1) and $\omega(r) \lesssim r^{\beta}$ for some $\beta > 0$, then $u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$.*

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- φ is assumed to be C^1 for t (In former regularity results in the case $\varphi(x,t) = \varphi(t)$, φ is always assumed to be $C^2((0,\infty))$).
  In fact, the assumption implies $W^{2,\infty}_{\text{loc}}((0,\infty))$.
  - Recall that

$$p-1 \le \frac{t\varphi''(t)}{\varphi'(t)} \le q-1 \quad \overset{\varphi \in C^2}{\Longleftrightarrow} \quad \varphi'(t): \text{ (Inc)}_{p-1} \text{ and (Dec)}_{q-1}.$$

  - For instance $\varphi(t) := \int_0^t \max\{s^{p-1}, s^{q-1}\}\, ds$ satisfies the assumptions in the above theorem, but is not C^2.

(A1) and (VA1)

Let $B_r = B_r(x_0)$. Since $\sup_{x,y\in B_r} |\varphi(x,t) - \varphi(y,t)| = \varphi^+_{B_r}(t) - \varphi^-_{B_r}(t)$,

(A1)

$$\sup_{x,y\in B_r} \frac{|\varphi(x,t) - \varphi(y,t)|}{\varphi(x_0,t)} \le \sup_{x,y\in B_r} \frac{|\varphi(x,t) - \varphi(y,t)|}{\varphi^-_{B_r}(t)} \le L - 1,$$

$$^\forall t > 0 \text{ with } \varphi^-_{B_r}(t) \in [1, |B_r|^{-1}].$$

(VA1)

$$\sup_{x,y\in B_r} \frac{|\varphi(x,t) - \varphi(y,t)|}{\varphi(x_0,t)} \le \omega(r), \ ^\forall t > 0 \text{ with } \ \varphi^-_{B_r}(t) \in [\omega(r), |B_r|^{-1}].$$

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$$^\forall t>0 \text{ with } \varphi^-_{B_r}(t) \in [1, |B_r|^{-1}].$$

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- If the above inequalities hold for all $t \in (0,\infty)$, they imply

$$\varphi(x,t) \approx a(x)\varphi(x_0,t) \quad \text{(perturbed Orlicz case)},$$

where $\omega_a(r) \lesssim L-1$ and $\omega_a(r) \lesssim \omega(r)$, respectively.

(Recall) Functional with p-growth

$$\omega(r) := \sup_{z\in\mathbb{R}^n\setminus\{0\}} \sup_{x,y\in B_r,\, B_r\Subset\Omega} \frac{|f(x,z)-f(y,z)|}{|z|^p}.$$

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- (No additional condition) $\implies$ $u \in C^\alpha$ for some $\alpha \in (0,1)$.

Continuous for x

- $\lim_{r\to0^+}\omega(r)=0 \implies u\in C^\alpha$ for any $\alpha\in(0,1)$.
- $\omega(r)\lesssim r^\beta \implies u\in C^{1,\alpha}$ for some $\alpha\in(0,1)$. (Maximal regularity)

If $f(x,z) \Rightarrow \varphi(x,|z|)$ and $|z|^p \Rightarrow \varphi(x_0,|z|)$,

$$\omega(r) = \sup_{t\in(0,\infty)} \sup_{x,y\in B_r,\, B_r\Subset\Omega} \frac{|\varphi(x,t)-\varphi(y,t)|}{\varphi(x_0,t)}$$

Non-autonomous functional with Uhlenbeck's structure

The previous theorem covers most known regularity results with continuity assumption for x in special cases.

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- Double phase problem

$$\varphi(x,t) = t^p + b(x)t^q, \quad b \in C^{0,\beta} \text{ and } 0 \le b(\cdot) \le L.$$

- If $\frac{q}{p} < 1 + \frac{\beta}{n}$, φ satisfies (VA1) with $\omega(r) \lesssim r^\gamma$, $\gamma = \beta - \frac{n(q-p)}{p} > 0$, hence $u \in C^{1,\alpha}_{\mathrm{loc}}$. (Colombo-Mingione (2015))

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- If $\frac{q}{p} \ge 1 + \frac{\beta}{n}$, φ does not satisfy (VA1).

