$\mathscr{A}$-quasiconvexity and partial regularity

Joint work with Sergio Conti (Bonn)

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The historical key problem

- Let $\Omega \subset \mathbb{R}^n$ be open,
- $F \in \mathrm{C}(\mathbb{R}^{N\times n})$ satisfy **standard growth assuptions** – i.e., for $p \in (1,\infty)$

$$|F(z)| \leq c(1+|z|^p) \qquad \text{for all } z \in \mathbb{R}^{N\times n}$$

and consider, for $u_0 \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$, the problem

$$\text{to minimise} \quad \mathscr{F}[u;\Omega] := \int_\Omega F(Du)\,\mathrm{d}x \quad \text{over } \mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)$$

- For the direct method in the Calculus of Variations, we require
  - (i) coerciveness.
  - (ii) sequential weak lower semicontinuity:

$$W^{1,p}_{u_0}(\Omega;\mathbb{R}^N) \ni u_j \rightharpoonup u \Rightarrow \mathscr{F}[u;\Omega] \leq \liminf_{j\to\infty} \mathscr{F}[u_j;\Omega].$$

Morrey's Quasiconvexity

Call $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ as above **quasiconvex** provided

$$F(z) \leq \fint_\Omega F(z+D\varphi)\,\mathrm{d}x \qquad \text{for all } z \in \mathbb{R}^{N\times n},\ \varphi \in \mathrm{C}^\infty_c(\Omega;\mathbb{R}^N).$$

The canonical setup

Aim: Partial regularity of (local) minima of integral functionals

$$v \mapsto \int F(Dv)\,\mathrm{d}x$$

subject to

(H1) $F \in \mathrm{C}^2(\mathbb{R}^{N\times n})$ (smoothness)

(H2) $|F(z)| \lesssim c(1+|z|^p)$ (growth bound)

(H3) $F - \ell V_p$ is QC (SWLSC + coercivity)

Metaprinciple

These assumptions **not only guarantee existence of minima**,

but also their **partial regularity.**

Many contributors: – among others Evans, Acerbi, Fusco, Pasarelli di Napoli, Carozza, Mingione, Kristensen, Duzaar, Schmidt, Diening, Fuchs, Stroffolini ... and **many**, **many** others

Differential conditions

- In applications, variational problems often depend on differential operators of maps:

$$v \mapsto \int_\Omega F(\mathbb{A}v)\,\mathrm{d}x,$$

where, for V, W finite dimensional vector spaces and $\mathbb{A}_j \colon V \to W$ linear,

$$\mathbb{A}v := \sum_{j=1}^{n} \mathbb{A}_j \partial_j v \qquad \text{for } v\colon \Omega \to V.$$

Example (The symmetric gradient)

$V = \mathbb{R}^n$, $W = \mathbb{R}^{n\times n}_{\mathrm{sym}}$, $\mathbb{A}v := \varepsilon(v) := \frac{1}{2}(Du + Du^\top)$

Example (The trace-free symmetric gradient)

$V = \mathbb{R}^n$, $W = \mathbb{R}^{n\times n}_{\mathrm{tf,sym}}$, $\mathbb{A}v := \varepsilon^D(v) := \varepsilon(u) - \frac{1}{n}\mathrm{div}(u)E_n$

$\rightsquigarrow$ Existence and regularity of minima?

The historical development

- **Idea:** $Q = (0,1)^n$, $T : Q \to \mathbb{R}^{N\times n}$ and $\operatorname{curl}(T) = 0 \Rightarrow T = \nabla u$.
- **General setup:** V, W, Z finite dimensional vector spaces, and $\mathbb{A}$, $\mathscr{A}$ are differential operators

$$\mathbb{A} = \sum_{j=1}^{n} \mathbb{A}_j \partial_j, \quad \mathscr{A} = \sum_{|\alpha|=k} \mathscr{A}_\alpha \partial^\alpha,$$

with $\mathbb{A}_j \colon V \to W$, $\mathscr{A}_\alpha \colon W \to Z$.

