Gradient Estimates for weak solutions of Elliptic PDE's

Giuseppe Di Fazio

Dipartimento di Matematica e Informatica
Università di Catania

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PLAN

- Statement of problem
- Existing literature
- Harmonic Analysis tools
- Gradient estimates

The problem: Data

- Ω a bounded domain in $\mathbb{R}^n$, $n \geq 3$ with smooth boundary
- $A(x)$ a symmetric uniformly elliptic matrix with bounded measurable entries
- The uniformly elliptic equation $Lu \equiv \mathrm{div}(A(x)\nabla u) = \mathrm{div} f$ in Ω for suitable f and Ω.

Concept of solution

Given the divergence type equation

$$\mathrm{div}(A(x)\nabla u) = \mathrm{div} f$$

we mean by solution a function u such that

$$\int_\Omega A(x)\nabla u \nabla\varphi \, dx = \int_\Omega f \cdot \nabla\varphi \, dx \qquad \forall \varphi \in W_0^{1,p'}(\Omega)$$

where $1 < p < \infty$, $1/p + 1/p' = 1$.

The problem: Aim

- A priori estimate of the following kind

$$\|\nabla u\|_{L^p(\Omega)} \leq c \left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right)$$

for generalized solutions of the equation

$$Lu = \operatorname{div} f$$

- Existence and uniqueness for BVPs related to our equation
- Well posedness and regularity of solution of BVPs

THE HISTORY

The result is known to be true for the Laplace operator and for continuous coefficients operators.

THE HISTORY

No hope for general discontinuous coefficients

COUNTEREXAMPLE (MEYERS -1963)

If the coefficients $A(x)$ belong to L^∞ - in particular if they are discontinuous functions - $f \in L^p$ does not imply $\nabla u \in L^p$ for any $1 < p < \infty$.

In general the implication

$$f \in L^p \Rightarrow \nabla u \in L^p$$

is false for arbitrary values of $1 < p < +\infty$

The history

Theorem (Meyers)

There exists $p_0 > 2$ depending on dimension and ellipticity such that

$$f \in L^p \Rightarrow \nabla u \in L^p$$

for any $1 < p < p_0$ and that it is false for $p = p_0$.
Moreover $p_0 \to 2$ as the ellipticity ratio degenerates.

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THE COUNTEREXAMPLE

Let us consider the equation

$$Lu = (au_x + bu_y)_x + (bu_x + cu_y)_y = 0$$

where μ is a constant $0 < \mu < 1$ and

$$\begin{cases} a = 1 - (1-\mu^2)\dfrac{y^2}{x^2+y^2} \\ b = (1-\mu^2)\dfrac{xy}{x^2+y^2} \\ c = 1 - (1-\mu^2)\dfrac{x^2}{x^2+y^2} \end{cases}$$

THE COUNTEREXAMPLE

The operator is uniformly elliptic with constants μ^2 and 1.
The function $u(x, y) = x(x^2 + y^2)^{(\mu-1)/2}$ satisfies the equation and

$$\nabla u \in L^p \qquad \forall 1 \leq p < p_\mu \equiv \frac{2}{1-\mu}$$

but it does not belong to L^{p_μ}.

Note that $p_\mu \to 2$ as $\mu \to 0^+$.

THE HISTORY

QUESTION

What kind of extra assumption could we add in order to get

$$f \in L^p \Rightarrow \nabla u \in L^p$$

for any $1 < p < +\infty$?

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Remarkable cases

1. Constant coefficients operators
2. Continuous coefficients operators

The case of Laplace operator

The proof is based on a representation formula for derivatives in terms of the fundamental solution.

For any $u \in \mathcal{D}(B)$ such that

$$-\Delta u = f_0 + \mathrm{div} f$$

we have

$$u(x) = \int_B \Gamma_j(x-y) f_j(y)\,dy + \int_B \Gamma(x-y) f_0(y)\,dy \quad \forall x \in B.$$

By differentiation we obtain

$$u_{x_i}(x) = P.V. \int_B \Gamma_{ij}(x-y) f_j(y)\,dy + \delta_{ij} f_j(x) + \int_B \Gamma_i(x-y) f_0(y)\,dy$$

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THE CASE OF LAPLACE OPERATOR

Now Harmonic Analysis ends the game by boundedness properties of potentials and singular integral operators between Lebesgue spaces.
We obtain

$$\|\nabla u\|_{L^p(B)} \leq c\,\left(\|f\|_{L^p(B)} + \|f_0\|_{L^{p_*}(B)}\right)$$

where p_* is such that

$$\frac{1}{p_*} = \frac{1}{p} + \frac{1}{n}$$

and $c = c(n,p)$.
Standard covering and flattening arguments yield the result on bounded domains with sufficiently smooth boundary.

