Symmetrization on fully anisotropic elliptic equations

Giuseppina di Blasio

Dipartimento di Matematica e Fisica,
Università degli Studi della Campania "L. Vanvitelli" - Caserta - Italy
giuseppina.diblasio@unicampania.it

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Joint project with *A. Alberico* and *F. Feo*

Introduction

Nonlinear equations with general growth

Including:

- Non-power type growth
- Anisotropic growth (i.e. not depending just on $|\nabla u|$)
- No Δ_2-condition
- Non-reflexive Sobolev type spaces

Nonlinearities governed by an arbitrary n-dimensional Young function

Introduction

Let Ω be an open bounded set of $\mathbb{R}^n$, $n \geqslant 2$

$$(*) \qquad \begin{cases} -\text{div}\,(a(x,u,\nabla u)) + b(u) = f\,(x) - \text{div}\,(g(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

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$$a(x,\eta,\xi) \cdot \xi \geqslant \Phi(\xi) \quad \forall (\eta,\xi) \in \mathbb{R} \times \mathbb{R}^n,$$

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Φ: $\mathbb{R}^n \to [0,+\infty)$ n-dimensional Young function

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2. $b : \mathbb{R} \to \mathbb{R}$,$f : \Omega \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^n$ satisfy suitable assumptions

Introduction

- Problem: to obtain sharp estimates and regularity results for solutions u of

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in terms of the sources f and g, using symmetrization methods.

- Solution: to use symmetrization techniques to compare the solution to anisotropic problem $(*)$ with the solution to a suitable isotropic symmetric problem defined in a ball $\Omega^\star$ of the same measure as Ω.

Rearrangements

Let u be a measurable function (continued by 0 outside its domain Ω) fulfilling

$$|\{x \in \mathbb{R}^n : |u(x)| > t\}| < +\infty \quad \text{for every } t > 0$$

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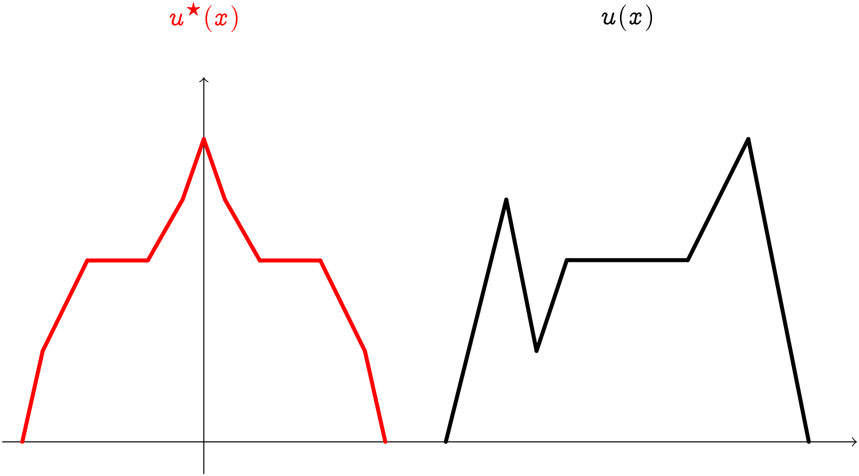
- The *spherical decreasing rearrangement* of u is defined by

$$u^\star(x) = u^*(\omega_n |x|^n), \qquad x \in \mathbb{R}^n,$$

where ω_n denote the volume of the unit ball in $\mathbb{R}^n$.

Rearrangements



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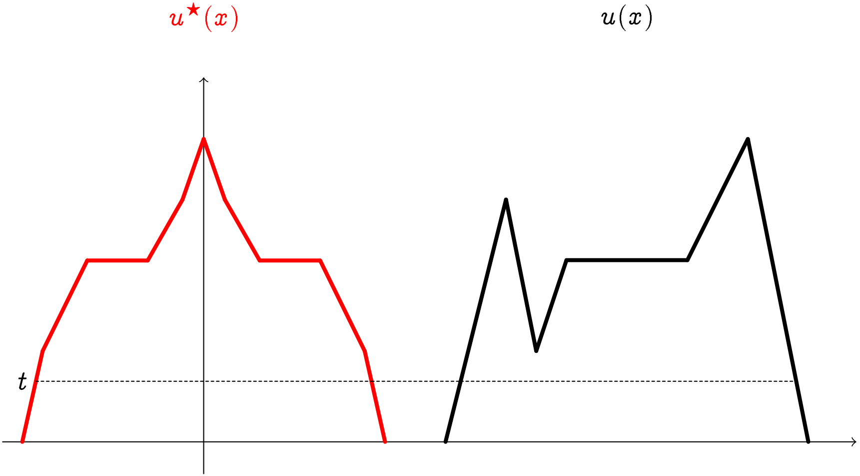

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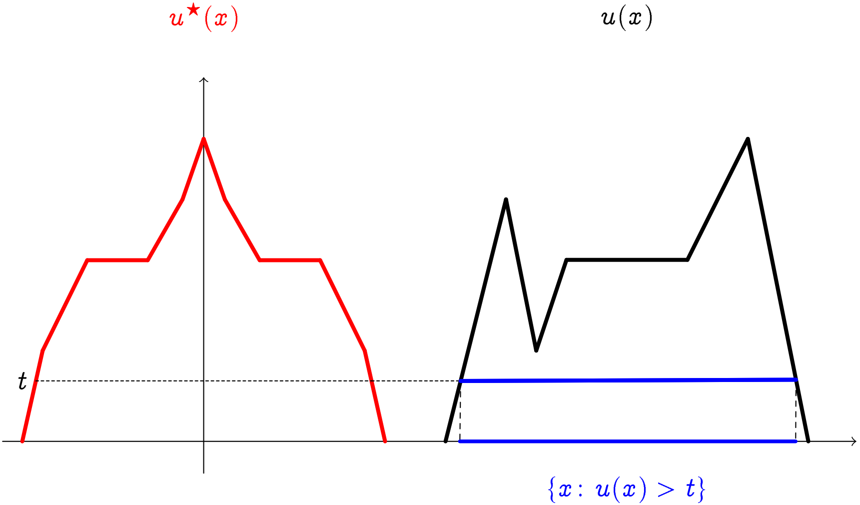

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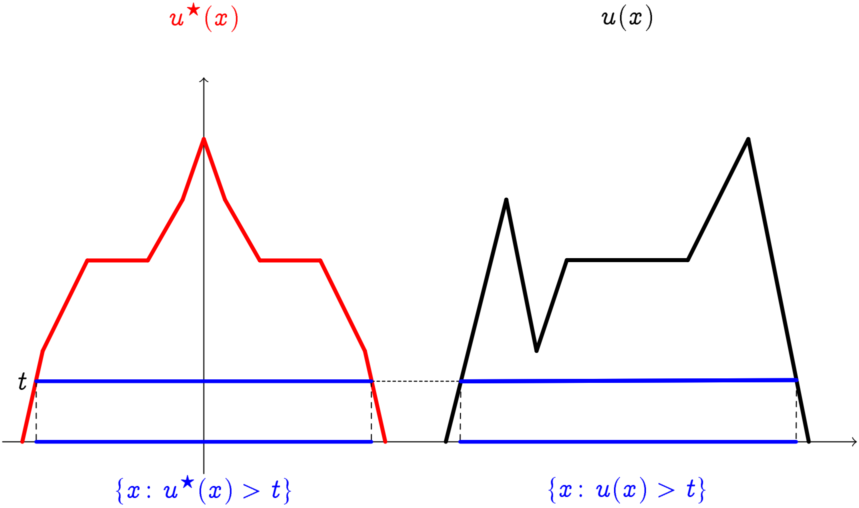

