Bounded weak solutions to elliptic PDE with data in Orlicz spaces

David V. Cruz-Uribe, OFS

University of Alabama

The Nonstandard Seminar

Acknowledgments

Joint work with Scott Rodney, Cape Breton University

Basic notation

Hereafter:

- Ω bounded, open, connected set in $\mathbb{R}^n$
- $Q = Q(x)$ $n \times n$, positive semidefinite, measurable matrix function
- v a weight: non-negative, measurable function
- $Lip_0(\Omega)$ Lipschitz functions with compact support in Ω

Trudinger's theorem

Theorem (Trudinger, 1973)

Let $f \in L^q(\Omega)$, $q > \frac{n}{2}$, Q uniformly elliptic, and let u be a non-negative weak (sub)solution of

$$\begin{cases} -\operatorname{Div}(Q\nabla u) = f & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Then

$$\|u\|_{L^\infty(\Omega)} \leqslant C\|f\|_{L^q(\Omega)}.$$

Trudinger's theorem

Theorem (Trudinger, 1973)

Let $f \in L^q(\Omega)$, $q > \frac{n}{2}$, Q uniformly elliptic, and let u be a non-negative weak (sub)solution of

$$\begin{cases} -\operatorname{Div}(Q\nabla u) = f & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Then

$$\|u\|_{L^\infty(\Omega)} \leqslant C\|f\|_{L^q(\Omega)}.$$

The proof

Standard proof (e.g. Gilbarg & Trudinger) uses Moser iteration.

Critical exponent $\frac{n}{2} = \left(\frac{n}{n-2}\right)'$ is dual of the "gain" in classical Sobolev inequality.

The proof

Standard proof (e.g. Gilbarg & Trudinger) uses Moser iteration.

Critical exponent $\frac{n}{2} = \left(\frac{n}{n-2}\right)'$ is dual of the "gain" in classical Sobolev inequality.

Sharpness

Theorem is sharp: take $Q = I$ and let $q = \frac{n}{2}$. Then

$$f(x) = |x|^{-2}\log(e + |x|^{-1})^{-1} \in L^q(B(0,1))$$

but solution u unbounded at origin.

Our questions

- Sharper version of Trudinger's theorem in scale of Orlicz spaces
- Improve size of bound
- Extend to degenerate elliptic operators

Our questions

- Sharper version of Trudinger's theorem in scale of Orlicz spaces
- Improve size of bound
- Extend to degenerate elliptic operators

Our questions

- Sharper version of Trudinger's theorem in scale of Orlicz spaces
- Improve size of bound
- Extend to degenerate elliptic operators

Our questions

- Sharper version of Trudinger's theorem in scale of Orlicz spaces
- Improve size of bound
- Extend to degenerate elliptic operators

First generalization

Theorem (DCU-SR 2020, Cianchi 1999)

Let $f \in L^A(\Omega)$, $A(t) = t^{\frac{n}{2}} \log(e+t)^q$, $q > \frac{n}{2}$, Q uniformly elliptic, and let u be a non-negative weak (sub)solution of

$$\begin{cases} -\operatorname{Div}(Q\nabla u) = f & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Then

$$\|u\|_{L^\infty(\Omega)} \leqslant C\|f\|_{L^A(\Omega)}.$$

First generalization

Theorem (DCU-SR 2020, Cianchi 1999)

Let $f \in L^A(\Omega)$, $A(t) = t^{\frac{n}{2}} \log(e+t)^q$, $q > \frac{n}{2}$, Q uniformly elliptic, and let u be a non-negative weak (sub)solution of

$$\begin{cases} -\operatorname{Div}(Q\nabla u) = f & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Then

$$\|u\|_{L^\infty(\Omega)} \leqslant C\|f\|_{L^A(\Omega)}.$$

Result almost sharp

Above counter-example satisfies $f \in L^A(B(0,1))$ for $q < \frac{n}{2} - 1$.

Cianchi (1999) showed this bound is sharp.

So our proof has a gap.

Result almost sharp

Above counter-example satisfies $f \in L^A(B(0,1))$ for $q < \frac{n}{2} - 1$.

Cianchi (1999) showed this bound is sharp.

So our proof has a gap.

Result almost sharp

Above counter-example satisfies $f \in L^A(B(0,1))$ for $q < \frac{n}{2} - 1$.

