Regularity results for scalar integrals with p, q and general slow growth conditions

Cantor, Doctoral thesis (1867): In re mathematica ars proponendi quaestionem pluris facienda est quam solvendi..... ....In mathematics the art of proposing a question must be held of higher value than solving it.....

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Monday's Nonstandard Seminar

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Quasilinear Elliptic Equations

Quasi-Linear Elliptic Equations

$$\sum_{i=1}^{n} D_{x_i}\left(A_i(x,u,Du)\right) = B(x,u,Du)$$

- $\Omega \subset \mathbb{R}^n$ open bounded, $n \geq 2$
- $A_i : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$

Particular case: variational integrals

$$\mathcal{F}(u) = \int_\Omega f\left(x, u(x), Du(x)\right)\, dx$$

Minimizers of I satisfy the **Euler-Lagrange equation**

$$\sum_{i=1}^{n} D_{x_i}\left(f_\xi\left(x, u(x), Du(x)\right)\right) = f_s\left(x, u(x), Du(x)\right)$$

which is elliptic if f is **convex** with respect to Du

Weak Solutions

$$\sum_{i=1}^{n} D_{x_i}\left(A_i(x,u,Du)\right) = B(x,u,Du)$$

$u \in W^{1,1}(\Omega;\mathbb{R}^m)$ is a **distributional solution** if

$$\sum_i \int_\Omega \boxed{A_i(x,u,Du)}\,\varphi_{x_i}\,dx + \int_\Omega \boxed{B(x,u,Du)}\,\varphi\,dx = 0$$

for all test function $\boxed{\varphi \in C_0^\infty(\Omega)}$

The definition of **weak solution** leads to assign growth assumptions on A_i and B (counterexample Serrin, 1964)

Regularity of weak solutions

Given u, a weak solution or a local minimizer of an integral functional, which are the additional regularity properties of u in the interior of Ω?

Historical Notes: De Giorgi theorem, 1957

Linear elliptic equation (and quadratic functionals)

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x)\, u_{x_j} \right) = 0$$

- (a_{ij}) symmetric, $a_{ij}(x) \in L^\infty$
- $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2, \; \nu > 0$

every weak solution $\mathbf{u} \in \mathbf{W^{1,2}}$ is locally Hölder continuous

The proof relies on a sophisticated application of the maximum principle

Historical Notes: Some generalization

- Stampacchia 1958-1960
- Serrin 1964-1965 complete analysis in nonlinear case
- Ladyzhenskaya-Ural'tseva 1968, papers and book
- Giaquinta-Giusti, 1983-1985: Quasi-minimizer

$$\mathcal{F}(u,\Omega) = \int_\Omega f(x,u,Du)\,dx,$$

u is a quasi-minimizer when for every $\varphi \in W^{1,1}(\Omega)$ with $\operatorname{supp}\varphi \Subset \Omega$

$$\mathcal{F}(u, \operatorname{supp}\varphi) \le Q\,\mathcal{F}(u, \operatorname{supp}\varphi), \quad Q \ge 1.$$

under p-growth conditions

$$\boxed{|Du|^p \le f(x,u,Du) \le C(1+|Du|)^p, \;\; p > 1},$$

No Euler-Lagrange equation is available

Integral functional $I(u) = \int_\Omega f(x, u, Du)\, dx$

The direct methods and the regularity theory generally works when f satisfies p growth

$$\boxed{|Du|^{\mathbf{p}} \leq f(x, u, Du) \leq C(1 + |Du|)^{\mathbf{p}}, \;\; p > 1}$$