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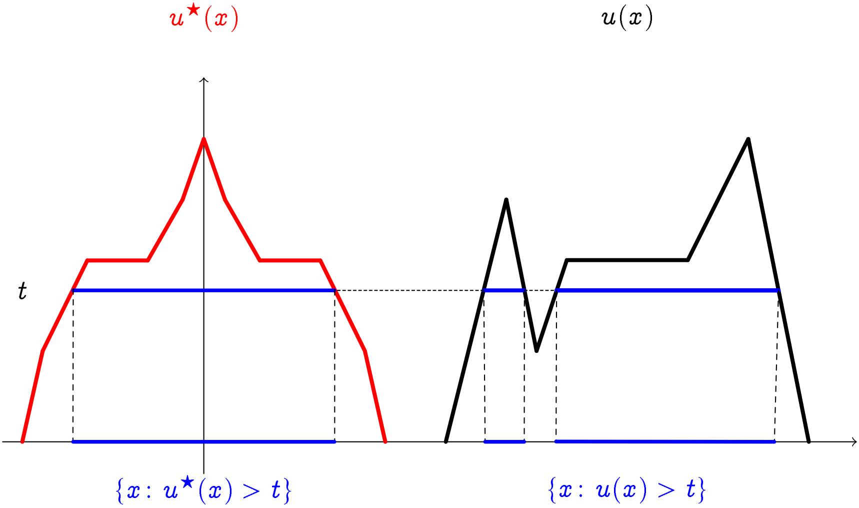

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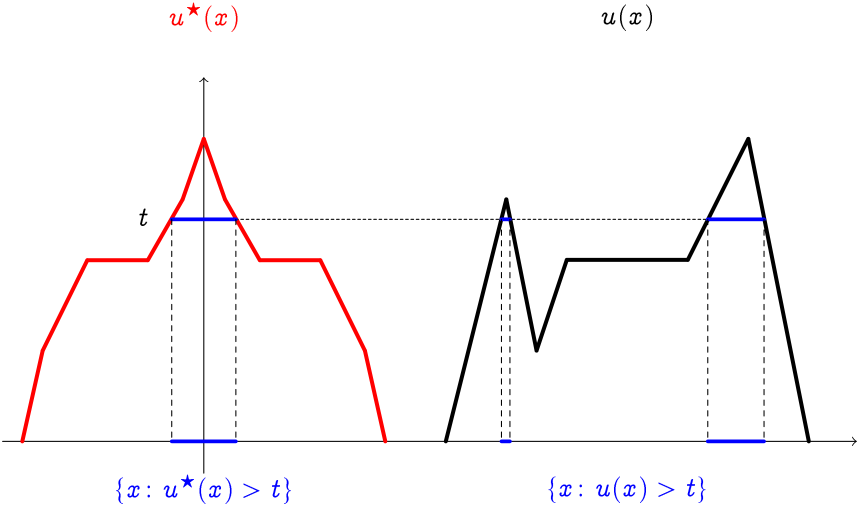

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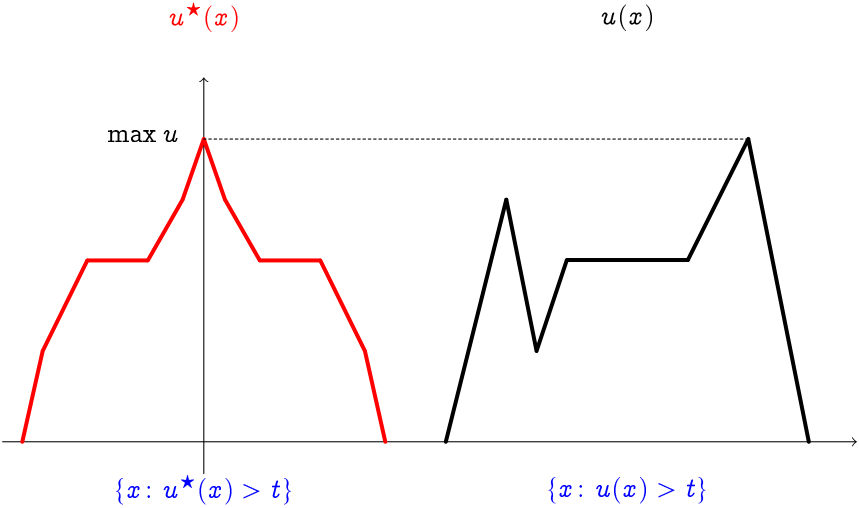

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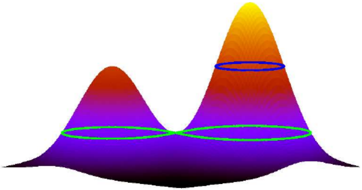

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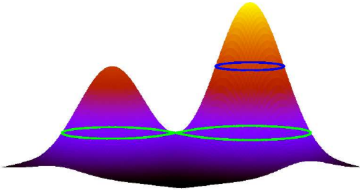

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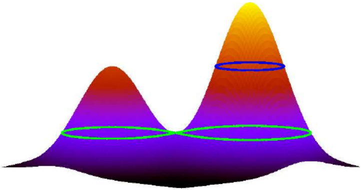

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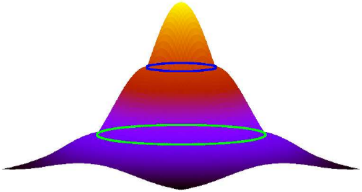

$u^\star$

We have that

$$\mu_u(t) = \mu_{u^*}(t) = \mu_{u^\star}(t).$$

Moreover the level set $\{x \in \Omega^\star : u^\star(x) > t\}$ is a ball centered in 0 whose measure is $\mu_u(t)$

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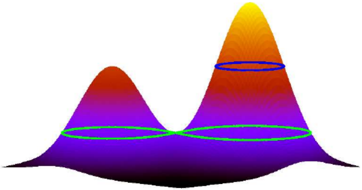

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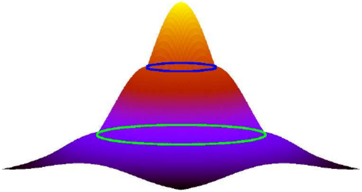

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Symmetric increasing rearrangement of h:

$$h_\star(x) = \sup\{t > 0 : |\{x \in \mathbb{R}^n : |h(x)| < t\}| < \omega_n |x|^n\}, \qquad x \in \mathbb{R}^n$$

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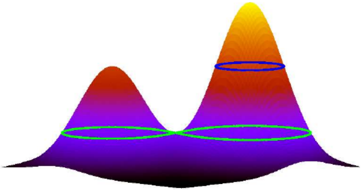

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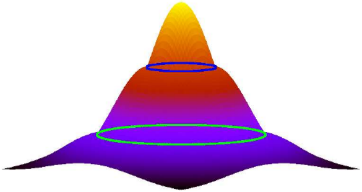

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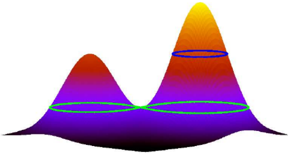

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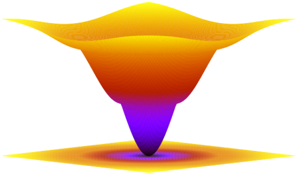

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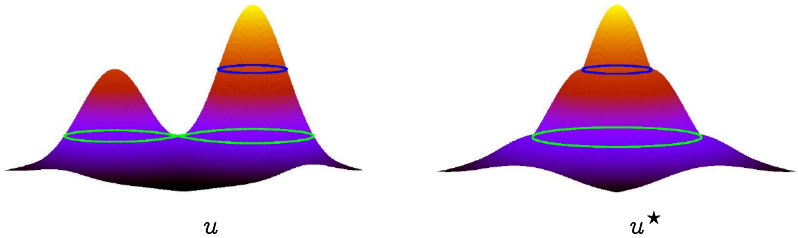