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THE CASE OF CONTINUOUS COEFFICIENTS OPERATORS

The argument is based on pointwise perturbation. Let us consider $u \in \mathcal{D}$ such that

$$Lu = f_0 + \mathrm{div} f.$$

Let L_0 be the operator L with the coefficients frozen at some point x_0. We have

$$L_0 u(x) = (L_0 - L)u(x) + Lu(x)$$

and then apply the previous estimate.
Then we obtain

$$\|\nabla u\|_p \leq c\,(\|(L_0 - L)u\|_p + \|f\|_p + \|f_0\|_{p_*})$$

The case of continuous coefficients operators

Note that, we use the continuity assumption here because

$$\|\nabla u\|_p \le c\left(\max_B |a_{ij}(x) - a_{ij}(x_0)|\, \|\nabla u\|_p + \|f\|_p + \|f_0\|_{p_*}\right)$$

can be made small as we wish if the radius of the balls is suitably small.

Constant vs continuous

By comparison of techniques we note that

1. Harmonic Analysis yields the result for constant coefficients.
2. Pointwise perturbation & constant coefficients case imply continuous coefficients case.

In the continuous coefficients case there is no explicit use of Harmonic Analysis

Towards *VMO*

We study the case of variable and discontinuous coefficients case by using Harmonic Analysis in an explicit way.

Towards *VMO*

We start as in the case of continuous coefficients
Let L_0 be the operator L with the coefficients frozen at some point x_0, i.e.

$$L_0 u = \left(a_{ij}(x_0) u_{x_i}\right)_{x_j}$$

We have

$$L_0 u(x) = (L_0 - L)u(x) + Lu(x)$$

We do not need any continuity assumption now because we use a representation formula for an operator with constant coefficients.

Towards *VMO*

For any solution $u \in \mathcal{D}(B)$ we have

$$u(x) = \int_B \Gamma_j(x_0, x-y)\{(a_{ij}(x_0) - a_{ij}(y))u_{x_i}(y) - f_j(y)\}\, dy \\ - \int_B \Gamma(x_0, x-y) f_0(y)\, dy \quad \forall x \in B.$$

TOWARDS *VMO*

Now we do not apply the estimates obtained for the constant coefficients case.
We proceed as for the Laplace operator by differentiating representation formula.

$$u_{x_k}(x) = PV\int_B \Gamma_{jk}(x_0, x-y)\{(a_{ij}(x_0) - a_{ij}(y))u_{x_i}(y) + \\ -f_j(y)\}\, dy + \int_B \Gamma_i(x_0, x-y) f_0(y)\, dy + \\ + \left[(a_{ij}(x_0) - a_{ij}(x))u_{x_i}(x) - f_j(x)\right] \int_{|t|=1} \Gamma_i(x_0, t)\, t_j\, d\sigma(t)$$

for any $x \in B$.

Towards *VMO*

By taking $x_0 = x$ we get

Representation formula

$$u_{x_k}(x) = PV\int_B \Gamma_{ij}(x, x-y)\{(a_{hj}(x) - a_{hj}(y))u_{x_h}(y) + \\ -f_j(y)\}\,dy + \int_B \Gamma_i(x, x-y) f_0(y)\,dy + f_j(x)\int_{|t|=1} \Gamma_i(x,t)\,t_j\,d\sigma(t)$$

for all $x \in B$

where $c_{ij}(x) = \int_{|t|=1} \Gamma_i(x,t)\,t_j\,d\sigma(t)$ are bounded functions.

Singular integrals and commutators

We look at the formula as an implicit representation one.

$$\nabla u = \underbrace{[K, A]\,\nabla u}_{\color{red}{Commutator}} + \underbrace{K(Lu)}_{\color{blue}{SIO}} + \underbrace{I(Lu)}_{\color{green}{Rieszpotential}} + c\,Lu$$

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SINGULAR INTEGRALS AND COMMUTATORS

Singular integral of non-convolution type.

$$K(f)(x) \equiv PV \int_B \Gamma_{ij}(x, x-y)\, f(y)\, dy$$

Commutator between singular integral of non-convolution type and multiplication operator

$$[\Gamma_{ij}, a_{hk}](f)(x) \equiv P.V. \int_B \Gamma_{ij}(x, x-y)(a_{hk}(x) - a_{hk}(y)) f(y)\, dy$$

Singular integrals and commutators

The parameter can be handled following a procedure due to A.P. Calderón.
The idea is to develop the kernel in a series of spherical harmonics.