- We say that $\mathscr{A}$ is an **annihilator** for $\mathbb{A}$, and $\mathbb{A}$ is a **potential** for $\mathscr{A}$ if

$$V \xrightarrow{\mathbb{A}[\xi]} W \xrightarrow{\mathscr{A}[\xi]} Z \text{ is exact for any } \xi \in \mathbb{R}^n \setminus \{0\}.$$

- Based on Dacorogna (80s), Fonseca & Müller defined

$\mathscr{A}$-quasiconvexity

An integrand $F \colon W \to \mathbb{R}$ is called $\mathscr{A}$-quasiconvex provided

$$F(z) \leq \int_{(0,1)^n} F(z+\psi)\,\mathrm{d}x$$

holds for all $z \in W$, $\psi \in \mathrm{C}^\infty(\mathbb{T}^n; W)$ with $(\psi)_{(0,1)^n} = 0$ and $\mathscr{A}\psi = 0$.

Lower semicontinuity

Call $\mathscr{A}$ a **constant-rank operator** provided $\dim(\mathscr{A}[\xi](W))$ does not depend on $\xi \in \mathbb{R}^n \setminus \{0\}$.

Metatheorem a lá Fonseca & Müller SIAM '99

If F is $\mathscr{A}$-quasiconvex and of p-growth, the associated integral functional

$$v \mapsto \int_\Omega F(v)\,\mathrm{d}x$$

is weakly lower semicontinuous along sequences (v_j) with $\mathscr{A} v_j = 0$.

$\Downarrow$

Since '99 largerly preferred viewpoint: View gradients as curl-free fields

$\Downarrow$

The above results yields the existence of minima for functionals depending on $\mathbb{A}u$ (!). Is this the same?

$\Downarrow$

Put differenty, does any reasonable $\mathscr{A}$ possess a potential $\mathbb{A}$?

A paradigm shift: The Raita theorem

Theorem (Raita, Calc Var PDE '19)

Any constant rank operator $\mathscr{A}$ has a potential $\mathbb{A}$.

Upshot: The regularity theory for $\mathscr{A}$-quasiconvex problems can be fully reduced to that for functionals depending on $\mathbb{A}u$.

Aim: Partial regularity of (local) minima $u \in X$ of integral functionals

$$v \mapsto \int F(\mathbb{A}v)\,\mathrm{d}x$$

subject to

(H1) $F \in \mathrm{C}^2(W)$ (smoothness)

(H2) $|F(z)| \lesssim c(1+|z|^p)$ for all $z \in W$ (growth bound, $1 < p < \infty$)

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Main theorem

Theorem (Conti & FXG, '20)

Let $\mathbb{A}$ be an elliptic differential operator of order one and $F\colon W \to \mathbb{R}$ satisfy

(H1) $F \in \mathrm{C}^2(W)$,

(H2) $|F(z)| \lesssim c(1+|z|^p)$ *for all* $z \in W$ *(growth bound,* $1 < p < \infty$*)*,

(H3) $F - \ell V_p$ *is* $\mathscr{A}$*-QC.*

Then any local minimiser of the integral functional

$$v \mapsto \int F(\mathbb{A}v)\,\mathrm{d}x$$

is partially regular.

- higher order equally possible, here first order for simplicity
- completely resolves the matter of partial regularity.
- *partial* partial regularity for non-elliptic operators is not included here but is currently investigated.

Elliptic differential operators

Ellipticity a lá Spencer & Hörmander

Call a differential operator $\mathbb{A} := \sum_{j=1}^{n} \mathbb{A}_j \partial_j$ with $\mathbb{A}_j \colon V \to W$ **elliptic** provided

$$\mathbb{A}[\xi] = \sum_{j=1}^{n} \xi_j \mathbb{A}_j \colon V \to W \qquad \text{is injective for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

- For the partial regularity as above, this is a necessary requirement: If $\mathbb{A}$ is not elliptic, then

$$\exists \xi \in \mathbb{R}^n \setminus \{0\} \, \exists v \in V \setminus \{0\} \colon \; \mathbb{A}[\xi]v = 0,$$

and we consider the **plane waves** $w(x) := h(\langle x, \xi \rangle)v$.