There are many integral functionals (and the related Euler-Lagrange equations) which satisfy $\boxed{p, q}$ growth

$$\boxed{|Du|^{\mathbf{p}} \leq f(x, u, Du) \leq C(1 + |Du|)^{\mathbf{q}}, \;\; 1 < p < q}$$

p, q, anisotropic and general growth

- small perturbation of polynomial growth

$$f(\xi) = |\xi|^{p} \log^{\alpha}(1 + |\xi|), \qquad p \geq 1, \ \alpha > 0$$

- anisotropic growth

$$f(\xi) = \sum_i |\xi_i|^{p_i} \qquad p_i \geq p, \ \forall i = 1, \ldots, n$$

- double phase functional

$$f(x, \xi) = |\xi|^{p} + a(x)|\xi|^{q}, \qquad 0 \leq a(x)$$

Model for strongly anisotropic materials Zhikov,1986

p, q, anisotropic and general growth

- variable exponents

$$f(x,\xi) = |\xi|^{\mathbf{p(x)}}, \quad f(x,\xi) = [h(|\xi|)]^{\mathbf{p(x)}}, \quad p \le p(x) \le q$$

Model proposed for electrorheological fluids Rajagopal- Růžička 2001, image denoising Chen et al. 2006, growth of heterogeneous sandpiles Bocea et al. 2015

- anisotropic variable exponents

$$f(x,\xi) \sim \sum_{i=1}^{n} |\xi_i|^{p_i(x)}, \quad \xi_i = (\xi_i^1, \ldots.., \xi^m), \quad p \le p_i(x) \le q$$

- exponential growth

$$f(x,\xi) \sim e^{|\xi|^{p(x)}}$$

Model proposed in the **gas reaction** Aris, 1975 and **combustion theory** Mosely, 1983

- general growth: There exist h_1 and h_2 convex (no power functions) such that

$$h_1(|\xi|) - c_1 \leq f(x,\xi) \leq c_2(1 + h_2(|\xi|))$$

and/or there exist g_1 and g_2 such that

$$\boxed{g_1(|\xi|)\,|\lambda|^2 \leq \sum_{i,j=1}^{n} f_{\xi_i\xi_j}(x,\xi)\,\lambda_i\lambda_j \leq g_2(|\xi|)\,|\lambda|^2\,, \quad \forall\, \lambda,\xi \in \mathbb{R}^n}$$

Control of the quadratic form of the second derivatives of f

Regularity under p, q, anisotropic and general growth

p, q growth (eventually anisotropic)

- To get regularity the gap $\frac{p}{q}$ cannot differ to much from 1 and q and p constrained together by a condition depending on n

$$\frac{q}{p} \leq c(n) \rightarrow 1 \text{ as } \; n \rightarrow \infty$$

Counterexamples Marcellini, 1989, 1991

Survey on the regularity under non standard growth, Mingione 2006:
Regularity of minima: an invitation to the dark side of the calculus of variations

Non complete list of references

- Marcellini, 1989, 1991, 1993, 1996, ...
- Mascolo-Papi, 1994, 1996
- Esposito-Leonetti-Mingione 1999, 2002, 2004
- Dall'Aglio-Mascolo, 1998 [$f = g(x, |Du|)$, bounded solution]
- Coscia-Mingione 1999 [$|Du|^{p(x)}$]
- Acerbi-Mingione 2000-2001
- Mascolo-Migliorini 2003 [$g(x, |Du|) \simeq e^{|Du|^{p(x)}}$]
- Leonetti-Mascolo-Siepe 2001, 2003
- Bildhauer-Fuchs (et al.) 2002, 2003
- Cupini-Guidorzi-Mascolo 2003 [Local Lipschitz continuity]
- Marcellini-Papi 2006

Some recent results: $f = f(x, Du)$

- Colombo-Mingione 2014, 2015 [$f(x,\xi) = |\xi|^{\mathbf{p}} + a(x)|\xi|^{\mathbf{q}}$]
- Baroni-Colombo-Mingione 2014, 2015, 2018 [Functionals with double phase]
- Cupini-Marcellini-Mascolo 2015, 2017, 2018 [Local Boundedness]
- Bögelein-Duzaar-Marcellini 2013, 2014, 2015 [Parabolic equations and systems under p, q growth]
- Cupini-Leonetti-Mascolo 2015, 2017 + Focardi 2019 [Polyconvex functionals+ Quasi convex]
- Eleuteri-Marcellini-Mascolo 2016, 2019 [Sobolev dependence]
- Mingione-Beck, 2020 [$f = f(x, u, Du)$ Lipschitz and local estimates for general nonuniformly elliptic integrals]
- Mingione-De Filippis, 2020 [$f = f(x, u, Du)$ Lipschitz regularity for vector-valued nonautonomous variational problems]
- De Marco-Marcellini, 2020 [Elliptic systems under either slow or fast growth conditions]