The rearrangement preserves the L^p norms:

If $1 \leqslant p \leqslant +\infty$,

$$\|u^*\|_{L^p(0,|\Omega|)} = \|u^\star\|_{L^p(\Omega^\sharp)} = \|u\|_{L^p(\Omega)}$$

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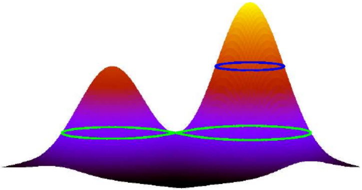

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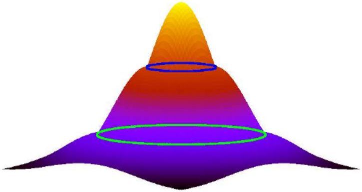

$u^\star$

Hardy-Littlewood inequality:

If $u, v : \Omega \to \mathbb{R}$ are measurable, then

$$\int_\Omega |uv|\, dx \leqslant \int_{\Omega^\star} u^\star v^\star\, dx$$

- Symmetrization methods in a priori estimates for solution to isotropic elliptic equations are well known

   Maz'ya 69, Talenti 76,...

Talenti's Theorem

Let Ω be a open bounded set of $\mathbb{R}^n$, $n > 2$ and $f \in L^{\frac{2n}{n+2}}(\Omega)$

$$(**) \qquad \begin{cases} -\Delta u = f & in\ \Omega \\ \\ u = 0 & on\ \partial\Omega, \end{cases}$$

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Question: When does problem $(**)$ have the "largest" solution under the assumptions that the rearrangement of f and $|\Omega|$ are fixed?

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Then

$$u^\star(x) \leqslant v(x), \quad \forall x \in \Omega^\star$$

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Remark

The above inequality basically says that the functional over the sets Ω of fixed measure

$$\sup_f \frac{\|u\|_{L^q}}{\|f\|_{L^p}}$$

attains the maximum value when Ω is a ball.

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A priori estimate

- The comparison result is the starting point to obtain sharp a priori estimate

$$u^\star(x) \leqslant v(x) = \frac{1}{n^2\omega_n^{2/n}} \int_{\omega_n|x|^n}^{\Omega} r^{-2+2/n} \int_0^r f^*(s)\,ds\,dr$$

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$$(**) \quad \begin{cases} -\displaystyle\sum_{i,j=1}^{n} (a_{ij}(x) u_{x_j})_{x_i} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta v = f^\star & \text{in } \Omega^\star \\ v = 0 & \text{on } \partial\Omega^\star. \end{cases}$$

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$$\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geqslant |\xi|^2.$$

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- Talenti, 1976-1979
- Alvino-L.Lions-Trombetti, 1990
- Ferone-Posteraro, 1991
- Betta-Ferone-Mercaldo 1994

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where

$$\sum_{i=1}^{n} a_i(\xi)\xi_i \geqslant F^2(\xi)$$

with F a sufficiently smooth nonnegative convex function, positively homogeneous of degree 1.

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where

$$\sum_{i=1}^{n} a_i(\xi)\xi_i \geqslant F^2(\xi)$$

with F a sufficiently smooth nonnegative convex function, positively homogeneous of degree 1. Then

$$u^\bullet(x) \leqslant w(x) \quad \forall x \in \Omega^\bullet,$$

where $u^\bullet(x) = u^*(\kappa_n F^\circ(x)^n)$ and $\Omega^\bullet$ is a set homothetic to the Wulff shape $\mathcal{W} = \{\xi \in \mathbb{R}^n : F^\circ(\xi) < 1\}$ having the same measure of Ω.

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- Symmetrization methods in a priori estimates for solution to fully anisotropic elliptic equations are treat in

  Cianchi 2007, Alberico-Cianchi 2008, Alberico 2011, Aberico-di B.-Feo 2017-2019, Barletta-Cianchi 2017, Alberico-Chlebicka-Cianchi-Zatorska-Goldstein 2019 ...

Construction of symmetrized problem

$$\Omega \mapsto \Omega^\star$$

$\Omega^\star$ is a ball of $\mathbb{R}^n$ centered at the origin s.t. $|\Omega^\star| = |\Omega|$

Construction of symmetrized problem

$$\Omega \mapsto \Omega^\star$$

$$f(x) \mapsto f^\star(x)$$

$f^\star(x)$ is the symmetric decreasing rearrangement of f

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$$\operatorname{div}\left(a(x, u, \nabla u)\right) \mapsto ?$$

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$$\Omega \mapsto \Omega^\star$$

$$f(x) \mapsto f^\star(x)$$

$$\text{div}\,(a(x,u,\nabla u)) \mapsto ?$$

Key tools:

- $\Phi_\blacklozenge(|\xi|)$ is a suitable symmetrization of $\Phi(\xi)$
- $\Phi_\bullet(\xi)$ is the *Young conjugate* of $\Phi(\xi)$

Klimov symmetrization of Φ

Let Φ be a n-dimensional Young function, that means

$$\text{a convex, even function s.t. } \Phi(0) = 0 \text{ and } \lim_{|\xi| \to +\infty} \Phi(\xi) = +\infty$$

Klimov symmetrization of Φ

Let Φ be a n-dimensional Young function, we consider

$\Phi_\blacklozenge \colon \mathbb{R} \to [0, +\infty)$ defined as

$$\Phi_\blacklozenge(|\xi|) = \Phi_{\bullet\star\bullet}(\xi) \qquad \text{with } \xi \in \mathbb{R}^n,$$

Klimov, 1974

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$$\Phi_\bullet(\eta) = \sup\{\xi \cdot \eta - \Phi(\xi) : \xi \in \mathbb{R}^n\} \quad \text{with } \eta \in \mathbb{R}^n.$$

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$\Phi_\bullet$ is a n-dimensional Young function if $\displaystyle\lim_{|\xi| \to +\infty} \frac{\Phi(\xi)}{|\xi|} = +\infty$

Klimov, 1974

Klimov symmetrization of Φ

Let Φ be a n-dimensional Young function, we consider

$\Phi_\blacklozenge \colon \mathbb{R} \to [0, +\infty)$ defined as

$$\Phi_\blacklozenge(|\xi|) = \Phi_{\bullet\star\bullet}(\xi) \qquad \text{with } \xi \in \mathbb{R}^n,$$

where $\Phi_\star : \mathbb{R}^n \to [0, +\infty)$ is the symmetric increasing rearrangement of Φ and $\Phi_\bullet \colon \mathbb{R}^n \to [0, +\infty)$ is the Young conjugate of Φ defined as

$$\Phi_\bullet(\eta) = \sup\{\xi \cdot \eta - \Phi(\xi) : \xi \in \mathbb{R}^n\} \quad \text{with } \eta \in \mathbb{R}^n.$$

$\Phi_\bullet$ is a n-dimensional Young function if $\displaystyle\lim_{|\xi|\to+\infty} \frac{\Phi(\xi)}{|\xi|} = +\infty$

$\Phi_\blacklozenge$ is a 1-dimensional Young function.

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$\Phi_\blacklozenge$ and $\Phi_\star$ are not equal in general, unless Φ is radial.

