Hölder regularity
for nonlocal double phase equations

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Based on a joint work[1] with
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[1] http://palatucci.altervista.org

We deal with a class of possible degenerate and singular integro-differential equations whose leading operator switches between **two different types of fractional elliptic phases**, according to the zero set of a **modulating coefficient $a{=}a(.,.)$**.

The model case is driven by

$$\mathcal{L}u(x) := \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\,\mathrm{d}y + \int_{\mathbb{R}^n} a(x,y)\frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))}{|x-y|^{n+tq}}\,\mathrm{d}y,$$

where $q \geq p$ and $a(\cdot,\cdot) \geqq 0$.

More in general, we will deal with inhomogeneous equations, for very general classes of measurable kernels.

The nonlocal double phase operator $\mathcal{L}$ can be plainly seen as the nonlocal analog of the classical double phase functional,

$$\mathcal{F}(u) := \int \Big(|Du|^p + a(x)|Du|^q\Big)\,\mathrm{d}x, \quad 1 < p \leq q,$$

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Non-uniformly elliptic functionals

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From a regularity point of view, even without the presence of the modulating coefficient $a(\cdot)$, such functional presents very interesting features, falling in the class of the non-uniformly elliptic ones having (p,q)-growth conditions. Basically, one can prove that

$$\frac{q}{p} < 1 + o(n)$$

is a sufficient [Marcellini, *JDE* 1991] and necessary [Giaquinta, *Manu. Math.* 1987] condition for regularity.

First several fundamental contributions on non-uniformly elliptic operators: Hong, Fusco-Sbordone, Leon Simon, Lieberman, Uraltseva-Urdaletova; and more recently Fiscella, Fonseca, Maly, Mingione, Pucci, Radulescu, and many others.

Non-autonomous functionals

$$\mathcal{F}(u) := \int \left(|Du|^p + a(x)|Du|^q \right) \mathrm{d}x$$

Because of the modulating coefficient, the functional $\mathcal{F}$ is the prototype of a bad kind of interplay between a coefficient in x and the (p,q)-growth, since it brings a change of ellipticity occurring on the set $\{a = 0\}$:

- in the points where $a > 0$, $\mathcal{F}$ reduces to a non-standard (p,q)-growth functional, which exhibits a q-growth in the gradient (in the relevant case when $q > p$).

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Indeed, it was introduced by Zhikov in order to describe strongly anisotropic materials whose hardening properties drastically change with the point: the regulation of the mixture between two different materials, with p and q hardening, is modulated by the coefficient $a(\cdot)$.

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The first result in this spirit was recently due to Colombo and Mingione [*ARMA* 2015, and *ARMA* 2015]: *If the modulating coefficient $a(\cdot)$ is Hölder continuous, the weak solutions to the double phase equations are Hölder continuous as well,*

by assuming that $1 \leq q/p \leq 1 + \alpha/n$, where $\alpha \in (0, 1]$ is the Hölder exponent of $a(\cdot)$.

*A first (counter-)example by Fonseca-Maly-Mingione [*ARMA* 2004].

- Baroni-Colombo-Mingione [*Nonlinear Anal.* 2015, *Calc.Var PDE* 2018]: **Harnack inequalities**, general classes of double phase functionals.
- Byun-Oh [*JDE* 2017, *Anal. PDE* 2020]: **Gradient estimates** for the borderline case; BMO coefficients in nonsmooth domains; **Generalized double phase functional**.
- Chlebicka-De Filippis [*AMPA* 2019]: Removability of the singularities; **obstacle** problems.
- Colombo-Mingione [*JFA* 2016]; De Filippis-Mingione [*J. Geom. Anal.* 2019]: **Calderon-Zygmund** Theory, classical and borderline setting.
- De Filippis-Oh [*JDE* 2019]: **Multiphase** (different rates of ellipticity with Hölder continuous coefficients).
- De Filippis-Mingione [*JDG* 2019, *JDG* 2020]: **Manifold constrained problems; Vectorial case** and critical systems.
- Hästö-Ok [*Preprints* 2019]: Maximal regularity for local minimizers; Calderón-Zygmund estimates in Orlicz setting.
- Ok [*Nonlinear Anal.* 2018]: Partial regularity for double phase **Systems**.
- Papageorgiou-Radulescu-Repovs [*Proc.AMS* 2019, *ZAMP* 2019]: Discontinuity of the spectrum; **Nodal solutions**.
- Bahrouni-Radulescu-Repovs [*Calc. Var. PDE* 2019]: **Nonlinear patterns** and stationary waves.
- Balci, Carozza, Cho, Cupini, Eleuteri, Fiscella, Mascolo, Harjulehto, Karppinen, Kim, Leonetti, Liu, Pinamonti, Ragusa, Repovs, Scheven, Stroffolini, Surnachev, Tachikawa, Verde, Yao, Zhang, Zheng, and **many many** others.