p, q-growth: **some recent results**: $f = f(x, Du)$

- ..... **many others authors and papers**
- Cupini-Marcellini-Mascolo-Passarelli, 2021: Lipschitz regularity for degenerate elliptic integrals $2 \leq p \leq q$:

$$\boxed{a(x)\,(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \leq \langle f_{\xi\xi}(x,\xi)\lambda,\lambda\rangle \leq L\,(1+|\xi|^2)^{\frac{q-2}{2}}|\lambda|^2}$$

- Eleuteri-Marcellini-Mascolo 2020: Scalar integrals with Sobolev dependence and without structure conditions.
- Eleuteri-Marcellini-Mascolo-Perrotta 2021: Energy integrals with general slow growth.

In $f = f(x, Du)$ the presence of the x-variables **cannot** be treated as a simple perturbation and the regularity problem is not yet fully resolved

still open issues

Scalar case: Lipschitz continuity

$$\mathcal{F}(u) = \int_\Omega f(x, Du)\, dx, \quad u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$$

- $f_{\xi\xi}$, $f_{\xi x}$ Carathéodory functions
- $M_1\, |\xi|^{p-2} |\,\lambda|^2 \leq \sum_{i,j} f_{\xi_i \xi_j}(x, \xi)\lambda_i \lambda_j,$ $\boxed{|\xi| \geq M_0}$
- $|f_{\xi\,\xi}(x, \xi)| \leq M_2\, |\xi|^{q-2},$ $\boxed{|\xi| \geq M_0}$, $1 < p < q$
- $|f_{\xi,x}(x\,\xi)| \leq h(x)\, |\xi|^{q-1},$ $h \in L^r(\Omega)$, $r > n$ $\boxed{|\xi| \geq M_0}$

Examples: no structure conditions

- $f(x, \xi) = |\xi|^p + c(x)\, |\xi|^s + |\xi_n|^q$, $s \leq \frac{p+q}{2}$
- $f(x, \xi) = a(x)\, |\xi|^p + |\xi_n|^q$

Asymptotically regular problems

We consider assumptions only at infinity: the assumptions of convexity, structure and further properties are made only for $|\xi| \to +\infty$

- Chipot- Evans 1986
- Fonseca-Fusco- Marcellini 2002 [Applications to non convex problems]
- Cupini- Guidorzi- Mascolo 2003
- Kristensen-Taheri 2003
- Celada- Cupini- Guidorzi 2007
- Leone- Passarelli di Napoli- Verde 2007
- Foss- Passarelli di Napoli- Verde (2008), 2011
- Diening- Stroffolini- Verde 2011
- Cupini- Giannetti-Giova Passarelli di Napoli 2016, 2017
- Eleuteri- Marcellini- Mascolo 2016, 2017, 2018

Eleuteri-Marcellini-Mascolo 2020

A-priori estimate: Let u be a **smooth** local minimizer and

$$\boxed{\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}}$$

then, there exist positive constants C, β, γ such that

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)} \leq C \left(\|1+h\|_{L^r(\Omega)}\right)^{\beta\gamma} \left(\int_{B_R} \{1 + |Du|^p\}\, dx\right)^{\frac{\gamma}{p}}$$

Bound on p, q, r, n Eleuteri-Marcellini-Mascolo, 2016

First we obtain an a-priori estimate with the $W^{1,q}(\Omega)$-norm of u then by interpolation arguments we get the estimate with the norm in $W^{1,p}(\Omega)$ Marcellini, 1991