Klimov, 1974

Comparison result: simpler case

Let Ω be an open bounded set of $\mathbb{R}^n$, $n \geqslant 2$

$$(P) \qquad \begin{cases} -\text{div}\,(a(x,u,\nabla u)) + b(u) = f\,(x)\, - \text{div}\,(g(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where

$$a(x,\eta,\xi)\cdot\xi \geqslant \Phi(\xi) \quad \forall(\eta,\xi) \in \mathbb{R}\times\mathbb{R}^n.$$

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Final aim: A pointwise estimate of a weak solution to the anisotropic problem (P) in term of the solution to a suitable isotropic problem.

Weak solution

Let us consider the space

$$V_0^{1,\Phi}(\Omega) = \Big\{ u : u \text{ real-valued funct. in } \Omega \text{ whose continuation by } 0 \text{ outside } \Omega \text{ is weakly diff. in } \mathbb{R}^n \text{ and } \int_\Omega \Phi(\nabla u)\,dx < +\infty \Big\}.$$

Definition

A function $u \in V_0^{1,\Phi}(\Omega)$ is called a weak solution to (P) if

$$\int_\Omega a(x,u,\nabla u)\cdot \nabla\varphi\,dx = \int_\Omega f\varphi\,dx \quad \forall \varphi \in V_0^{1,\Phi}(\Omega).$$

Cianchi's result

Let $u \in V_0^{1,\Phi}$ be a weak solution to problem (P), $s^{1/n} f^{**}(s) \in L^{\Phi_\blacklozenge \bullet}$ and let v be the spherically symmetric solution to the following symmetrized problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\Phi_\blacklozenge(|\nabla v|)}{|\nabla v|^2} \nabla v \right) = f^\star(x) & \text{in} \quad \Omega^\star \\ v = 0 & \text{on} \quad \partial\Omega^\star \end{cases}$$

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Then

$$u^\star(x) \leqslant v(x) \qquad \text{for } x \in \Omega^\star,$$

Moreover

$$\int_\Omega \Phi(\nabla u)\, dx \leqslant \int_{\Omega^\star} \Phi_\blacklozenge(|\nabla v|)\, dx.$$

Cianchi, 2007.

A possible prototype

$$\Phi(\xi) = \sum_{i=1}^{n} \alpha_i |\xi_i|^{p_i} \quad \text{with} \quad \alpha_i > 0, p_i > 1$$

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$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \left(\alpha_i \left| u_{x_i} \right|^{p_i - 2} u_{x_i} \right)_{x_i} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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$$\Phi(\xi) = \sum_{i=1}^{n} \alpha_i |\xi_i|^{p_i} \quad \Longrightarrow \quad \Phi_\blacklozenge(|\xi|) = \Lambda |\xi|^{\overline{p}} \quad \text{for suitable } \Lambda \in \mathbb{R}$$

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where $\overline{p}$ is the harmonic mean of the exponents $p_1, \dots, p_n$ and $f \in L^{(\overline{p}^*)', \overline{p}'}(\Omega)$.

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Skech of proof: Fixed t and $\kappa > 0$. We consider the following function

$$u_{\kappa,t}(x) = \begin{cases} 0 & \text{if } |u(x)| \leqslant t, \\ (|u(x)| - t)\,\mathrm{sign}(u(x)) & \text{if } t < |u(x)| \leqslant t + \kappa \\ \kappa\,\mathrm{sign}(u(x)) & \text{if } t + \kappa < |u(x)|, \end{cases}$$

as test function in the weak formulation.

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$$\int_{t<|u|<t+\kappa} \Phi(\nabla u)\,dx \geqslant \int_{t<u^\star<t+\kappa} \Phi_\blacklozenge(|\nabla u^\star|)\,dx$$

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Letting $\kappa \to 0^+$, we obtain

$$-\frac{d}{dt}\int_{u^{\star}>t} \Phi_{\blacklozenge}\left(\left|\nabla u^{\star}\right|\right) dx \leqslant \int_0^{\mu_u(t)} f^*(s)\, ds.$$

- An application of Jensen's inequality
- $\mu_{u^\star} = \mu_u$

$$\Phi_\blacklozenge\left(\frac{\frac{1}{\kappa}\int_{\{t<u^\star<t+\kappa\}} |\nabla u^\star| \, dx}{\frac{\mu_u(t)-\mu_u(t+\kappa)}{\kappa}}\right) \leqslant \frac{\frac{1}{\kappa}\int_{\{t<u^\star<t+\kappa\}} \Phi_\blacklozenge\left(|\nabla u^\star|\right) dx}{\frac{\mu_u(t)-\mu_u(t+\kappa)}{\kappa}}$$

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- By Coarea formula

$$\int_{\{t<u^\star<t+\kappa\}}|\nabla u^\star|dx=\int_t^{t+\kappa}P(\{u^\star>\tau\})d\tau=\int_t^{t+\kappa}n\omega_n^{1/n}\mu_u(r)^{\frac{1}{n'}}dr$$

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Letting $\kappa \to 0^+$, we get for a.e. $t > 0$

$$1 \leqslant \frac{-\mu_u'(t)}{n\omega_n^{1/n}\left(\mu_u(t)\right)^{1/n'}}\Psi_\blacklozenge^{-1}\left(\frac{-\frac{d}{dt}\int_{u^\star>t}\Phi_\blacklozenge\left(|\nabla u^\star|\right)dx}{n\omega_n^{1/n}\left(\mu_u(t)\right)^{1/n'}}\right)$$

where $\Psi_\blacklozenge(s) = \dfrac{\Phi_\blacklozenge(s)}{s}, \quad s > 0.$

Combing the inequalities

$$-\frac{d}{dt}\int_{u^{\star}>t}\Phi_{\blacklozenge}\left(\left|\nabla u^{\star}\right|\right)dx \leqslant \int_0^{\mu_u(t)} f^{*}(s)\,ds \tag{1}$$

Combing the inequalities

(1) $$-\frac{d}{dt}\int_{u^\star > t} \Phi_\blacklozenge\left(\left|\nabla u^\star\right|\right)dx \leqslant \int_0^{\mu_u(t)} f^*(s)\,ds$$

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that is

$$u^\star(x) \leqslant v(x) = \int_{\omega_n|x|^n}^{|\Omega|}\frac{1}{n\omega_n^{1/n}r^{1/n'}}\Psi_\blacklozenge^{-1}\left(\frac{\int_o^r f^*(\sigma)d\sigma}{r^{1/n'}n\omega_n^{1/n}}\right)dr$$

Generalization 1

Let Ω be an open bounded set of $\mathbb{R}^n$, $n \geqslant 2$

$$(P_g) \qquad \begin{cases} -\text{div}\,(a(x,u,\nabla u)) = f\,(x) - \text{div}\,(g(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

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$$\text{div}\,(g(x)) \mapsto ?$$

Pseudo-rearrangement

To say that $G : (0, |\Omega|) \to \mathbb{R}$ is a pseudo-rearrangement of $h \in L^1(\Omega)$ with respect to the measurable function u means, easily speaking, that

$$G(s) = \frac{d}{ds} \int_{\{|u(x)| > u^*(s)\}} h(x)\, dx \qquad \text{for } s \in (0, |\Omega|).$$

We will use it with $h = \Phi_\bullet(c\, g)$ for a positive constant c.

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We will use it with $h = \Phi_\bullet(c\, g)$ for a positive constant c.

$$\Phi_\bullet(c\, g) \in L^p(\Omega) \quad \Rightarrow \quad G \in L^p(0, |\Omega|)$$

$$\|G\|_{L^p(0,|\Omega|)} \leqslant \|\Phi_\bullet(c\, g)\|_{L^p(\Omega)} \text{ for } 1 < p \leqslant +\infty$$

The same property holds for norm in rearrangement invariant spaces.