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- Double phase problem

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- If $\frac{q}{p} < 1 + \frac{\beta}{n}$, φ satisfies (VA1) with $\omega(r) \lesssim r^\gamma$, $\gamma = \beta - \frac{n(q-p)}{p} > 0$, hence $u \in C^{1,\alpha}_{\text{loc}}$. (Colombo-Mingione (2015))

- If $\frac{q}{p} \ge 1 + \frac{\beta}{n}$, φ does not satisfy (VA1).

- However, if $\frac{q}{p} = 1 + \frac{\beta}{n}$, $u \in C^{1,\alpha}_{\text{loc}}$. (Baroni-Colombo-Mingione (2018))

Non-autonomous functional with Uhlenbeck's structure

u is a minimizer of

$$v \;\mapsto\; \int_\Omega \varphi(x, |Dv|)\, dx$$

if and only if it is a weak solution to

$$\operatorname{div}\left(\frac{\varphi'(x,|Du|)}{|Du|} Du\right) = 0. \qquad (4)$$

- If we regard u as a weak solution to (4) and do not return to a variational problem, (as far as we have checked) the approach used in the proof implies the same regularity results except for replacing the inequality in (VA1) by

$$(\varphi')^+_{B_r}(t) \le (1+\omega(r))(\varphi')^-_{B_r}(t) \quad {}^\forall t > 0 \text{ with } \varphi^-_{B_r}(t) \in [\omega(r), |B_r|^{-1}].$$

Note The above inequality is not comparable to the original one.

Non-autonomous functional with Uhlenbeck's structure

(wVA1): weak (VA1)

For any $\epsilon > 0$, there exists a non-decreasing, bounded, continuous function $\omega = \omega_\epsilon : [0, \infty) \to [0, 1]$ with $\omega(0) = 0$ such that for any small ball $B_r \Subset \Omega$,

$$\varphi^+_{B_r}(t) \le (1 + \omega(r))\varphi^-_{B_r}(t) + \omega(r) \quad \text{with} \quad \varphi^-_{B_r}(t) \in [\omega(r), |B_r|^{-1+\epsilon}].$$

- (VA1) $\implies$ (wVA1) $\implies$ (A1).

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- (VA1) $\implies$ (wVA1) $\implies$ (A1).
- $\varphi(x,t) = t^p + b(x)t^q$ ($b \in C^{0,\beta}$, $0 \le b(\cdot) \le L$)
  If $\frac{q}{p} \le 1 + \frac{\beta}{n}$, φ satisfies (wVA1) with $\omega_\epsilon(r) \lesssim r^{\gamma_\epsilon}$,
  $\gamma_\epsilon = \beta - \frac{n(1-\epsilon)(q-p)}{p} > 0$.
- The following inequality implies the inequality in (wVA1) (with different ω).

$$(\varphi')^+_{B_r}(t) \le (1+\omega(r))(\varphi')^-_{B_r}(t) + \omega(r) \quad \text{with} \quad \varphi^-_{B_r}(t) \in [\omega(r), |B_r|^{-1+\epsilon}]$$

Non-autonomous functional with Uhlenbeck's structure

Theorem (Hästö-Ok, to appear in JEMS)

Let $\varphi(x,\cdot) \in C^1([0,\infty))$ for every $x \in \Omega$ with φ' satisfying (A0), (Inc)$_{p-1}$ and (Dec)$_{q-1}$ for some $1 < p \le q$.

(1) *If φ satisfies (wVA1), then $u \in C^{\alpha}_{loc}(\Omega)$ for any $\alpha \in (0,1)$.*

(2) *If φ satisfies (wVA1) with $\omega_\epsilon(r) \lesssim r^{\beta_\epsilon}$ for some $\beta_\epsilon > 0$, then $u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$.*

- (As far as we have checked) The above theorem covers all previous regularity results with continuity assumptions (w.r.t. x) for special cases: standard growth case, $p(x)$-growth case, double phase case, ...
- (wVA1) can be replaced by the combination of (A1) and (wVA1) with fixed small ϵ that depends on the structure contants.