We consider the following inhomogeneous nonlocal double phase equation,

$$\mathcal{L}u = f,$$

where f is bounded and the integro-differential operator $\mathcal{L}$ is given by

$$\begin{aligned}\mathcal{L}u(x) \ := \ & P.V.\int_{\mathbb{R}^n} |u(x)-u(x+y)|^{p-2}(u(x)-u(x+y))K_{sp}(x,y)\,\mathrm{d}y \\ & + P.V.\int_{\mathbb{R}^n} a(x,y)|u(x)-u(x+y)|^{q-2}(u(x)-u(x+y))K_{tq}(x,y)\,\mathrm{d}y.\end{aligned}$$

For $s,t \in (0,1)$ and $p,q > 1$, the measurable kernels K_{sp} and K_{tq} essentially behave like (s,p) and (t,q)-kernels, respectively. More precisely, there exists a positive constant Λ such that

$$\begin{cases} \Lambda^{-1}|y|^{-n-sp} \leq K_{sp}(x,y) \leq \Lambda|y|^{-n-sp}, \\ K_{sp}(x,y) = K_{sp}(x,-y), \end{cases} \quad \text{and} \quad \begin{cases} \Lambda^{-1}|y|^{-n-tq} \leq K_{tq}(x,y) \leq \Lambda|y|^{-n-tq}, \\ K_{tq}(x,y) = K_{tq}(x,-y). \end{cases}$$

Our main result is the following

Theorem [De Filippis-Palatucci, *J. Differential Equations* 2019]

Let $p, q > 1$ *be such that*

$$p > \frac{1}{1-s} \text{ if } p < 2, \qquad q > \frac{1}{1-t},$$

and

$$1 \leq q/p \leq \min\left\{\frac{s}{t}, 1+s\right\},$$

and let f *be in* $L^\infty(B_2)$.

Assume that the modulating coefficient a *is a measurable function such that* $0 \leq a(x,y) \leq M$ *for a. e.* $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. *If* u *is a bounded viscosity solution to*

$$\mathcal{L}u = f \quad in \quad B_2,$$

then $u \in C^{0,\gamma}(B_1)$ *for some* $\gamma = \gamma(n,p,q,s,t,M,\Lambda,\|u\|_{L^\infty},\|f\|_{L^\infty}) \in (0,1)$.

A few observations are in order

• **local vs nonlocal**: in the local case, regularity results for bounded weak solutions are achieved provided that $1 \leq q/p \leq 1+\alpha$, with $a \in C^{0,\alpha}$.
In the nonlocal case, assuming only $a(\cdot) \in L^\infty$, we have $1 \leq q/p \leq \min\left\{\frac{s}{t}, 1+s\right\}$.

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• **nonlocal nonlinear nonstandard**: The nonlocal equation inherits both the difficulties newly arising from the double phase problems and those naturally arising from the fractional integro-differential operators.

• To our knowledge, this is **the very first regularity result for solutions to nonlocal double phase** equations. Even in the very special case when $s = t$ and $p = q$, no related results involving a modulating coefficient could be found in the literature. It is worth mentioning the fine Hölder estimates in a relevant paper by Kassmann, Rang and Schwab [*Indiana J.* 2014], for elliptic integro-differential operators with kernels satisfying lower bounds on conic subsets, thus strongly directionally dependent.