Comparison result

Theorem [A.Alberico - G. di B. - F. Feo, *Math. Nachr.* 2017]

Let $u \in V_0^{1,\Phi}(\Omega)$ be a weak solution of the anisotropic problem (P_g). Then

$$u^\star(x) \leqslant v(x) \qquad \text{for } x \in \Omega^\star,$$

where v is the weak solution of the following isotropic problem

$$\begin{cases} -\text{div}\left(\dfrac{\Phi_\blacklozenge(|\nabla v|)}{|\nabla v|^2}\nabla v\right) = C\left(f^\star(x) - \text{div}\left(\Phi_{\blacklozenge\bullet}^{-1}\left(G(\omega_n|x|^n)\right)\dfrac{x}{|x|}\right)\right) & \text{in } \Omega^\star \\ v = 0 & \text{on } \partial\Omega^\star \end{cases}$$

with $C \in \mathbb{R}$, $s^{1/n} f^{**}(s) \in L^{\Phi_{\blacklozenge\bullet}}$ and $\int_\Omega \Phi_\bullet(g)\,dx < \infty$. Moreover

$$\int_\Omega \Phi(\nabla u)\,dx \leqslant \int_{\Omega^\star} \Phi_\blacklozenge(|\nabla v|)\,dx.$$

A priori bounds for u

$$u^\star(x) \leqslant v(x) \qquad \text{for } x \in \Omega^\star$$

where

$$v(x) = \int_{\omega_n |x|^n}^{|\Omega|} \frac{F(r)}{n\omega_n^{1/n} r^{1/n'}} \, dr,$$

$$F(r) = \Psi_\blacklozenge^{-1}\left(C \, \frac{\int_o^r f^*(\sigma)\,d\sigma(s)}{r^{1/n'} n\omega_n^{1/n}} + C \, \Phi_{\blacklozenge\bullet}^{-1}\left(G(r)\right)\right).$$

$$\Downarrow$$

norm estimates for u in terms of f and g

Orlicz space

The *Orlicz space* $L_A(\Omega)$, associated with the 1−dimensional Young function A, is the set of all measurable functions g in Ω for which the Luxemburg norm

$$\|g\|_{L_A(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega A\left(\frac{|g(x)|}{\lambda}\right)\,dx \leqslant 1\right\}$$

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is finite.

$\|g\|_{L^A(\Omega)}$ is a rearranged invariant norm, i.e.:

$$\|g\|_{L^A(\Omega)} = \|g^*\|_{L^A(0,|\Omega|)}$$

Orlicz Results

Suppose $F \in L^A(0, |\Omega|)$ with $\int_0 \left(\frac{s}{A(\tau)}\right)^{\frac{1}{n-1}} d\tau < +\infty.$

$$u^\star(x) \leqslant v(x) = \int_{\omega_n |x|^n}^{|\Omega|} \frac{F(r)}{n\omega_n^{1/n} r^{1/n'}} dr$$

$$+$$

Anisotropic Orlicz embedding [Cianchi, 2000]

$$\Downarrow$$

$$u \in L^{A_n}(\Omega)$$

where

$$A_n(s) = A(H_A^{-1}(|s|)), \quad s \in \mathbb{R} \quad ; H_A(r) = \left(\int_0^r \left(\frac{s}{A(s)}\right)^{\frac{1}{N-1}} ds\right)^{\frac{1}{N'}}$$

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Anisotropic Orlicz embedding [Cianchi, 2000]

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It is possible to prove that u enjoys a stronger summability which is given by the finiteness of the norm in the Orlicz-Lorentz spaces.

Alberico-di B.-Feo, 2017.

Generalization 2

Let Ω be an open bounded set of $\mathbb{R}^n$, $n \geqslant 2$

$$(P_b) \qquad \begin{cases} -\text{div}\,(a(x,u,\nabla u)) + b(u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where

$$a(x,\eta,\xi)\cdot\xi \geqslant \Phi(\xi) \quad \forall(\eta,\xi) \in \mathbb{R}\times\mathbb{R}^n$$

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Difficulty: Existence result for the symmetrized problem

- Φ not necessarily fulfil the Δ_2-condition

Then the Orlicz-Sobolev spaces could not be reflexive.

The Δ_2–condition

An n–dimensional Young function Φ is said to satisfy the Δ_2*–condition* near infinity if there exist constants $C > 1$ and $K \geqslant 0$ such that

$$\Phi(2\xi) \leqslant C\,\Phi(\xi) \quad \text{for} \quad |\xi| > K\,.$$

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Example of function which satisfy Δ_2−condition is given by

$$\Phi(\xi) = \sum_{i=1}^{N} \lambda_i\,|\xi_i|^{p_i} \qquad \text{for } \xi \in \mathbb{R}^N,$$

for some $\lambda_i > 0$ and $p_i > 1$, for any $i = 1, \ldots, N$.

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An extension is given by

$$\Phi\,(\xi) = \sum_{i=1}^{N} \Upsilon_i(\xi_i) \qquad \text{for } \xi \in \mathbb{R}^N,$$

where Υ_i, for $i = 1, \ldots, N$, are 1−dimensional Young functions vanishing only at zero.

The Δ_2–condition

An n–dimensional Young function Φ is said to satisfy the Δ_2*–condition* near infinity if there exist constants $C > 1$ and $K \geqslant 0$ such that

$$\Phi(2\xi) \leqslant C\,\Phi(\xi) \quad \text{for} \quad |\xi| > K\,.$$

An extension is given by

$$\Phi\,(\xi) = \sum_{i=1}^{N} \Upsilon_i(\xi_i) \qquad \text{for } \xi \in \mathbb{R}^N,$$

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$$\Phi \text{ satisfies } \Delta_2 - \mathit{condition} \iff \Upsilon_i \text{ satisfies } \Delta_2 - \mathit{condition}$$

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$$\Upsilon_i(s) = |s|^{p_i}\,(\log\,(c + |s|))^{\alpha_i} \qquad \text{for } s \in \mathbb{R}\; i = 1, \ldots, N$$

where either $p_i > 1$ and $\alpha_i \in \mathbb{R}$ or $p_i = 1$ and $\alpha_i \geqslant 0$, and the constant c is large enough for Υ_i to be convex.

Orlicz-Sobolev spaces

The *Orlicz space* $L_A(\Omega)$, associated with the 1−dimensional Young function A, is the set of all measurable functions g in Ω for which the Luxemburg norm

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We define the space $E_A(\Omega)$ as the closure of $L^\infty(\Omega)$ in $L_A(\Omega)$. Note that

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$$< f, g >= \int_\Omega f \; g \; dx$$

for $f \in L_A(\Omega)$ and $g \in E_{A_\bullet}(\Omega)$.

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$$W^1L_A(\Omega) = \{u \text{ measurable s.t. } u \text{ and } |\nabla u| \text{ belongs to } L_A(\Omega)\}$$

equipped with the norm

$$\|u\|_{W^1L_A(\Omega)} = \|u\|_{L_A(\Omega)} + \||\nabla u|\|_{L_A(\Omega)}.$$

We define $W_0^1L_A(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ in $W^1L_A(\Omega)$ with respect to the weak topology $\sigma(L_A(\Omega), E_{A_\bullet}(\Omega))$.