Examples (variable exponent type)

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$$\varphi(x,t) = t^{p(x)}$$

- $\lim_{r\to 0} \omega_p(r)\ln(1/r) = 0 \iff \varphi$ satisfies (VA1) $\implies u \in C^\alpha \ \ \forall \alpha.$
  (Acerbi-Mingione(2001))
- $\omega_p(r) \lesssim r^{\tilde{\beta}} \iff \varphi$ satisfies (VA1) with $\omega(r) \lesssim r^\beta \implies u \in C^{1,\alpha}.$
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- $\omega_p(r) \lesssim r^{\tilde{\beta}} \iff \varphi$ satisfies (VA1) with $\omega(r) \lesssim r^{\beta} \implies u \in C^{1,\alpha}.$
(Cosica-Mingione(1999))

$$\varphi(x,t) = t^{p(x)} + t^{q(x)}, \qquad p(\cdot) \le q(\cdot)$$

- $\lim_{r\to 0} \omega_p(r) = 0$, $\lim_{r\to 0} \omega_q(r) \ln(1/r) = 0 \implies \varphi$ satisfies (VA1)
$\implies u \in C^{\alpha} \quad \forall \alpha.$
- $\omega_p(r), \omega_q(r) \lesssim r^{\tilde{\beta}} \implies \varphi$ satisfies (VA1) with $\omega(r) \lesssim r^{\beta}.$
$\implies u \in C^{1,\alpha}.$

Examples (double phase type)

$\frac{\xi}{\psi}$ is almost increasing, $a(\cdot), b(\cdot) \in C^0$, $a(\cdot), b(\cdot) \geq 0$ and $a(\cdot) + b(\cdot) \approx 1$.

$$\varphi(x,t) := a(x)\psi(t) + b(x)\xi(t),$$

Define $\omega_\epsilon(r) := \omega_a(r) + \omega_b(r) r^{n(1-\epsilon)} \xi\big(\psi^{-1}(r^{-n(1-\epsilon)})\big), \quad \epsilon \in [0,1)$.

- $\lim_{r\to 0} \omega_\epsilon(r) = 0 \implies \varphi$ satisfies (wVA1). $\implies u \in C^\alpha \ \forall \alpha \in (0,1)$.
- Moreover, $\omega_\epsilon(r) \lesssim r^{\beta_\epsilon} \implies u \in C^{1,\alpha}$ for some $\alpha \in (0,1)$.

If ψ and ξ are general Orlicz functions,

- it is natural to assume that $b \in C^{\omega_b} \ \Leftrightarrow \ \sup_{x,y} \frac{|b(x)-b(y)|}{\omega_b(|x-y|)} < \infty \ \Leftrightarrow \ |b(x) - b(y)| \lesssim \omega_b(|x-y|)$.
- we can distinguish C^α-regularity for any $\alpha \in (0,1)$ and $C^{1,\alpha}$-regularity.

Proof(main steps)

Step 1. Higher integrability

There exists $\sigma_0 = \sigma_0(n,p,q,L) > 0$ and $c_1 = c_1(n,p,q,L) \geq 1$ such that

$$\left(\fint_{B_r} \varphi(x,|Du|)^{1+\sigma_0}\,dx \right)^{\frac{1}{1+\sigma_0}} \leq c_1 \left(\fint_{B_{2r}} \varphi(x,|Du|)\,dx + 1 \right)$$

for any $B_{2r} \Subset \Omega$ with $\int_{B_{2r}} \varphi(x,|Du|)\,dx \leq 1$. Hence $\varphi(\cdot,|Du|) \in L^{1+\sigma_0}_{\mathrm{loc}}(\Omega)$.

- We have the following reverse Hölder and Jensen type inequalities:

$$\fint_{B_r} \varphi(x,|Du|)\,dx \leq c_t \left[\left(\fint_{B_{2r}} \varphi(x,|Du|)^t\,dx \right)^{\frac{1}{t}} + 1 \right], \qquad t \in (0,1].$$

$$\fint_{B_r} \varphi(x,|Du|)\,dx \leq c\varphi^-_{B_{2r}} \left(\fint_{B_{2r}} |Du|\,dx + 1 \right).$$

Proof(main steps)

Step 2. Construction of a regular function

Let $B = B_{2r}(x_0)$, $t_1 := (\varphi_B^-)^{-1}(\omega(2r))$ and $t_2 := (\varphi_B^-)^{-1}(|B|^{-1})$.