Definition [viscosity subsolutions]

Let $\Omega \subset \mathbb{R}^n$ be an open subset and $\mathcal{L}$ be the nonlocal double phase functional. An upper semicontinuous function $u \in L^\infty_{\rm loc}(\Omega)$ is a subsolution of $\mathcal{L}(\cdot) = C$ in Ω, and we write

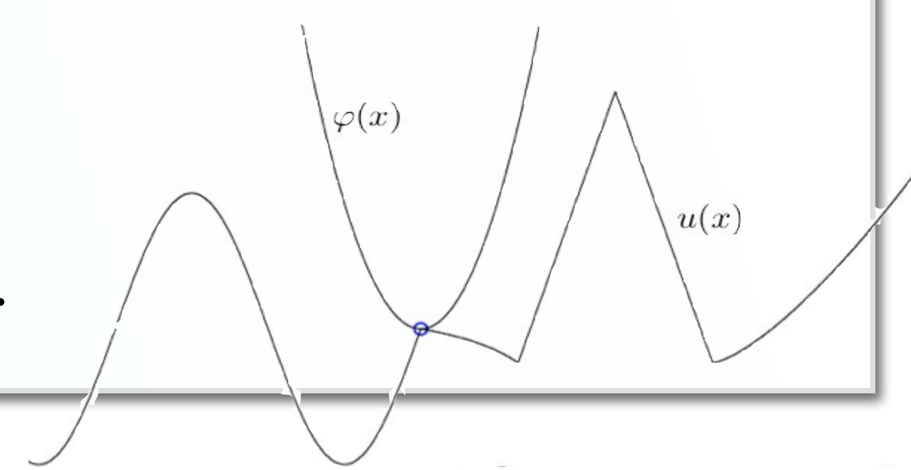
"u is such that $\mathcal{L}(\mathbf{u}) \leq \mathbf{C}$ in $\boldsymbol{\Omega}$ in the viscosity sense"

if the following statement holds: whenever $x_0 \in \Omega$ and $\varphi \in C^2(B_\varrho(x_0))$ for some $\varrho > 0$ are so that

$$\varphi(x_0) = u(x_0), \quad \varphi(x) \geq u(x) \ \textit{ for all } x \in B_\varrho(x_0) \Subset \Omega,$$

then we have $\mathcal{L}\varphi_\varrho(x_0) \leq C$, where

$$\varphi_\varrho := \begin{cases} \varphi & \textit{in } \ B_\varrho(x_0) \\ u & \textit{in } \ \mathbb{R}^n \setminus B_\varrho(x_0). \end{cases}$$



A ***viscosity supersolution*** is defined in an analogous fashion,

and a ***viscosity solution*** is a function which is both a subsolution and a supersolution.

As soon as we can touch a viscosity subsolution with a C^2-function, then it behaves as a classical subsolution.

Proposition [De Filippis-Palatucci, *J. Differential Equations* 2019]

Suppose that $\mathcal{L}u \leq C$ *in* B_1 *in the viscosity sense. If* $\varphi \in C^2(B_\varrho(x_0))$ *is such that*

$$\varphi(x_0) = u(x_0), \quad \varphi(x) \geq u(x) \quad \textit{in} \quad B_\varrho(x_0) \Subset B_1,$$

for some $0 < \varrho < 1$, *then* $\mathcal{L}u$ *is defined in the pointwise sense at* x_0 *and* $\mathcal{L}u(x_0) \leq C$.

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Proof. We plainly extend to the double phase problems a by-now classical approach, as firstly seen in Caffarelli-Silvestre [*CPAM* 2009] for fully nonlinear integro-differential operators, and successfully applied even for the fractional p-Laplace equation by Lindgren [*NoDEA* 2016]. □

Basically we extend the the approach of Silvestre in *Indiana J.* (2006), where he shows the Hölder continuity of fractional harmonic functions, via a purely analytical proof which goes back to De Giorgi. Not for free, because of the nonstandard (p, q)-growth, and the zero set of $a(\cdot, \cdot)$.

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Sketch of the proof (*be aware: lots of cheating*)

For $\sigma > 0$, let $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_\sigma$ be a suitable scaling of our functional, and let φ be any radial map which is C^2-regular, vanishes outside B_1, and it is non-increasing along rays from the origin.

Step 1 (controlling the energy of smooth maps).

$\forall \varepsilon > 0\ \exists \kappa \in (0,1/2]$ such that $\tilde{\mathcal{L}}\varphi \lesssim \varepsilon\sigma/\kappa^{q-1}$,

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Step 2 (refining).