$\rightsquigarrow$ crucial for the *partial regularity*.

- Partial regularity proofs are sometimes lengthy ...

...here we argue **via reduction**

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$\mathbb{A}$ versus ∇ – weighted Korn-type inequalities I

Theorem (Weighted Korn for elliptic operators)

Let $1 < p < \infty$ and let $\mathbb{A}$ be an elliptic operator of order one. Then for any $\omega \in A_p$, where

$$\omega \in A_p \Leftrightarrow [\omega]_{A_p} := \sup_Q \left(\fint_Q \omega \, \mathrm{d}x \right) \left(\fint_Q \omega^{1-p'} \, \mathrm{d}x \right)^{p-1} < \infty,$$

there exists $c = c(\mathbb{A}, p, [\omega]_{A_p}) > 0$ such that

$$\|\nabla u\|_{\mathrm{L}^p_\omega(\mathbb{R}^n)} \leq c \|\mathbb{A}u\|_{\mathrm{L}^p_\omega(\mathbb{R}^n)}$$

holds for all $u \in \mathrm{C}_c^\infty(\mathbb{R}^n; V)$.

- Write for $u \in \mathrm{C}_c^\infty(\mathbb{R}^n; V)$ and $j \in \{1, ..., n\}$:

$$\partial_j u = c_n \mathscr{F}^{-1}\Big(\underbrace{\xi_j (\mathbb{A}[\xi]^* \mathbb{A}[\xi])^{-1} \mathbb{A}[\xi]^*}_{= m_j(\xi)} \mathscr{F}[\mathbb{A}u] \Big)$$

$\longrightarrow$ $m_j \in \mathrm{C}^\infty(\mathbb{R}^n \setminus \{0\}; \mathscr{L}(W; V))$ and is homogeneous of degree zero.

$\longrightarrow$ Apply **Theorem of Mihlin-Hörmander / Calderón-Zygmund** + sharp weight bounds for such operators. ∎

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Let $\varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an N-function, i.e.,

- φ is right-continuous, differentiable, convex, and

$$\lim_{t\searrow 0} \frac{\varphi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty.$$

- φ is said to be Δ_2 if $\varphi(2t) \leq c\varphi(t)$ for all $t > 0$.
- φ is said to be ∇_2 if $\varphi^*(t) := \sup_{s>0} st - \varphi(s)$ is Δ_2.

Miracle of extrapolation – a lá Rubio de Francia

Let $1 < p < \infty$. If for any $\omega \in A_p$ there exists a constant $c = c(\ldots, p, [\omega]_p) > 0$ such that

$$\int_{\mathbb{R}^n} |f|^p \omega \,\mathrm{d}x \leq c \int_{\mathbb{R}^n} |g|^p \omega \,\mathrm{d}x \qquad \text{for all } (f,g) \in \mathcal{F}.$$

Then for any N-function $\varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ of class $\Delta_2 \cap \nabla_2$ there exists $c(p, \Delta_2(\varphi), \nabla_2(\varphi)) > 0$ such that

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A quick digression I

- Even for smooth Ω, ellipticity is **not sufficient** to yield the full Korn estimate

$$\|Du\|_{\mathrm{L}^p(\Omega)} \lesssim (\|u\|_{\mathrm{L}^p(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^p(\Omega)}) \qquad \text{for all } u \in \mathrm{C}^\infty(\overline{\Omega}; V). \qquad (\clubsuit)$$

- ($\clubsuit$) implies that

$$\mathrm{W}^{\mathbb{A},p}(\Omega) := \{v\colon\ \|v\|_{\mathrm{L}^p(\Omega)} + \|\mathbb{A}v\|_{\mathrm{L}^p(\Omega)} < \infty\} \approx \mathrm{W}^{1,p}(\Omega; V).$$

Peetre-Tartar Lemma

- $(X_1, \|\cdot\|_{X_1})$ Banach, $(X_2, \|\cdot\|_{X_2})$, $(X_3, \|\cdot\|_{X_3})$ normed spaces
- $A \in \mathscr{L}(X_1, X_2)$, and a compact $B \in \mathscr{L}(X_1, X_3)$,
- $\|x\|_{X_1} \simeq \|Ax\|_{X_2} + \|Bx\|_{X_3}$ for $x \in X_1$.