We construct $\tilde{\varphi}$ s.t.

(1) $\tilde{\varphi} \in C^1([0,\infty)) \cap C^2((0,\infty))$ and $t\tilde{\varphi}''(t) \approx \tilde{\varphi}'(t)$.

(2) $0 \le \tilde{\varphi}(t) - \varphi(x_0,t) \le c(r\varphi_B^-(t) + \omega(2r))$, $\forall t \in [t_1, t_2]$.

(3) $\theta_0(x,t) := \varphi(x, \tilde{\varphi}^{-1}(t))$ satisfies (A0), (aInc)$_1$, (aDec)$_{q/p}$ and (A1).

$$\psi_B(t) := \begin{cases} a_1\left(\frac{t}{t_1}\right)^{p-1} & \text{if } 0 \le t < t_1, \\ \varphi'(x_0,t) & \text{if } t_1 \le t \le t_2, \\ a_2\left(\frac{t}{t_2}\right)^{p-1} & \text{if } t_2 < t < \infty, \end{cases} \qquad \varphi_B(t) := \int_0^t \psi_B(s)\,ds,$$

where $a_1 = \varphi'(x_0,t_1)$ and $a_2 = \varphi'(x_0,t_2)$, and

$$\tilde{\varphi}(t) := \int_0^\infty \varphi_B(t\sigma)\eta_r(\sigma-1)\,d\sigma = \int_1^{1+r} \varphi_B(t\sigma)\eta_r(\sigma-1)\,d\sigma,$$

$\eta \in C_0^\infty(\mathbb{R}^+)$ with $\mathrm{supp}\,\eta \subset (0,1)$ and $\|\eta\|_1 = 1$, and $\eta_r(t) := \frac{1}{r}\eta(\frac{t}{r})$.

Proof(main steps)

Step 3. Regularity results for the regularized problem

Let $v \in W^{1,\tilde{\varphi}}(B_r)$ be the minimizer of the functional

$$u + W_0^{1,\tilde{\varphi}}(B_r) \ni v \mapsto \int_{B_r} \tilde{\varphi}(|Dv|)\, dx,$$

equivalently, v is a weak solution to

$$\operatorname{div}\left(\frac{\tilde{\varphi}'(|Dv|)}{|Dv|} Dv\right) = 0 \ \text{ in } \ B_r, \quad v = u \ \text{ on } \ \partial B_r.$$

Proof(main steps)

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$C^{1,\alpha}$-regularity

$u \in C^{1,\alpha}_{\mathrm{loc}}(B_r)$ for some $\alpha \in (0,1)$. For any $B_\rho(y) \subset B_r$

$$\sup_{B_{\rho/2}(y)} |Dv| \le c \fint_{B_\rho(y)} |Dv|\, dx$$

and, for any $\tau \in (0,1)$,

$$\fint_{B_{\tau\rho}(y)} |Dv - (Dv)_{B_{\tau\rho}(y)}|\, dx \le c\tau^\alpha \fint_{B_\rho(y)} |Dv|\, dx.$$

Proof(main steps)

Calderón-Zygmund type estimates

Suppose $\theta = \theta(x,t)$ satisfies (A0), (aInc)$_{p_1}$, (aDec)$_{q_1}$ and (A1) for some $1 < p_1 < q_1$. Then

$$\|\tilde{\varphi}(|Dv|)\|_{L^\theta(B_r)} \le c\|\tilde{\varphi}(|Du|)\|_{L^\theta(B_r)}.$$

This implies that

$$\fint_{B_r} \theta(x, \tilde{\varphi}(|Dv|))\,dx \le c\left(\fint_{B_r} \theta(x, \tilde{\varphi}(|Du|))\,dx + 1\right),$$

provided that $\int_{B_r} \theta(x, \tilde{\varphi}(|Du|))\,dx \le 1$.