If u is such that $|B_1 \cap \{u(x) < 0\}| > 0$ and $\begin{cases} \tilde{\mathcal{L}}u \leq \sigma \text{ in } B_1 \\ u \leq 1 \text{ in } B_1, \end{cases}$ then $u \leq 1 - \kappa$ in $B_{1/2}$,

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Step 3 (iterating).

Let $\tilde{u} := \left(\frac{1}{\|u\|_{L^\infty} + (\|f\|_{L^\infty(B_2)}/\sigma)^{1/(p-1)}} \right) u$. We have $\operatorname{osc} \tilde{u} < 1$ and $\tilde{\mathcal{L}}\tilde{u} = \tilde{f}$, for a suitable $\tilde{f}$.

By suitably choosing ε and σ in the previous steps, we can start an iteration, to get

$$\operatorname*{osc}_{B_\varrho(x_0)} u \leq c(\mathtt{data}) \left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_2)}^{\frac{1}{p-1}} \right) \varrho^\gamma,$$

for some $\gamma = \gamma(\mathtt{data}) \in (0,1)$, which implies, by covering, $u \in C^{0,\gamma}(B_1)$, as desired. □

Let $u \in L^\infty(\mathbb{R}^n)$ be a viscosity solution to $\mathcal{L}u = f$.
We rescale and blow u around $x_0 \in B_1$ as follows. For $\lambda, \mu > 0$ and $x \in B_1$, we define the map

$$u^{(\lambda)}_{\mu,x_0}(x) := \lambda u(\mu x + x_0).$$

Such a function satisfies

$$\hat{\mathcal{L}} u^{(\lambda)}_{\mu,x_0}(x) := \hat{f}(x) \quad \text{in} \quad B_1,$$

where

$$\begin{aligned}\hat{\mathcal{L}} v(x) \; := \; & \int_{\mathbb{R}^n} |v(x) - v(x+y)|^{p-2}(v(x) - v(x+y)) \hat{K}_{sp}(x,y)\,\mathrm{d}y \\ & + \int_{\mathbb{R}^n} \hat{a}(x,y) |v(x) - v(x+y)|^{q-2}(v(x) - v(x+y)) \hat{K}_{tq}(x,y)\,\mathrm{d}y\end{aligned}$$

and

$$\hat{f}(x) := \lambda^{p-1}\mu^{sp} f(\mu x + x_0).$$

The modulating coefficient and the kernels appearing above are defined as

$$\hat{a}(x,y) := \lambda^{p-q}\mu^{sp-tq} a(\mu x + x_0, \mu y)$$

and

$$\begin{cases} \hat{K}_{sp}(x,y) := \mu^{n+sp} K_{sp}(\mu x + x_0, \mu y) \\ \hat{K}_{tq}(x,y) := \mu^{n+tq} K_{tq}(\mu x + x_0, \mu y) \end{cases},$$

respectively.

Related open problems

- Whether or not, and under which assumptions on the structural quantities, the viscosity solutions to nonlocal double phase equations are indeed **fractional harmonic functions** and/or weak solutions, and vice versa (see, e. g. [Korvempaa-Kuusi-Lindgren, *JMPA* 2019] for the fractional p-Laplace equation).

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- In the same spirit of Baroni-Colombo-Mingione [*Nonlinear Anal.* 2015, *Calc.Var. PDE* 2018], one would expect **higher differentiability** and regularity results for the bounded solutions to nonlocal double phase equations (see, e. g., Brasco-Lindgren-Schikorra [*Adv. Math.* 2018] for the fractional p-Laplace equation).
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- **Harnack-type inequalities**. Preliminary results for weak supersolutions have been proven in De Filippis-Palatucci [*Preprint* 2021], namely by dealing with the resulting error term as a right hand-side (a nonlocal tail), and proving local Boundedness, a Caccioppoli Inequality with tail, and a weak Harnack, in the same flavour of the works by Brasco, Chen, Kassmann, Kuusi, Iannizzotto, Lindgren, Silvestre, Squassina, et Al. (that is, in the spirit of the De Giorgi-Nash-Moser theory).

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- Both in the local and in the nonlocal double phase theory, nothing is known about the regularity for solutions to **parabolic double phase equations**.

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thank you

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