$\Longrightarrow$ $\dim(\ker(A)) < \infty$.

$\longrightarrow$ ($\clubsuit$) implies $\dim(\ker(\mathbb{A})) < \infty$.

A digression II

- By Smith '70, Kalamajska '94 and Breit, Diening & FXG '20:

$\dim(\ker(\mathbb{A}))<\infty \Longleftrightarrow \mathbb{C}$-ellipticity of $\mathbb{A}$

Call $\mathbb{A}$ $\mathbb{C}$-**elliptic** provided

$$\mathbb{A}[\xi]\colon V+\mathrm{i}V\to W+\mathrm{i}W \quad \text{is injective for all } \xi\in\mathbb{C}^n\setminus\{0\}.$$

Example (The trace-free symmetric gradient for $n=2$)

The operator $\varepsilon^D(u):=\varepsilon(u)-\frac{1}{2}\operatorname{div}(u)\mathbf{I}_{2\times 2}$ is elliptic, but not $\mathbb{C}$-elliptic:

$$\varepsilon^D(u)\overset{!}{=}0 \Longrightarrow \begin{cases}\partial_1 u_1 &= \partial_2 u_2\\ \partial_2 u_1 &= -\partial_1 u_2\end{cases} \qquad \text{Cauchy-Riemann!}$$

$\longrightarrow$ $\mathrm{W}^{\mathbb{A},p}(\Omega)\simeq \mathrm{W}^{1,p}(\Omega;V) \Longleftrightarrow \mathbb{A}$ is $\mathbb{C}$-elliptic.

but: $\mathrm{W}^{\mathbb{A},p}_{\mathrm{loc}}(\Omega)=\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega;V) \Longleftrightarrow \mathbb{A}$ is elliptic, since

$$\int_{\mathrm{B}_r}|Du|^p\,\mathrm{d}x\leq\int_{\mathrm{B}_R}|D(\rho u)|^p\,\mathrm{d}x\lesssim\int_{\mathrm{B}_R}|\mathbb{A}(\rho u)|^p\,\mathrm{d}x\lesssim\int_{\mathrm{B}_R}|u\otimes_{\mathbb{A}}\nabla\rho|^p+|\rho\mathbb{A}u|^p\,\mathrm{d}x.$$

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A digression III

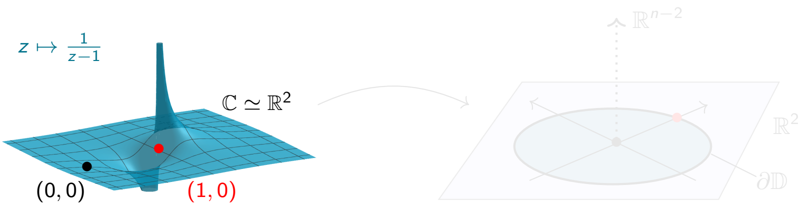
$\mathbb{A}$ not $\mathbb{C}$-elliptic

↓

$\mathbb{A}$ contains a copy of the two-dimensional ε^D

↓

shift singularity in $\mathbb{C} \simeq \mathbb{R}^2$ along $\{0\} \times \mathbb{R}^{n-2}$



$u \in \mathrm{W}^{\mathbb{A},p}(\mathbb{D} \times (-1,1)^{n-2})$ for $1 \leq p < 2$, $\int_{\partial\mathbb{D}(0,1)\times(-1,1)^{n-2}} |u| \, \mathrm{d}\mathscr{H}^{n-1} = +\infty$