Cianchi (1999) showed this bound is sharp.

So our proof has a gap.

Second generalization

Theorem (DCU-SR 2020, X. Xu 2011)

Let $f \in L^A(\Omega)$, $A(t) = t^{\frac{n}{2}} \log(e+t)^q$, $q > \frac{n}{2}$, Q uniformly elliptic, and let u be a non-negative weak (sub)solution of

$$\begin{cases} -\operatorname{Div}(Q\nabla u) = f & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Then

$$\|u\|_{L^\infty(\Omega)} \leqslant C\|f\|_{L^{\frac{n}{2}}(\Omega)}\left(1 + \log\left(1 + \frac{\|f\|_{L^A(\Omega)}}{\|f\|_{L^{\frac{n}{2}}(\Omega)}}\right)\right)$$

Second generalization

Theorem (DCU-SR 2020, X. Xu 2011)

Let $f \in L^A(\Omega)$, $A(t) = t^{\frac{n}{2}} \log(e+t)^q$, $q > \frac{n}{2}$, Q uniformly elliptic, and let u be a non-negative weak (sub)solution of

$$\begin{cases} -\operatorname{Div}(Q\nabla u) = f & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Then

$$\|u\|_{L^\infty(\Omega)} \leqslant C\|f\|_{L^{\frac{n}{2}}(\Omega)}\left(1 + \log\left(1 + \frac{\|f\|_{L^A(\Omega)}}{\|f\|_{L^{\frac{n}{2}}(\Omega)}}\right)\right)$$

Entropy bump

The quantity

$$\frac{\|f\|_{L^A(\Omega)}}{\|f\|_{L^{\frac{n}{2}}(\Omega)}}$$

is an "*entropy bump*" (Treil & Volberg); measures how close/far f is from an $L^{\frac{n}{2}}$ function.

General PDE

We consider the Dirichlet problem

$$\begin{cases} -v^{-1}\operatorname{Div}(Q\nabla u) = f & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Weight v "cancels" bad behavior of Q. First explicit in

$$L_w u = -w^{-1}\operatorname{Div}(wQ\nabla u)$$

(DCU-Rios 2008)

General PDE

We consider the Dirichlet problem

$$\begin{cases} -v^{-1}\operatorname{Div}(Q\nabla u) = f & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Weight v “cancels” bad behavior of Q. First explicit in

$$L_w u = -w^{-1}\operatorname{Div}(wQ\nabla u)$$

(DCU-Rios 2008)

Key hypotheses

- $v \in L^1(\Omega)$
- $|Q(x)|_{op} = \sup\{|Q(x)\xi| : \xi \in \mathbb{R}^n, |\xi| = 1\} \leqslant kv(x)$
- for all $\psi \in Lip_0(\Omega)$

$$\left(\int_\Omega |\psi(x)|^{2\sigma}\, v(x)dx\right)^{\frac{1}{2\sigma}} \leqslant C_0 \left(\int_\Omega \left|\sqrt{Q(x)}\nabla\psi(x)\right|^2 dx\right)^{\frac{1}{2}}$$

Key hypotheses

- $v \in L^1(\Omega)$
- $|Q(x)|_{op} = \sup\{|Q(x)\xi| : \xi \in \mathbb{R}^n, |\xi| = 1\} \leqslant kv(x)$
- for all $\psi \in Lip_0(\Omega)$

$$\left(\int_\Omega |\psi(x)|^{2\sigma}\, v(x)dx\right)^{\frac{1}{2\sigma}} \leqslant C_0 \left(\int_\Omega \left|\sqrt{Q(x)}\nabla\psi(x)\right|^2 dx\right)^{\frac{1}{2}}$$

Key hypotheses

- $v \in L^1(\Omega)$
- $|Q(x)|_{op} = \sup\{|Q(x)\xi| : \xi \in \mathbb{R}^n, |\xi| = 1\} \leqslant kv(x)$
- for all $\psi \in Lip_0(\Omega)$

$$\left(\int_\Omega |\psi(x)|^{2\sigma}\, v(x)dx\right)^{\frac{1}{2\sigma}} \leqslant C_0 \left(\int_\Omega \left|\sqrt{Q(x)}\nabla\psi(x)\right|^2\, dx\right)^{\frac{1}{2}}$$