Function test: $\boxed{\Phi \simeq \Phi((|Du| - M_0)_+)}$

- By the growth assumption $\mathcal{F}$ is defined if $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ but not uniquely defined for $u \in W^{1,p}(\Omega) \setminus W^{1,q}(\Omega)$.
- The lack of structure and the x-dependence, do not exclude

Lavrentiev phenomenon

$$\inf\left\{\mathcal{F}(w), w \in u_0 + W^{1,p}(\Omega)\right\} < \inf\left\{\mathcal{F}(w), w \in u_o + W^{1,q}(\Omega)\right\}$$

for datum u_0 (even smooth)

Lavrentiev phenomenon: not complete list of references

- Zhikov 1995,
- Belloni-Buttazzo 1995
- Esposito-Leonetti-Mingione 2002
- Esposito-Leonetti-Petricca 2019
- De Filippis-Mingione 2019

Extension of the integral energy

Let $u_0 \in W_0^{1,q}(\Omega)$, for $u \in W_0^{1,p}(\Omega)$ define

$$\mathcal{I}(u) = \inf\left\{\liminf_k \mathcal{F}(u_k),\ \ u_k \in W_0^{1,q}(\Omega) + u_0,\ \ u_k \overset{w}{\rightharpoonup} u \text{ in } W^{1,p}(\Omega)\right\}$$

Extension

- Serrin 1959: pionieristic paper
- Ioffe, Ioffe-Timorov 1968: definition of extension
- De Giorgi-Franzoni 1975: $\Gamma-$convergence

For each $v \in W_0^{1,p}(\Omega) + u_0$, there exists $v_k \in W_0^{1,q}(\Omega) + u_0$ such that $v_k \overset{w}{\rightharpoonup} v$ in $W^{1,p}(\Omega)$ and

$$\mathcal{I}(v) = \lim_{k\to+\infty} \mathcal{F}(v_k)$$

Eleuteri-Marcellini-Mascolo 2020

Existence and regularity

Let $\mathcal{I}$ be the extension of $\mathcal{F}$ and $u_0 \in W^{1,q}(\Omega)$, then there exists at least one locally Lipschitz continuous solution of the Dirichlet problem:

$$\min\{\mathcal{I}(u) : u \in W_0^{1,p}(\Omega) + u_0\}$$

No Lavrentiev phenomenon

$$f(x, Du) = \sum_{i=1}^{N} a_i(x)\, g_i(Du)$$

with $a_i(x) > 0$ a.e., $a_i \in W^{1,r}(\Omega)$, $r > n$, g_i convex and strictly convex for $|Du| \geq M_0$

Treat sepately the variables x and Du and apply on each $g_i = g_i(Du)$ the Jensen inequality: every local minimimizers is locally Lipschitz continuous

General growth

$$\boxed{\mathcal{F}(u) = \int_\Omega f(Du(x))\,dx, \quad u : \Omega \to \mathbb{R}}$$

Control of the quadratic form of of the second derivatives

$$g_1(|\xi|)\,|\lambda|^2 \le \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(\xi)\,\lambda_i \lambda_j \le g_2(|\xi|)\,|\lambda|^2\,, \quad \forall\, \lambda, \xi \in \mathbb{R}^n.$$

$$g_1, g_2 : [0, +\infty) \to [0, +\infty)$$

- If g_1 is positive and there exists a constant $M \ge 1$ such that

$$g_2(t) \le M\, g_1(t) \text{ for all } t \in [0, +\infty)$$

we are dealing with an **uniformly elliptic** problem.

General growth

Particular case: $f(\xi) = g(|\xi|)$

$$g_1(t) = \min\left\{g''(t), \frac{g'(t)}{t}\right\}, \quad g_2(t) = \max\left\{g''(t), \frac{g'(t)}{t}\right\}$$

- Complete analysis in Marcellini-Papi, 2006

$p-$Laplacian

$$f(\xi) = |\xi|^p, \; p > 1$$

$$g(t) = t^p \implies \frac{g'(t)}{t} = p\,t^{p-2} = \frac{1}{p-1}\,g''(t)$$

then the energy integrand is uniformly elliptic.