Set $\theta(x,t) = \theta_0(x,t)^{1+\sigma} = \varphi(x, \tilde{\varphi}^{-1}(t))^{1+\sigma}$.

$$\begin{aligned}\fint_{B_r} \varphi(x, |Dv|)^{1+\sigma}\,dx &= \fint_{B_r} \theta(x, \tilde{\varphi}(|Dv|))\,dx \\ &\le c\left(\fint_{B_r} \theta(x, \tilde{\varphi}(|Du|))\,dx + 1\right) = c\left(\fint_{B_r} \varphi(x, |Du|)^{1+\sigma}\,dx + 1\right).\end{aligned}$$

In particular, $v \in W^{1,\varphi}(B_r)$ and $\varphi(\cdot, |Dv|) \in L^{1+\sigma}(B_r)$.

Proof(main steps)

extrapolation

If

$$\|f\|_{L^p_w(B_r)} \le c([w]_p)\|g\|_{L^p_w(B_r)}$$

for some $1 < p < \infty$ and for all weight $w \in A_p$, then

$$\|f\|_{L^\theta(B_r)} \le c\|g\|_{L^\theta(B_r)}$$

for any $\theta = \theta(x,t)$ satisfying (A0), (aInc)$_{p_1}$, (aDec)$_{q_1}$ and (A1).

- $f = \tilde{\varphi}(|Dv|)$ and $g = \tilde{\varphi}(|Du|)$.

Proof(main steps)

Step 4. Approximating estimate

- Use that

$$\int_{B_r} \varphi(x,|Du|)\,dx \le \int_{B_r} \varphi(x,|Dv|)\,dx, \quad \int_{B_r} \tilde{\varphi}(|Dv|)\,dx \le \int_{B_r} \tilde{\varphi}(|Du|)\,dx$$

- Separate $B_{r/2}$ into the three regions:

$$\{|Du| \le t_1\}, \qquad \{t_1 < |Du| \le t_2\}, \qquad \{|Du| > t_2\},$$

  where $t_1 := (\varphi^-_{B_{2r}})^{-1}(\omega(2r))$ and $t_2 := (\varphi^-_{B_{2r}})^{-1}(|B_{2r}|^{-1})$, and estimate integrals over the above regions independently.

- By applying reverse type estimates, we obtain L^1 comparison estimate.

$$\fint_{B_{r/2}} |Du - Dv|\,dx \lesssim \tilde{\omega}(r) \fint_{B_{2r}} |Du|\,dx,$$

where $\tilde{\omega}(r) = (\omega(r) + r)^{(\text{power})}$.

Proof(main steps)

Step 5. Iteration

By standard iteration arguments with B_{r_j} with $r_j = 2 \cdot 4^{-j} r$, we obtain

Morrey type estimate

For each $\alpha \in (0,1)$

$$\int_{B_r} |Du|\, dx \lesssim r^{n-1+\alpha} \quad \text{for any small ball } B_r,$$

which imply $u \in C^{\alpha}_{\mathrm{loc}}$.

Campanato type estimate

Suppose $\omega_\epsilon(r) \lesssim r^{\beta_\epsilon}$. For some $\alpha \in (0,1)$

$$\int_{B_r} |Du - (Du)_{B_r}|\, dx \lesssim r^{n+\alpha} \quad \text{for any small ball } B_r$$

which implies $u \in C^{1,\alpha}_{\mathrm{loc}}$.

Hölder continuous or bounded minimizers

Hölder continuous or bounded minimizers

(VA1-s) : vanishing (A1-s)

Let $s > 0$. There exists a non-decreasing, bounded, continuous function $\omega : [0, \infty) \to [0, 1]$ with $\omega(0) = 0$ such that for any small ball $B_r \Subset \Omega$,

$$\varphi^+_{B_r}(t) \leq (1 + \omega(r))\varphi^-_{B_r}(t) + \omega(r) \quad \text{for all} \quad t^s \in [\omega(r), |B_r|^{-1}].$$

(wVA1-s) : weak (VA1-s)

Let $s > 0$. For any small $\epsilon > 0$, there exists a non-decreasing, bounded, continuous function $\omega = \omega_\epsilon : [0, \infty) \to [0, 1]$ with $\omega(0) = 0$ such that for any small ball $B_r \Subset \Omega$,

$$\varphi^+_{B_r}(t) \leq (1 + \omega(r))\varphi^-_{B_r}(t) + \omega(r) \quad \text{for all} \quad t^s \in [\omega(r), |B_r|^{-1+\epsilon}].$$

- If φ satisfies (wVA1-s), then it does (VA1-$\tilde{s}$) for any $\tilde{s} > s$.