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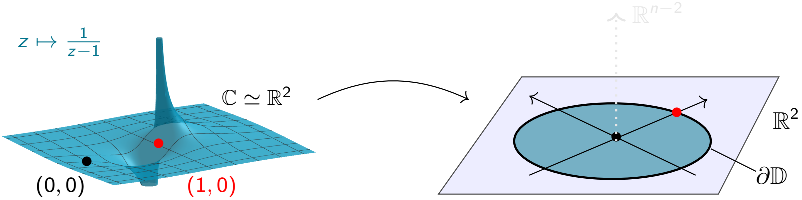

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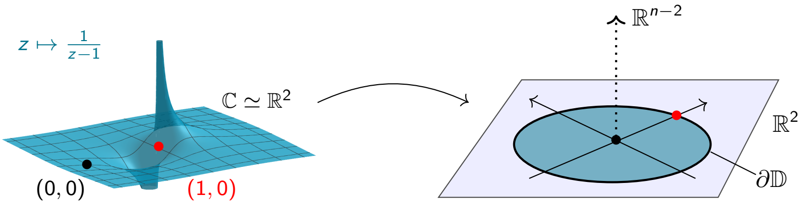

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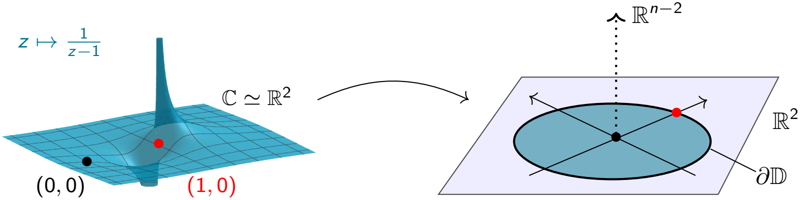
↓

shift singularity in $\mathbb{C} \simeq \mathbb{R}^2$ along $\{0\} \times \mathbb{R}^{n-2}$



$u \in \mathrm{W}^{\mathbb{A},p}(\mathbb{D} \times (-1,1)^{n-2})$ for $1 \leq p < 2$, $\int_{\partial\mathbb{D}(0,1)\times(-1,1)^{n-2}} |u| \, \mathrm{d}\mathscr{H}^{n-1} = +\infty$

Proof outline

The essential cone and span of $\mathbb{A}$

For a differential operator $\mathbb{A}$, define

$$v \otimes_{\mathbb{A}} \xi := \sum_{j=1}^{n} \xi_j \mathbb{A}_j v, \qquad v \in V, \, \xi \in \mathbb{R}^n.$$

We then define the

- **essential cone** by $\mathscr{C}(\mathbb{A}) := \{v \otimes_{\mathbb{A}} \xi\colon \; v \in V, \xi \in \mathbb{R}^n\}$.
- **essential span** by $\mathscr{R}(\mathbb{A}) := \mathrm{span}(\mathscr{C}(\mathbb{A})) \subset W$.

Upshot: If $N := \dim(V)$, then $\mathscr{R}(\mathbb{A}) \hookrightarrow \mathbb{R}^{N \times n}$.

$\longrightarrow$ upon identification, we may assume that $W = \mathscr{R}(\mathbb{A}) \subset \mathbb{R}^{N \times n}$.

For $F\colon W \to \mathbb{R}$ $\mathscr{A}$-quasiconvex, now define

$$G(z) := F(\Pi_{\mathbb{A}}(z)), \qquad z \in \mathbb{R}^{N \times n},$$

with $\Pi_{\mathbb{A}}\colon \mathbb{R}^{N \times n} \to \mathscr{R}(\mathbb{A})$ such that $\Pi_{\mathbb{A}}[\nabla v] = \mathbb{A} v$.

The case $p \geq 2$: Properties of $G = F \circ \Pi_{\mathbb{A}}$

(H1') $G \in C^2$ if $F \in C^2$.

(H2') $|G(z)| \lesssim (1 + |z|^p)$ since F satisfies this estimate.