Eleuteri-Marcellini-Mascolo-Perrotta, 2021

$f = f(\xi)$: general **slow** growth

- $f \in \mathcal{C}(\mathbb{R}^n) \cap \mathcal{C}^2(\mathbb{R}^n - B_{M_0}(0))$, $M_0 \geq 0$, f convex;
- Control of the quadratic form for $|\xi| \geq M_0$ and $\lambda \in \mathbb{R}^n$

$$g_1(|\xi|)\,|\lambda|^2 \leq \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(\xi)\,\lambda_i\lambda_j \leq g_2(|\xi|)\,|\lambda|^2$$

- g_1, g_2 continuous function such that:

$$g_2 \text{ decreasing and } t\,g_2 \text{ increasing}$$

- Connection with g_1 and g_2: for $t \geq M_0$

$$(g_2(t))^{\frac{n-2}{n}} \leq C_1\, t^{2\beta} g_1(t)\,, \quad \frac{1}{n} < \beta < \frac{2}{n}$$

- Connection with g_1, g_2 and f : for $t \geq M_0$:

$$g_2(|\xi|)\,|\xi|^2 \leq C_2\,[1 + f(\xi)]^{\alpha} \quad \alpha > 1$$

Regularity Theorem

$$\mathcal{F}(u) = \int_\Omega f(Du(x))\, dx, \quad u \subset \mathbb{R}^n : \Omega \to \mathbb{R}$$

Eleuteri-Marcellini-Mascolo-Perrotta, 2021

Assume

$$\boxed{2 - \alpha(n\beta) > 0}$$

$$\boxed{f(\xi)/|\xi| \to +\infty \text{ as } |\xi| \to \infty}$$

Then local minimizer $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ of $\mathcal{F}$ is **locally Lipschitz continuous**

Interpolation Lemma and *a-priori estimate*

Interpolation Lemma

Let $v \in L^{\infty}_{\mathrm{loc}}(\Omega)$ and assume that for $\vartheta \geq 1$, $c > 0$ and $0 < \rho < R$

$$\|v\|^{\frac{1}{\vartheta}}_{L^{\infty}(B_{\varrho})} \leq \frac{c}{(R-\varrho)^{n}} \int_{B_R} |v| \, dx$$

Then, for every $\lambda \in \left(\frac{\vartheta-1}{\vartheta}, 1\right)$ there exists c_{λ} such that

$$\|v\|^{\frac{1-\vartheta(1-\lambda)}{\vartheta}}_{L^{\infty}(B_{\varrho})} \leq \frac{c_{\lambda}}{(R-\varrho)^{n}} \int_{B_R} |v|^{\lambda} \, dx.$$

- For **regular** minimizers of $\mathcal{F}$ we prove an **a-priori estimate** of the L^{∞}-norm of the gradient

$$\|Du\|_{L^{\infty}(B_{\rho})} \leq C \left\{ \frac{1}{(R-\rho)^{n}} \int_{B_R} (1 + f(Du)) \, dx \right\}^{\tau}$$

with C and τ depending only on the costants in the assumptions.

Approximation methods

- Construct a sequence of smooth strictly convex functions f_k such that $f_k \simeq f$ for large $|\xi|$

- Let u be a local minimizer of $\mathcal{F}$ and we consider in $\overline{B}_R \subset\subset \Omega$ the variational problems:

$$\inf \left\{ \int_{B_R} f_k(Dv)\, dx, \quad v = u_\epsilon \text{ on } \partial B_R \right\}$$

where $u_\epsilon = u * \varphi_\varepsilon$, where φ_ε are smooth mollifiers

- Since u_ϵ satisfies the *bounded slope condition*, the well know regularity result by Hartman-Stampacchia, 1966 ensures that each approximate problem has an unique Lipschitz continuous solution $v_{k,\epsilon}$.