Symmetrized problem

Let $\Omega^\star$ be the ball centered at the origin having the same measure as Ω

$$\begin{cases} -\operatorname{div}\left(\dfrac{\Phi_\blacklozenge\left(|\nabla v|\right)}{|\nabla v|^2}\nabla v\right) + b(v) = h(x) & \text{in } \Omega^\star \\ v = 0 & \text{on } \partial\Omega^\star \end{cases}$$

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Search solutions $v \in W_0^1 L_{\Phi_\blacklozenge}(\Omega^\star)$, s. t. $\dfrac{\Phi_\blacklozenge(|\nabla v|)}{|\nabla v|} \in L_{\Phi_{\blacklozenge\bullet}}(\Omega^\star)$, $b(v) \in L^1(\Omega^\star)$

$$\int_{\Omega^\star}\left[\frac{\Phi_\blacklozenge(|\nabla v|)}{|\nabla v|^2}\nabla v\cdot\nabla\varphi + b(v)\varphi\right]dx = \int_{\Omega^\star} h\,\varphi\,dx$$

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Since $\Phi_\blacklozenge$ does not necessarily fulfill the Δ_2-condition, then $\dfrac{\Phi_\blacklozenge(|\nabla v|)}{|\nabla v|}$ does not necessarily belong to the space $L_{\Phi_{\blacklozenge\bullet}(\Omega^\star)}$ for every $v \in W_0^1 L_{\Phi_\blacklozenge}(\Omega^\star)$

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Theorem [A.Alberico - G. di B. - F. Feo, *Nonlinear Anal.* 2019]

There exists a unique symmetric positive weak solution v to problem (P_s), which belongs to the space

$$\mathcal{W}_0^{1,\Phi_\blacklozenge}(\Omega^\star) = \left\{ v \in W_0^1 L_{\Phi_\blacklozenge}(\Omega^\star) : \frac{\Phi_\blacklozenge(|\nabla v|)}{|\nabla v|} \in L_{\Phi_{\blacklozenge\bullet}}(\Omega^\star) \right\}.$$

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$$V_0^{1,\Phi_\blacklozenge}(\Omega^\star) = \Big\{ u : u \text{ real-valued funct. in } \Omega^\star \text{ whose continuation by } 0 \text{ outside } \Omega^\star \text{ is weakly diff. in } \mathbb{R}^n \text{ and } \int_{\Omega^\star} \Phi_\blacklozenge(|\nabla u|)\,dx < +\infty \Big\}.$$

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We want solution in the class $V_0^{1,\Phi^\blacklozenge}(\Omega^\star)$.

- We require more summability on datum h in order to assure that the weak solution $v \in \mathcal{W}_0^{1,\Phi^\blacklozenge}(\Omega^\star)$ belongs to the class $V_0^{1,\Phi^\blacklozenge}(\Omega^\star)$. If

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- either $\lim_{r\to+\infty} \Psi_\blacklozenge(r) = +\infty$ or $\frac{s^{1/n}}{n\omega_n^{1/n}} h^{**}(s) < \lim_{r\to+\infty} \Psi_\blacklozenge(r)$, $s > 0$
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then

$$\int_{\Omega^\star} \Phi_\blacklozenge(|\nabla v|)\, dx = \int_0^{|\Omega^\star|} \Phi_\blacklozenge\left(\Psi_\blacklozenge^{-1}\left(\frac{\int_0^r [h^*(s) - b(v^*(s))]\, ds}{n\omega_n^{1/n} r^{1/n'}}\right)\right) dr$$

$$\leqslant \int_0^{|\Omega^\star|} \Phi_\blacklozenge\left(\Psi_\blacklozenge^{-1}\left(\frac{r^{1/n}h^{**}(r)}{n\omega_n^{1/n}}\right)\right) dr < +\infty.$$

Mass comparison results

Let us consider

$$(P_b) \qquad \begin{cases} -\text{div}\,(a(x,u,\nabla u)) + b(u) = f\,(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with f a nonnegative function and $a(x,\eta,\xi)\cdot\xi \geqslant \Phi(\xi)$ for all $(\eta,\xi) \in \mathbb{R} \times \mathbb{R}^n$

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Theorem [A.Alberico - G. di B. - F. Feo, *Nonlinear Anal.* 2019]

Let u be a nonnegative weak solution to (P_b) and v the nonnegative weak solution to

$$(P_s) \qquad \begin{cases} -\,\text{div}\left(\dfrac{\Phi_\blacklozenge(|\nabla v|)}{|\nabla v|^2}\nabla v\right) + b(v) = h(x) & \text{in } \Omega^\star \\ v = 0 & \text{on } \partial\Omega^\star \end{cases}$$

If

$$\int_0^s f^*(t)\,dt \leqslant \int_0^s h^*(t)\,dt \qquad \textit{for any } s\in[0,|\Omega|],$$

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$$\int_0^s b(u^*(t))\,dt \leqslant \int_0^s b(v^*(t))\,dt \qquad \textit{for any } s\in[0,|\Omega|].$$

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Corollary [A.Alberico - G. di B. - F. Feo, *Nonlinear Anal.* 2019]

Let $A : [0,+\infty) \to [0,+\infty)$ be a convex function such that $A(0) = 0$. We get

$$\int_\Omega A(b(u(x)))\,dx \leqslant \int_{\Omega^\star} A(b(v(x)))\,dx.$$

Moreover,

$$\|u\|_{L^\infty(\Omega)} \leqslant \|v\|_{L^\infty(\Omega^\star)}.$$

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$$\begin{cases} \partial_t u - \sum_{i=1}^{N} \left(\alpha_i \, |\partial_{x_i} u|^{p_i - 2} \, \partial_{x_i} u \right)_{x_i} = f(x,t) & \text{in } Q_T := \Omega \times (0, T) \\ u(x,0) = u_0(x) & \text{in } \Omega \\ u(x,t) = 0 & \text{on } \Gamma_T := \partial\Omega \times (0, T), \end{cases}$$

$\Phi(\xi) = \sum_{i=1}^{n} \alpha_i |\xi_i|^{p_i}$: Parabolic problem

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- Ω is an open, bounded subset of $\mathbb{R}^N$ with Lipschitz continuous boundary
- $N \geqslant 2$, $T > 0, \alpha_i > 0$ and $p_i \geqslant 1$ for $i = 1, \ldots, N$ such that their harmonic mean $\bar{p} > 1$
- the data f and u_0 have a suitable summability

Main idea

Our goal is to compare the solutions u, v to the following problems

$$\begin{cases} \partial_t u - \left(\alpha_i \, |\partial_{x_i} u|^{p_i-2} \, \partial_{x_i} u\right)_{x_i} = f & \text{in } Q_T \\ u(x,0) = u_0(x) & \text{in } \Omega \\ u(x,t) = 0 & \text{on } \Gamma_T, \end{cases} \qquad \begin{cases} v_t - \operatorname{div}\left(\Lambda |\nabla v|^{\bar{p}-2} \nabla v\right) = f^\star & \text{in } Q_T^\star \\ v(x,0) = u_0^\star(x) & \text{in } \Omega^\star \\ v(x,t) = 0 & \text{on } \Gamma_T, \end{cases}$$

where $Q_T := \Omega \times (0, T)$, $\Gamma_T := \partial\Omega \times (0, T)$, $Q_T^\star := \Omega^\star \times [0, T]$.

Alberico- di B.- Feo, *Rend. Atti Lincei*, 2017

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where $Q_T := \Omega \times (0, T)$, $\Gamma_T := \partial\Omega \times (0, T)$, $Q_T^\star := \Omega^\star \times [0, T]$.
We prove for a.e. $t \in (0, T)$ that

$$\int_0^s u^*(\sigma, t) d\sigma \leqslant \int_0^s v^*(\sigma, t) d\sigma \quad s \in (0, |\Omega|)$$

Alberico- di B.- Feo, *Rend. Atti Lincei*, 2017

The strategy: time discretization

The application of the method of implicit time discretization leads to consider, for each time $T > 0$, a decomposition of $[0, T]$ in n subintervals $(t_{k-1}, t_k]$ where $t_k = kh$, $k = 1, \ldots, n$, and $h = T/n$: then the evolution problem is reduced to a sequence of nonlinear elliptic problems of the form

$$-h\left(\alpha_i \left|\partial_{x_i} u_{h,k}\right|^{p_i-2} \partial_{x_i} u_{h,k}\right)_{x_i} = u_{h,k-1} + h f_{h,k}$$

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where $u_{h,0} = u_0$. Piecing altogether the functions $u_{h,k}$ we get a discrete approximate solution u_h. Then the problem reduces to study the elliptic problem

$$-\left(\alpha_i \left| \partial_{x_i} u \right|^{p_i - 2} \partial_{x_i} u \right)_{x_i} + u = f$$

and check if a comparison result holds. Then proving a suitable a priori estimate for u_h, it is possible to pass to the limite and get our aim.