Hölder continuous or bounded minimizers

Theorem (Hästö-Ok, in preparation)

(1) *If* φ *satisfies* *(VA1-$\frac{n}{1-\gamma'}$)* *and* $u \in C^{\gamma}(\Omega)$ *for some* $0 < \gamma' < \gamma < 1$, *then* $u \in C^{\alpha}_{loc}(\Omega)$ *for any* $\alpha \in (0, 1)$.

(2) *If* φ *satisfies* *(VA1-$\frac{n}{1-\gamma'}$)* *with* $\omega(r) \lesssim r^{\delta}$ *and* $u \in C^{\gamma}(\Omega)$ *for some* $0 < \gamma' < \gamma < 1$ *and* $\delta > 0$, *then* $u \in C^{1,\alpha}_{loc}(\Omega)$ *for some* $\alpha \in (0, 1)$.

- $\varphi(x, t) = t^p + b(x)t^q$ with $b \in C^{0,\beta}$.
  Baroni-Colombo-Mingione (2018) prove that if

$$q < p + \frac{\beta}{1-\gamma} \quad \text{with} \quad \gamma \in (0, 1) \tag{5}$$

  and $u \in C^{\gamma}(\Omega)$, then $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$.

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Theorem (Hästö-Ok, in preparation)

(1) *If φ satisfies (VA1-$\frac{n}{1-\gamma'}$) and $u \in C^{\gamma}(\Omega)$ for some $0 < \gamma' < \gamma < 1$, then $u \in C^{\alpha}_{loc}(\Omega)$ for any $\alpha \in (0,1)$.*

(2) *If φ satisfies (VA1-$\frac{n}{1-\gamma'}$) with $\omega(r) \lesssim r^{\delta}$ and $u \in C^{\gamma}(\Omega)$ for some $0 < \gamma' < \gamma < 1$ and $\delta > 0$, then $u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$.*

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  Note that (5) implies that φ satisfies (VA1-$\frac{n}{1-\gamma'}$) with $\omega(r) \lesssim r^{\delta}$, where $\gamma' \in (0,\gamma)$ is chosen to satisfy

$$\delta := \beta - (q-p)(1-\gamma') = \frac{\beta - (q-p)(1-\gamma)}{2} > 0.$$

Hölder continuous or bounded minimizers

Theorem (Hästö-Ok, in preparation)

(1) *If φ satisfies (wVA1-n) and $u \in L^\infty(\Omega)$, then $u \in C^\alpha_{loc}(\Omega)$ for any $\alpha \in (0,1)$.*

(2) *If φ satisfies (wVA1-n) with $\omega_\epsilon(r) \lesssim r^{\beta_\epsilon}$ for some $\beta_\epsilon > 0$ and $u \in L^\infty(\Omega)$, then $u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$.*

- This is a corollary of the preceding theorem.

$$\text{(wVA1-}n\text{)} \quad \Rightarrow \quad \left\{ \begin{array}{c} \text{(A1-}n\text{)} \;\Rightarrow\; u \in C^\gamma \text{ for some } \gamma \in (0,1) \\ \text{(VA1-}\gamma'\text{) for any small } \gamma' > 0 \end{array} \right\}$$

Hölder continuous or bounded minimizers

- $\varphi(x,t) = t^p + b(x)t^q$ with $b \in C^{0,\beta}$.
  Colombo-Mingione (2015) prove that if

$$q \le p + \beta \tag{6}$$

  and $u \in L^\infty(\Omega)$, then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.

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  $$\delta_\epsilon = \beta - (q-p)(1-\epsilon) > 0.$$

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  Note that (6) implies that φ satisfies (wVA1-n) with $\omega_\epsilon(r) \lesssim r^{\delta_\epsilon}$, where
  $$\delta_\epsilon = \beta - (q-p)(1-\epsilon) > 0.$$

Remark In the bounded or Hölder continuous minimizer case, we cannot take advantage of the extrapolation.