Key hypotheses

- $v \in L^1(\Omega)$
- $|Q(x)|_{op} = \sup\{|Q(x)\xi| : \xi \in \mathbb{R}^n, |\xi| = 1\} \leqslant kv(x)$
- for all $\psi \in Lip_0(\Omega)$

$$\left(\int_\Omega |\psi(x)|^{2\sigma}\, v(x)dx\right)^{\frac{1}{2\sigma}} \leqslant C_0 \left(\int_\Omega \left|\sqrt{Q(x)}\nabla\psi(x)\right|^2\, dx\right)^{\frac{1}{2}}$$

Muckenhoupt weights

Fabes, Kenig, Serapioni 1982:

$$[v]_{A_2} = \sup_B \frac{1}{|B|}\int_B v(x)\,dx \frac{1}{|B|}\int_B v(x)^{-1}\,dx < \infty,$$

$$\lambda v(x)|\xi|^2 \leqslant \langle Q(x)\xi,\xi\rangle \leqslant \Lambda v(x)|\xi|^2$$

Muckenhoupt weights

Fabes, Kenig, Serapioni 1982:

$$[v]_{A_2} = \sup_B \frac{1}{|B|}\int_B v(x)\,dx \frac{1}{|B|}\int_B v(x)^{-1}\,dx < \infty,$$

$$\lambda v(x)|\xi|^2 \leqslant \langle Q(x)\xi, \xi\rangle \leqslant \Lambda v(x)|\xi|^2$$

p-admissible pairs

Chanillo, Wheeden 1985:

$u(x) \leqslant v(x)$ a.e., v doubling, $u \in A_2$

there exists $\sigma > 1$ such that for $B_1 \subset B_2 \subset \Omega$,

$$\frac{r(B_1)}{r(B_2)}\left(\frac{v(B_1)}{v(B_2)}\right)^{\frac{1}{2\sigma}} \leqslant C\left(\frac{u(B_1)}{u(B_2)}\right)^{\frac{1}{2}}$$

$$u(x)|\xi|^2 \leqslant \langle Q(x)\xi, \xi\rangle \leqslant v(x)|\xi|^2$$

p-admissible pairs

Chanillo, Wheeden 1985:

$u(x) \leqslant v(x)$ a.e., v doubling, $u \in A_2$

there exists $\sigma > 1$ such that for $B_1 \subset B_2 \subset \Omega$,

$$\frac{r(B_1)}{r(B_2)}\left(\frac{v(B_1)}{v(B_2)}\right)^{\frac{1}{2\sigma}} \leqslant C\left(\frac{u(B_1)}{u(B_2)}\right)^{\frac{1}{2}}$$

$$u(x)|\xi|^2 \leqslant \langle Q(x)\xi,\xi\rangle \leqslant v(x)|\xi|^2$$

Solution space

$QH^1_0(v;\Omega)$ is the closure of $Lip_0(\Omega)$ with respect to norm

$$\begin{aligned}\|\psi\|_{QH^1_0(v;\Omega)} &= \|\psi\|_{L^2(v;\Omega)} + \|\nabla\psi\|_{L^2_Q(\Omega)} \\ &= \left(\int_\Omega |\psi|^2 v\,dx\right)^{\frac{1}{2}} + \left(\int_\Omega |\sqrt{Q(x)}\nabla\psi|^2\,dx\right)^{\frac{1}{2}}\end{aligned}$$

Weak gradients

Each element of $QH^1_0(v;\Omega)$ is equivalence class of Cauchy sequences.

Associate to each a unique pair $\mathbf{u} = (u, \mathbf{g})$ converging in $L^2(v;\Omega) \times L^2_Q(\Omega)$.

Define $\nabla u = \mathbf{g}$ to be weak gradient of u.

$(u, \nabla u)$ satisfy the degenerate Sobolev inequality.

Weak gradients

Each element of $QH_0^1(v;\Omega)$ is equivalence class of Cauchy sequences.

Associate to each a unique pair $\mathbf{u} = (u, \mathbf{g})$ converging in $L^2(v;\Omega) \times L^2_Q(\Omega)$.

Define $\nabla u = \mathbf{g}$ to be weak gradient of u.

$(u, \nabla u)$ satisfy the degenerate Sobolev inequality.