Approximation methods

- By applying the *a-priori estimate* to each $v_{k,\epsilon}$ we obtain a L^∞−local bound of $Dv_{k,\epsilon}$ independent of k and ϵ

- Passing to the limit: $v_{k,\epsilon}$ converges in $W_0^{1,1}(\Omega) + u$ and in $W^{1,\infty}_{\text{loc}}(\Omega)$ to a Lipschitz continuous local minimizer v of $\mathcal{F}$

- The strict convexity of f for $|\xi| \geq M_0$ allows to transfer the Lipschitz property to the original minimizer u.

Corollary: subquadratic $p, q-$growth

Ellipticity conditions with $1 < p \leq q \leq 2$

Assume

- for $\lambda, \xi \in \mathbb{R}^n$, $|\xi| \geq M_0$

$$|\xi|^{p-2}|\lambda|^2 \leq \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(\xi)\lambda_i \lambda_j \leq C\left(1+|\xi|^2\right)^{\frac{q-2}{2}} |\lambda|^2$$

- with p, q such that $\boxed{\frac{q}{p} < 1 + \frac{2}{n}}$

Every local minimizer $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is local Lipschitz continuous and

$$\|Du\|_{L^{\infty}(B_\rho;\mathbb{R}^n)} \leq C\left\{\frac{1}{(R-\rho)^n}\int_{B_R}\{1+f(Du)\}\,dx\right\}^{\frac{2}{(n+2)p-nq}} .$$

Examples: Logaritmic behaviour

- $f(\xi) = |\xi|(\log|\xi|)^a, \; a > 0, \quad |\xi| \geq t_0 \geq 1.$

$$g_1(t) = \frac{a}{2}\frac{(\log t)^{a-1}}{t}, \;\; g_2(t) = (1+a)\frac{(\log t)^a}{t}$$

- $f(\xi) = (|\xi| + 1)L_k(|\xi|)$ where

$$L_1(t) = \log(1+t), \qquad L_{k+1}(t) = \log(1 + L_k(t))$$

Mingione-Siepe, 1999, Fuchs-Mingione, 2002

Anisotropic growth

$$\boxed{f(\xi) = \sum_{i=1}^{n} |\xi_i|^{p_i}, \qquad \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n},$$

the quadratic form

$$\sum_{i,j=1}^{n} f_{\xi_i \xi_j}(\xi)\, \lambda_i \lambda_j = \sum_{i=1}^{n} p_i (p_i - 1) |\xi_i|^{p_i - 2} |\lambda_i|^2, \quad \forall\, \lambda, \xi \in \mathbb{R}^n.$$

is not positive definite

- Bousquet-Brasco 2019 proved that bounded minimizers are locally Lipschitz continuous under the condition $p_i \geq 2$ for all $i = 1, 2, \ldots, n$

Results of bounded minimizers for $p_i > 1$

- Fusco-Sbordone 1993
- Cupini-Marcellini-Mascolo 2009, 2017, 2018
- DiBenedetto-Gianazza-Vespri, 2016

Anisotropic case: No-singular model

$$\boxed{f(\xi) = \sum_{i=1}^{n} \left(1 + |\xi_i|^2\right)^{\frac{p_i}{2}}, \quad 1 < p_i < 2 \qquad \forall\, i = 1, \ldots, n}$$

$$g_1(|\xi|)\,|\lambda|^2 \le \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(\xi)\,\lambda_i \lambda_j \le g_2(|\xi|)\,|\lambda|^2, \quad \forall\, \lambda, \xi \in \mathbb{R}^n$$

with

$$g_1(t) = p(p-1)\left(1 + t^2\right)^{\frac{p-2}{2}}, \;\; p =: \min p_i, \quad g_2(t) = 2$$