Functional with generalized Orlicz growth

$$W^{1,\varphi}(\Omega) \ni v \quad \mapsto \quad \int_\Omega f(x, Dv)\, dx.$$

$$\begin{cases} z \mapsto f(x,z) \text{ is } C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}), \\ \varphi \in C^1([0,\infty)) \text{ and } \varphi' \text{ satisfies (A0), (Inc)}_{p-1} \text{ and (Dec)}_{q-1}. \\ \nu\varphi(x,|z|) \le f(x,z) \le L\varphi(x,|z|), \\ \nu\frac{\varphi'(x,|z|)}{|z|}|\lambda|^2 \le \partial^2 f(x,z)\lambda\cdot\lambda \le L\frac{\varphi'(x,|z|)}{|z|}|\lambda|^2. \end{cases}$$

(AF): version of (wVA1) in the functional setting

For any $\epsilon > 0$, there exists a non-decreasing, bounded, continuous function $\omega = \omega_\epsilon : [0,\infty) \to [0,1]$ with $\omega(0) = 0$ such that for any small ball $B_r \Subset \Omega$,

$$f^+_{B_r}(z) \le (1 + \omega(r)) f^-_{B_r}(z) + \omega(r)$$

for all $z \in \mathbb{R}^n$ with $\varphi^-_{B_r}(|z|) \in [\omega(r), |B_r|^{-1+\epsilon}]$.

Functional with generalized Orlicz growth

Theorem (Hästö-Ok, in preparation)

(1) *If* f *satisfies (AF), then* $u \in C^{\alpha}_{loc}(\Omega)$ *for any* $\alpha \in (0,1)$.

(2) *If* f *satisfies (AF) with* $\omega_\epsilon(r) \lesssim r^{\beta_\epsilon}$ *for some* $\beta_\epsilon > 0$*, then* $u \in C^{1,\alpha}_{loc}(\Omega)$ *for some* $\alpha \in (0,1)$.

PDE with generalized Orlicz growth

$$\operatorname{div} A(x, Du) = 0.$$

$$\begin{cases} z \mapsto A(x,z) \in \mathbb{R}^n \text{ is } C^1(\mathbb{R}^n \setminus \{0\}), \\ \varphi \in C^1([0,\infty)) \text{ and } \varphi' \text{ satisfies (A0), (Inc)}_{p-1} \text{ and (Dec)}_{q-1}. \\ |A(x,|z|)| + |z||\partial A(x,z)| \le L\varphi'(x,|z|), \\ \nu \frac{\varphi'(x,|z|)}{|z|}|\lambda|^2 \le \partial A(x,z)\lambda \cdot \lambda. \end{cases}$$

(AP): version of (wVA1) in PDE setting

For any $\epsilon > 0$, there exists a non-decreasing, bounded, continuous function $\omega = \omega_\epsilon : [0,\infty) \to [0,1]$ with $\omega(0) = 0$ such that for any small ball $B_r \Subset \Omega$,

$$|A(x,z) - A(y,z)| \le \omega(r)((\varphi')^-_{B_r}(|z|) + 1)$$

for all $x, y \in B_r$ and for all $z \in \mathbb{R}^n$ with $\varphi^-_{B_r}(|z|) \in [\omega(r), |B_r|^{-1+\epsilon}]$.

PDE with generalized Orlicz growth

Theorem (Hästö-Ok, in preparation)

(1) *If A satisfies (AP), then $u \in C^{\alpha}_{loc}(\Omega)$ for any $\alpha \in (0,1)$.*

(2) *If A satisfies (AP) with $\omega_\epsilon(r) \lesssim r^{\beta_\epsilon}$ for some $\beta_\epsilon > 0$, then $u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$.*

Remark In the general functional or PDE case we have to construct $\tilde{f}(z)$ or $\tilde{A}(z)$ with $\tilde{\varphi}$-growth, where $\tilde{\varphi} = \tilde{\varphi}(t)$ is the regular function.

THANK YOU.