Weak gradients

Each element of $QH^1_0(v;\Omega)$ is equivalence class of Cauchy sequences.

Associate to each a unique pair $\mathbf{u} = (u, \mathbf{g})$ converging in $L^2(v;\Omega) \times L^2_Q(\Omega)$.

Define $\nabla u = \mathbf{g}$ to be weak gradient of u.

$(u, \nabla u)$ satisfy the degenerate Sobolev inequality.

Weak gradients

Each element of $QH^1_0(v;\Omega)$ is equivalence class of Cauchy sequences.

Associate to each a unique pair $\mathbf{u} = (u, \mathbf{g})$ converging in $L^2(v;\Omega) \times L^2_Q(\Omega)$.

Define $\nabla u = \mathbf{g}$ to be weak gradient of u.

$(u, \nabla u)$ satisfy the degenerate Sobolev inequality.

Weak solutions

$\mathbf{u} = (u, \nabla u) \in QH_0^1(v;\Omega)$ is weak solution of Dirichlet problem if

$$\int_\Omega \nabla\psi(x) \cdot Q(x)\nabla u(x)\,dx = \int_\Omega f(x)\psi(x)v(x)\,dx$$

for every $\psi \in Lip_0(\Omega)$.

By approximation argument, can take $(h, \nabla h) \in QH_0^1(v;\Omega)$ as test functions

Weak solutions

$\mathbf{u} = (u, \nabla u) \in QH_0^1(v;\Omega)$ is weak solution of Dirichlet problem if

$$\int_\Omega \nabla\psi(x)\cdot Q(x)\nabla u(x)\,dx = \int_\Omega f(x)\psi(x)v(x)\,dx$$

for every $\psi \in Lip_0(\Omega)$.

By approximation argument, can take $(h, \nabla h) \in QH_0^1(v;\Omega)$ as test functions

Main result I

Theorem (DCU-SR 2020)

Let $f \in L^A(v;\Omega)$, $A(t) = t^{\sigma'} \log(e+t)^q$, $q > \sigma'$, *and let* $\mathbf{u} = (u, \nabla u) \in QH_0^1(v;\Omega)$ *be a non-negative weak (sub)solution of Dirichlet problem. Then*

$$\|u\|_{L^\infty(v;\Omega)} \leqslant C\|f\|_{L^A(v;\Omega)}.$$

Main result II

Theorem (DCU-SR 2020)

Let $f \in L^A(v;\Omega)$, $A(t) = t^{\sigma'} \log(e+t)^q$, $q > \sigma'$, *and let* $\mathbf{u} = (u, \nabla u) \in QH_0^1(v;\Omega)$ *be a non-negative weak (sub)solution of Dirichlet problem. Then*

$$\|u\|_{L^\infty(v;\Omega)} \leqslant \|f\|_{L^{\sigma'}(\Omega)} \left(1 + \log\left(1 + \frac{\|f\|_{L^A(\Omega)}}{\|f\|_{L^{\sigma'}(\Omega)}}\right)\right)$$

Why De Georgi iteration

Could not make Moser iteration work!

Open problem: make Moser iteration work in this setting.

Why De Georgi iteration

Could not make Moser iteration work!

Open problem: make Moser iteration work in this setting.

Step 1: Technical obstacle

Prove pairs $(u, \nabla u) \in QH^1_0(v;\Omega)$ behave like classical weak gradients.

Key place to use hypothesis $|Q|_{op} \leqslant kv$.

Step 1: Technical obstacle

Prove pairs $(u, \nabla u) \in QH^1_0(v;\Omega)$ behave like classical weak gradients.

Key place to use hypothesis $|Q|_{op} \leqslant kv$.

Step 2: Apply Sobolev inequality

For each $r > 0$, define

$$\phi = \phi_r(u) = (u - r)_+.$$

Let $S(r) = \{x : u(x) > r\}$. Then

$$(\phi, \nabla\phi) = ((u - r)_+, \chi_{S(r)} \nabla u) \in QH^1_0(\Omega).$$

By Sobolev inequality and definition of weak (sub)solution with ϕ as test function

$$\|\phi\|^2_{L^{2\sigma}(v;\Omega)} \leqslant C_0 \|f\|_{L^{(2\sigma)'}(v;\Omega)} \|\phi\|_{L^{2\sigma}(v;S(r))}$$