Then the series of operators is estimated in a very clever way.

As a result, both the singular operators and the commutators are bounded between L^p classes.
Regarding the commutator the result has been proven assuming the a_{hk} belong to the John & Nirenberg space BMO.

SINGULAR INTEGRALS AND COMMUTATORS

Summarizing, Chiarenza,Frasca and Longo proved that

$$\|Kf\|_{L^p} \leq c\|f\|_{L^p}, \quad \|C[a,f]\|_{L^p} \leq c\|a\|_*\|f\|_{L^p}$$

where $\|a\|_*$ is the *BMO* seminorm of a.

HA ⇒ PDE ESTIMATES

Now we apply the boundedness property of the singular operators and commutators (a localized version of) taking norms on both sides of representation formula and we get

$$\|\nabla u\|_{L^p(B)} \leq c\left(\|a\|_* \|\nabla u\|_{L^p(B)} + \|u\|_{L^p(B)} + \|f\|_{L^p(B)} + \|f_0\|_{L^{p_*}(B)}\right).$$

VMO TIME

Now a very important subclass of *BMO* comes into play, the class *VMO*.

DEFINITION

For any locally integrable function f we consider

$$\eta(r) \equiv \sup_{\substack{x\in\mathbb{R}^n \\ \varrho<r}} \fint_{B_\varrho} |f - f_\varrho|\, dy$$

η *bounded means* $f \in BMO$, *while* $\eta \to 0$ *with r means* $f \in VMO$.

EXAMPLE

1. continuous functions are *VMO*
2. $W^{1,n} \subset VMO$

VMO TIME

REMARK

It is very important to note the difference between BMO and VMO. For VMO functions, by taking suitable small ball B we can make η small as we please. The last feature is very important for us.

REMARK

The VMO assumption on the leading coefficients means that

we avoid jump discontinuity.
This is compatible with Meyer's counterexample

HA + VMO ⇒ A PRIORI ESTIMATES

The result is now a consequence of the *VMO* character of the leading coefficients.
Let us consider an arbitrary positive number ε. Then, there exists r_0 such that, for any $0 < r < r_0$ we get

$$\|\nabla u\|_{L^p(B_r)} \leq c\left(\varepsilon\|\nabla u\|_{L^p(B_r)} + \|u\|_{L^p(B_r)} + \|f\|_{L^p(B_r)} + \|f_0\|_{L^{p_*}(B_r)}\right)$$

so, if $\varepsilon = \dfrac{1}{2c}$ we obtain

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Estimate on small balls

$$\|\nabla u\|_{L^p(B_r)} \leq c \left(\|u\|_{L^p(B_r)} + \|f\|_{L^p(B_r)} + \|f_0\|_{L^{p_*}(B_r)}\right)$$

that is the estimate on sufficiently small balls i.e. for any $0 < r < r_0$ with constant $c = c(n, p, \nu, \eta)$.

From local to global

Now standard arguments i.e. covering the domain and flattening its boundary give the estimate on the whole domain

$$\|\nabla u\|_{L^p(\Omega)} \leq c\ \left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|f_0\|_{L^{p_*}(\Omega)}\right)$$

where $c = c(n, p, \nu, \eta)$.

Boundary value problems

The estimate just obtained and an iteration argument yield existence and uniqueness of $W^{1,p}$ solution of the Dirichlet problem

Theorem (D)

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with sufficiently smooth boundary. Let the matrix $A(x)$ be bounded, measurable, uniform elliptic in Ω and VMO. Then, for any given f in $L^p(\Omega)$, $1 < p < \infty$, there exists a unique solution u to the problem

$$\begin{cases} Lu = \mathrm{div} f & \text{in } \Omega \\ u \in W_0^{1,p} \end{cases}$$

such that

$$\|\nabla u\|_{L^p(\Omega)} \leq c\,\|f\|_{L^p(\Omega)}$$

Boundary value problems

As an immediate consequence of the previous result and the linearity of the problem we have

Theorem (D)

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with sufficiently smooth boundary. Let the matrix $A(x)$ be bounded, measurable, uniform elliptic in Ω and VMO. Then, for any given f in $L^p(\Omega)$, $\varphi \in W^{1,p}(\Omega)$, $1 < p < \infty$, there exists a unique solution u to the problem

$$\begin{cases} Lu = \operatorname{div} f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

such that

$$\|\nabla u\|_{L^p(\Omega)} \leq c \left(\|f\|_{L^p(\Omega)} + \|\varphi\|_{W^{1,p}(\Omega)} \right)$$

Thanks

Thank you very much for your attention