The regularity theorem gives the local Lipschitz minimizers when

$$\boxed{\frac{2}{p} < 1 + \frac{2}{n} \;\;\Leftrightarrow\;\; p > \frac{2n}{n+2}}$$

New examples of anisotropic energy density

$$f(\xi) = \sqrt{\sum_{i=1}^{n} \left(1 + |\xi_i|^2\right)^{p_i}}, \quad 1 < p_i, \quad \forall\, i = 1, \ldots, n$$

Growth of the function f: $p = \min p_i$, $q = \max p_i \leq 2$

$$\frac{1}{\sqrt{n^p}} \left(1 + |\xi|^2\right)^{\frac{p}{2}} \leq f(\xi) \leq \sqrt{n} \left(1 + |\xi|^2\right)^{\frac{q}{2}},$$

Control of the quadratic form

$$\sum_{i,j=1}^{n} f_{\xi_i \xi_i}(\xi)\, \lambda_i \lambda_j \geq c|\xi|^{2p-q-2}|\lambda|^2$$

$$\sum_{i,j=1}^{n} f_{\xi_i \xi_i}(\xi)\, \lambda_i \lambda_j \leq C|\xi|^{\frac{p}{q}(q-2)}|\lambda|^2$$

New examples of anisotropic energy functions

Let

$$r = 2p - q \quad \text{and} \quad s = \frac{p}{q}(q-2) + 2$$

we obtain

$$m\,|\xi|^{r-2}\,|\lambda|^2 \leq \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(\xi)\,\lambda_i\,\lambda_j \leq M\left(1 + |\xi|^2\right)^{\frac{s-2}{2}}\,|\lambda|^2$$

and then the functions g_1 and g_2 are

$$g_1(t) = c\,t^{r-2} \quad \text{and} \quad g_2(t) = C\,t^{s-2}$$

Regularity Theorem

$$\mathcal{F}(u) = \int_{\Omega} \sqrt{\sum_{i=1}^{n} \left(1 + |u_{x_i}|^2\right)^{p_i}}\, dx$$

By the assumptions on α and β in the regularity result we get the local Lipschitz continuous when

$$\boxed{\frac{q}{p} < 1 + \frac{2}{n} - 2\left(\frac{1}{p} - \frac{1}{q}\right)} \iff \boxed{s < r + \frac{2}{n}p}$$

If we apply the corollary under the r, s-growth without taking in account p and q the bound is

$$s < r + \tfrac{2}{n} r$$

which is a more restrictive condition with respect to

$$s < r + \tfrac{2}{n} p \quad \text{since} \quad r < p$$

More examples

$$F(u) = \int_\Omega |Du|^p + \sqrt{\sum_{i=1}^{n} |u_{x_i}|^{2\,p_i}}\, dx.$$

Regularity result applies when

$$\boxed{\frac{q}{p} < 1 + \frac{q}{n} \quad \Longleftrightarrow \quad q < p^* =: \frac{np}{n-p}}$$

$$f(\xi) = |\xi|(\log|\xi|)^a + \sqrt{\sum_{i=1}^{n-1} |\xi_i|^2 + |\xi_n|^{2q}} \quad a > 0, \quad 1 < q \le 2$$

$$f(\xi) = |\xi| L_k(|\xi|) + \sqrt{\sum_{i=1}^{n-1} |\xi_i|^2 + |\xi_n|^{2q}}, \quad 1 < q \le 2$$

The last and/or the first slide

J. Serrin, The solvability of boundary value problems: Hilbert's Problem 19, Proc. of Symposia in Pure Mathematics, Providence 197

.....what in 1900 was a shy branch has blossomed in the twentieth century and developed in ways that Hilbert could never have imagined and now covers such a vast area of researches that just a few years ago would have seemed amazing ...

Ennio De Giorgi

...Un bel problema, anche se non lo risolvi, ti fa compagnia se ci pensi ogni tanto.....

..... A nice problem, even if you don't solve it, it keeps you company when you think of it every now and then .....