(H3') As a consequence of the p-strong $\mathscr{A}$-quasiconvexity, with $Q = (0,1)^n$,

$$\nu \int_Q (1 + |z|^2 + |\mathbb{A}\varphi|^2)^{\frac{p-2}{2}} |\mathbb{A}\varphi|^2 \,dx \leq \int_Q F(z + \mathbb{A}\varphi) - F(z)\,dx.$$

Thus with $\phi(t) := t^2 + t^p$,

$$\int_Q |D\varphi|^2 + |D\varphi|^p \,dx \lesssim \int_Q |\mathbb{A}\varphi|^2 + |\mathbb{A}\varphi|^p \,dx$$

$$\lesssim \int_Q F(\Pi_{\mathbb{A}}(z) + \mathbb{A}\varphi) - F(\Pi_{\mathbb{A}}(z))\,dx \lesssim \int_Q G(z + D\varphi) - G(z)\,dx$$

A note on $1 < p < 2$

More intricate, hinges on Diening's shifted ϕ-functions and

$$\int_Q (1 + |z|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 \,dx \lesssim \int_Q \underbrace{(1 + |\Pi_{\mathbb{A}}(z)|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2}_{\simeq \phi_{|\Pi_{\mathbb{A}}(z)|}(D\varphi)} \,dx$$

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The case $p \geq 2$ – and comments on the general case

Theorem (Partial regularity a lá Acerbi & Fusco)

If $G \in \mathrm{C}^2(\mathbb{R}^{N\times n})$ satisfies (H1'), (H2') and

$$\int_{(0,1)^n} |D\varphi|^2 + |D\varphi|^p \,\mathrm{d}x \leq c \int_{(0,1)^n} G(z + D\varphi) - G(z)$$

holds for all $z \in \mathbb{R}^{N\times n}$ and $\varphi \in \mathrm{C}_c^\infty((0,1)^n;\mathbb{R}^N)$, then any local minimiser $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ of the integral functional

$$v \mapsto \int G(Dv)\,\mathrm{d}x$$

is partially regular.

$\longrightarrow$ Since $\mathrm{W}^{\mathbb{A},p}_{\mathrm{loc}} = \mathrm{W}^{1,p}_{\mathrm{loc}}$, this concludes the proof in this growth regime. ■

- For $1 < p < 2$, invoke e.g. Carozza, Fusco & Mingione.

Some final comments

- Some results for (p,q)-growth, equally possible:

Theorem (Kristensen & FXG, to appear soon, generalising Schmidt '09)

Let $1 < p \leq q < \min\{\frac{np}{n-1}, p+1\}$ and let $F \in \mathrm{C}^{\infty}(\mathbb{R}^{N\times n})$ be an integrand that

- *is of q-growth: $0 \leq F(z) \leq c(1+|z|^q)$ for all $z \in \mathbb{R}^{N\times n}$,*
- *is p-strongly quasiconvex, i.e., $F - \ell V_p$ is quasiconvex.*

Then any weak local minimiser of the $\mathrm{W}^{1,p}_{\mathrm{loc}}$-relaxed functional

$$\overline{\mathscr{F}}[u,\omega] := \inf\left\{ \liminf_{j\to\infty} \int_{\omega} F(Du_j)\,\mathrm{d}x \colon \begin{array}{c} (u_j) \subset (\mathrm{W}^{1,q}_{\mathrm{loc}} \cap \mathrm{W}^{1,p})(\omega;\mathbb{R}^N) \\ u_j \rightharpoonup u \text{ in } \mathrm{W}^{1,p}(\omega;\mathbb{R}^N) \end{array} \right\}$$

is C^{∞}-partially regular.

- As above: This theorem 'self-improves' to a partial regularity result in the $\mathbb{A}$-framework.

Thank You! – & References

Many thanks for your attention!

References

- F.G. – J. Math. Pures Appl., 2020:
  Partial regularity for symmetric quasiconvex functionals on BD.
- S. Conti & F.G. – Preprint, 2020:
  $\mathscr{A}$-quasiconvexity and partial regularity
- F.G & J. Kristensen – in preparation:
  Partial regularity for quasiconvex functionals with (p, q)-growth

Thank You! – & References

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References

F.G. – J. Math. Pures Appl., 2020:
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