The strategy: Comparison result

Next step: derice a comparison result for solution to the anisotropic equation

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The strategy: Comparison result

Next step: derice a comparison result for solution to the anisotropic equation

$$-\left(\alpha_i \left|\partial_{x_i} u\right|^{p_i-2} \partial_{x_i} u\right)_{x_i} + u = f$$

Theorem [Alberico- di B.- Feo, *Rend. Atti Lincei*, 2017]

If w and z are the solution of the following problems

$$\begin{cases} -\left(\alpha_i \left|\partial_{x_i} w\right|^{p_i-2} \partial_{x_i} w\right)_{x_i} + \lambda w(x) = f(x) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} \operatorname{div}\left(\Lambda |\nabla z|^{\overline{p}-2} \nabla v\right) + \lambda z = f^\star(x) & \text{in } \Omega^\star \\ z = 0 & \text{on } \partial\Omega^\star \end{cases}$$

Then we have

$$\int_0^s w^*(\sigma)\, d\sigma \leqslant \int_0^s z^*(\sigma)\, d\sigma, \quad \forall s \in [0, |\Omega|]$$

that is

$$w^\star \prec z$$

- Symmetrization methods in a priori estimates for solution to isotropic elliptic equations are well known

  Maz'ya 69, Talenti 76,...

- Symmetrization methods in a priori estimates for solution to elliptic equations for a different type of anisotropy are also used in

  Alvino-Ferone-Trombetti-Lions 97, Belloni-Ferone-Kawohl 2003, Cianchi-Salani 2009, Wang-Xia 2012, Della Pietra-Gavitone 2013-2016,...

- Symmetrization methods in a priori estimates for solution to fully anisotropic elliptic equations are treat in

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- Other methods for elliptic equations governed by standard anisotropy, i.e.

$$\Phi(\xi) = \sum_{i=1}^{n} |\xi_i|^{p_i} \qquad \text{for } p_i \geqslant 1,$$

  are studied by

  Giaquinta 87, Marcellini 89, Acerbi-Fusco 94, Boccardo-Marcellini-Sbordone 90, Fusco-Sbordone 93, Stroffolini 93, Esposito-Leonetti-Mingione 2004, Fragalà-Gazzola-Kawohl 2004, Fragalà-Gazzola-Liebermann 2005, Antontsev-Chipot 2008, Di Castro 2009, Carozza-Moscariello-Passarelli di Napoli 2009, Cupini-Marcellini-Mascolo 2009, Di Nardo-Feo-Guibé 2013, DiBenedetto-Gianazza-Vespri, 2016, Bousquet-Brasco-Leone-Verde 2018-2020, Bousquet-Brasco 2020, ...

Symmetrization and Eigenvalue

Let $\Omega \subset \mathbb{R}^n$ bounded domain and $1 < p < +\infty$

$$\lambda_1(\Omega) = \min_{0 \neq u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}$$

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Faber -Krahn inequality

$$\lambda_1(\Omega) \geqslant \lambda_1(\Omega^\star)$$

where $\Omega^\star$ is the ball having the same measure of Ω.

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- If $\Phi(\xi_1, \ldots, \xi_n) = \sum_{i=1}^{n} |\xi|^p$, then

$$\Lambda_p = \min_{0 \neq u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^p dx}{\int_\Omega |u|^p \, dx}$$

Belloni-Kawohl 2004.

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- work in progress

Isotropic case

Let Ω be an open bounded subset in $\mathbb{R}^n$, $n \geqslant 2$

$$(P_p) \qquad \begin{cases} -\text{div}\,(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{q-2}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

with $1 < p < N$ and $1 < q < p^*$, where p^* is the Sobolev conjugate of p.

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- Equation in (P_p) is the Euler-Lagrange equation associated with the minimization problem

$$\min\left\{\int_\Omega |\nabla u|^p\,dx \,:\, u \in W_0^{1,p}(\Omega), \int_\Omega |u|^p\,dx = 1\right\}$$

Anisotropic case

Let Ω be an open bounded subset in $\mathbb{R}^n$, with $n \geqslant 2$,

$$(P_\Phi) \qquad \begin{cases} -\text{div}\,(\Phi_\xi(\nabla u)) = \lambda\, b(|u|)\,\text{sign}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

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An $n-$dimensional Young function Φ is called an $n-$dimensional $N-$function if it is a finite valued function, vanishes only at 0 and the following additional conditions are in force

$$\lim_{|\xi|\to+\infty}\frac{\Phi(\xi)}{|\xi|} = +\infty \qquad \lim_{|\xi|\to 0}\frac{\Phi(\xi)}{|\xi|} = 0$$

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- Φ: $\mathbb{R}^n \to [0,\infty)$ is a $N-$function belonging to $\mathcal{C}^1(\mathbb{R}^n)$
- λ is a positive real parameter
- b: $[0,\infty) \to [0,\infty)$ is an increasing, left-continuous function such that $b(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} b(t) = +\infty$.

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$$\inf\left\{\int_\Omega \Phi(\nabla u)\,dx : u \in W_0^1 L_{B,\Phi}(\Omega), \int_\Omega B(u)\,dx = r\right\}$$

- r is any positive real constant and $B(t) = \displaystyle\int_0^{|t|} b(\tau)d\tau$
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Φ and B not necessarily fulfil the Δ_2-condition

Theorem [A. Alberico - G. di B. - F. Feo, *J. Differ. Equ.* 2020]

Let Ω be an open bounded subset of $\mathbb{R}^n$ satisfying the segment property. Let $\Phi \in \mathcal{C}^1(\mathbb{R}^n)$ be an $n-$dimensional $N-$function such that

$$\int_0 \left(\frac{\tau}{\Phi_\star(\tau)} \right)^{\frac{1}{n-1}} \; d\tau < \infty.$$

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$\Phi_n : [0, \infty) \to [0, \infty]$ is the optimal Sobolev conjugate of Φ defined as

$$\Phi_n(t) = \Phi_\star(H^{-1}(t)) \qquad \text{for} \quad t \geqslant 0.$$

where $H : [0, \infty) \to [0, \infty)$ is given by

$$H(t) = \left(\int_0^t \left(\frac{\tau}{\Phi_\star(\tau)}\right)^{\frac{1}{n-1}} d\tau\right)^{\frac{n-1}{n}} \qquad \text{for} \quad t \geqslant 0,$$

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Then, for any $r > 0$ minimization problem

$$\inf \left\{ \int_\Omega \Phi(\nabla u) \, dx : u \in W_0^1 L_{B,\Phi}(\Omega), \int_\Omega B(u) \, dx = r \right\}$$

has at least one minimizer $u_r \in W_0^1 L_{B,\Phi}(\Omega)$.

Main Theorem [A. Alberico - G. di B. - F. Feo, *J. Differ. Equ.* 2020]

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Then, for any $r > 0$ there exist $\lambda_r > 0$ and $u_r \in \mathcal{W}_0^1 L_\Phi(\Omega) \cap L^\infty(\Omega)$ such that $\int_\Omega B(u_r)\, dx = r$ and u_r is a weak solution of problem (P_Φ) with $\lambda = \lambda_r$.

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Mustonen-Tienari, 1999 (1-dimensional Young function)

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Let Φ be an $n-$dimensional Young function. The anisotropic *Orlicz class* $\mathcal{L}_\Phi(\Omega; \mathbb{R}^n)$ is defined as

$$\mathcal{L}_\Phi(\Omega; \mathbb{R}^n) = \left\{ U : \Omega \to \mathbb{R}^n \text{ measurable s.t. } \int_\Omega \Phi(U) < +\infty \right\}.$$

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Note that $\mathcal{L}_\Phi(\Omega;\mathbb{R}^n)$ is a convex set of function and it need not be a linear space in general, unless Φ satisfies the Δ_2-condition near infinity.

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The *Orlicz space* $L_\Phi(\Omega;\mathbb{R}^n)$ is the linear hull of $\mathcal{L}_\Phi(\Omega;\mathbb{R}^n)$ and it is a Banach space with respect to the Luxemburg norm

$$\|U\|_\Phi = \inf\left\{ k > 0 : \int_\Omega \Phi\left(\frac{U}{k}\right) \leqslant 1 \right\}.$$

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The *Orlicz space* $L_\Phi(\Omega;\mathbb{R}^n)$ is the linear hull of $\mathcal{L}_\Phi(\Omega;\mathbb{R}^n)$. We denote by $E_\Phi(\Omega;\mathbb{R}^n)$ the closure in $L_\Phi(\Omega;\mathbb{R}^n)$ of the bounded measurable functions with compact support in $\overline{\Omega}$. In general

$$E_\Phi(\Omega;\mathbb{R}^n) \subset \mathcal{L}_\Phi(\Omega;\mathbb{R}^n) \subset L_\Phi(\Omega;\mathbb{R}^n),$$

and the equality holds if and only if Φ satisfies the Δ_2-condition near infinity

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$$\mathcal{W}_0^1 L_\Phi(\Omega) = \{u : \Omega \to \mathbb{R} : \text{ the continuation of u by } 0 \text{ outside } \Omega \text{ is weakly differentiable and } \nabla u \in L_\Phi(\Omega; \mathbb{R}^n)\}$$

equipped with the norm

$$\|u\|_{\mathcal{W}_0^1 L_\Phi(\Omega)} = \|\nabla u\|_{L_\Phi(\Omega;\mathbb{R}^n)} .$$

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$$W^1 L_{B,\Phi}(\Omega) = \{u \in L_B(\Omega) : u \text{ is weakly diff. in } \Omega \text{ and } \nabla u \in L_\Phi(\Omega; \mathbb{R}^n)\}$$

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$$W^1 L_{B,\Phi}(\Omega) = \{u \in L_B(\Omega) : u \text{ is weakly diff. in } \Omega \text{ and } \nabla u \in L_\Phi(\Omega; \mathbb{R}^n)\}$$

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$W^1 L_{B,\Phi}(\Omega)$ and $W^1 E_{B,\Phi}(\Omega)$ are Banach spaces equipped with the norm

$$\|u\|_{W^1 L_{B,\Phi}(\Omega)} = \|u\|_{L_B(\Omega)} + \|\nabla u\|_{L_\Phi(\Omega;\mathbb{R}^n)}$$

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$W_0^1 L_{B,\Phi}(\Omega)$ is the $\sigma(L_B \times L_\Phi, E_{B_\bullet} \times E_{\Phi_\bullet})$-closure of $\mathcal{D}(\Omega)$ in $W^1 L_{B,\Phi}(\Omega)$. Analogously, $W_0^1 E_{B,\Phi}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^1 L_{B,\Phi}(\Omega)$ with respect to the norm $\|u\|_{W^1 L_{B,\Phi}(\Omega)} = \|u\|_{L_B(\Omega)} + \|\nabla u\|_{L_\Phi(\Omega;\mathbb{R}^n)}$.

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- Anisotropic Orlicz and Orlicz-Sobolev spaces are not necessarily reflexive

Given a function $u \in W_0^1 L_{B,\Phi}(\Omega)$, the function obtained by extending u outside Ω by zero belongs to $W^1 L_{B,\Phi}(\mathbb{R}^n)$ and then

$$W_0^1 L_{B,\Phi}(\Omega) \subset \mathcal{W}_0^1 L_\Phi(\Omega) .$$

Both spaces, $W_0^1 L_{B,\Phi}(\Omega)$ and $\mathcal{W}_0^1 L_\Phi(\Omega)$, are reflexive if and only if $\Phi \in \Delta_2$ near infinity.

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A function $u \in \mathcal{W}_0^1 L_\Phi(\Omega)$ is a weak solution of (P_Φ) if $b(|u|) \in L_{B_\bullet}(\Omega)$, $\Phi_\xi(\nabla u) \in L_{\Phi_\bullet}(\Omega; \mathbb{R}^n)$ and

$$\int_\Omega \Phi_\xi(\nabla u) \cdot \nabla \varphi \, dx = \lambda \int_\Omega b(|u|) \operatorname{sign} u \, \varphi \, dx$$

for any $\varphi \in \mathcal{W}_0^1 L_\Phi(\Omega) \cap L^\infty(\Omega)$.

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Under the same assumptions as in Main Theorem if $u_r \in W_0^1 L_{B,\Phi}(\Omega)$ is a minimizer, then

(i) $\Phi_\xi(\nabla u_r) \in L_{\Phi_\bullet}(\Omega; \mathbb{R}^n)$;

(ii) $b(|u_r|) \in L_{B_\bullet}(\Omega)$.

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We consider the functionals $F : W_0^1 L_{B,\Phi}(\Omega) \to \overline{\mathbb{R}}$ and $G : W_0^1 L_{B,\Phi}(\Omega) \to \overline{\mathbb{R}}$ defined as

$$F(u) = \int_\Omega \Phi(\nabla u) \;\; dx \qquad G(u) = \int_\Omega B(u) \;\; dx$$

- F is a finite-valued functional on $W_0^1 L_{B,\Phi}(\Omega)$ if and only if Φ fulfils the Δ_2-condition.
- $G(u)$ is always finite for every $u \in W_0^1 L_{B,\Phi}(\Omega)$ for the compact embedding of $W_0^1 L_{B,\Phi}(\Omega)$ in $E_B(\Omega)$.

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We prove continuity of G and lower semicontinuity of F with respect the topology $\sigma(W_0^1 L_{B,\Phi}(\Omega), W^{-1} E_{B_\bullet,\Phi_\bullet}(\Omega))$.

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Let us define the functionals dF ad dG by

$$\langle dF, v\rangle = \int_\Omega \Phi_\xi(\nabla u_r)\cdot\nabla v \; dx \qquad \langle dG, v\rangle = \int_\Omega \frac{b(|u_r|)}{|u_r|} u_r v \; dx$$

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Abstract version of Lagrange Multipliers result assures the existence of $\lambda_r \in \mathbb{R}$, associated with the minimizer u_r, such that u_r is a weak solution of problem (P_Φ).

Zeidler 1985.

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Let X and K be real Banach spaces in duality with respect to continuous pairing $\langle\cdot,\cdot\rangle$, and let X_0 and K_0 be subspaces of X and K, respectively. Then $(X, X_0; K, K_0)$ represent a so-called complementary system if, by means of $\langle\cdot,\cdot\rangle$, the dual of X_0 can be identified to K and that of K_0 to K.

Thank you
for your attention!!