Step 2: Apply Sobolev inequality

For each $r > 0$, define

$$\phi = \phi_r(u) = (u - r)_+.$$

Let $S(r) = \{x : u(x) > r\}$. Then

$$(\phi, \nabla\phi) = ((u - r)_+, \chi_{S(r)}\nabla u) \in QH_0^1(\Omega).$$

By Sobolev inequality and definition of weak (sub)solution with ϕ as test function

$$\|\phi\|_{L^{2\sigma}(v;\Omega)}^2 \leqslant C_0\|f\|_{L^{(2\sigma)'}(v;\Omega)}\|\phi\|_{L^{2\sigma}(v;S(r))}$$

Step 3: Use Orlicz space properties

By generalized Hölder's inequality and definition of Orlicz norm, for $s > r$

$$\begin{aligned} v(S(s))^{\frac{1}{2\sigma}}(s-r) &\leqslant \|\phi\|_{L^{2\sigma}(v;\Omega)} \\ &\leqslant C\|f\|_{L^{(2\sigma)'}(v;S(r))} \\ &\leqslant C\|f\|_{L^A(v;\Omega)}\|\chi_{S(r)}\|_{L^B(v;\Omega)} \\ &\leqslant C\|f\|_{L^A(v;\Omega)} \frac{v(S(r))^{\frac{1}{2\sigma}}}{\log(e+(v(S(r)))^{-1})^{q\left(\frac{(2\sigma)'}{\sigma'}\right)}} \end{aligned}$$

Step 4: Moser iteration

Define

$$C_k = \tau_0 \|f\|_{L^A(v;\Omega)} \left(1 - \frac{1}{(k+1)^\epsilon}\right)$$

Goal to show

$$v(S(\tau_0\|f\|_{L^A(v;\Omega)})) = \lim_{k\to\infty} v(S(C_k)) = 0.$$

Step 5: Iteration formula

Let $m_k = -\log(v(S(C_k))$.

Let $s = C_{k+1}$, $r = C_k$. Fix $\epsilon = \frac{q}{\sigma'} - 1 > 0$

Then

$$m_{k+1} \geqslant \log\left(\frac{\epsilon \mathcal{T}_0}{C}\right) + \log\left(\frac{m_k}{k+2}\right)^{\frac{2\sigma q}{\sigma'}} + m_k$$

Step 5: Iteration formula

Let $m_k = -\log(v(S(C_k))$.

Let $s = C_{k+1}$, $r = C_k$. Fix $\epsilon = \frac{q}{\sigma'} - 1 > 0$

Then

$$m_{k+1} \geqslant \log\left(\frac{\epsilon \mathcal{T}_0}{C}\right) + \log\left(\frac{m_k}{k+2}\right)^{\frac{2\sigma q}{\sigma'}} + m_k$$

Step 6: Induction argument

By induction, there exists $\tau_0 > 0$ (very large) such that

$$m_k \geqslant m_0 + k$$

Therefore

$$\lim_{k\to\infty} m_k = \infty.$$

Step 6: Induction argument

By induction, there exists $\tau_0 > 0$ (very large) such that

$$m_k \geqslant m_0 + k$$

Therefore

$$\lim_{k\to\infty} m_k = \infty.$$

Step 1: Exponential integrability

Theorem (DCU-SR 2020, Xu 2011, Cianchi 1999)

Let $\|f\|_{L^{\sigma'}(v;\Omega)} = 1$ and let u be (bounded) solution of Dirichlet problem. Then for $\gamma > 0$ (very small)

$$\int_\Omega e^{\gamma u(x)} v(x)\, dx \leqslant M,$$

where $M = (\gamma, C_0, v(\Omega))$ is independent of u.

Need first theorem to apply this result!

Step 1: Exponential integrability

Theorem (DCU-SR 2020, Xu 2011, Cianchi 1999)

Let $\|f\|_{L^{\sigma'}(v;\Omega)} = 1$ and let u be (bounded) solution of Dirichlet problem. Then for $\gamma > 0$ (very small)

$$\int_\Omega e^{\gamma u(x)} v(x)\, dx \leqslant M,$$

where $M = (\gamma, C_0, v(\Omega))$ is independent of u.

Need first theorem to apply this result!

Step 2: Auxiliary Dirichlet problem

Let $h = e^{\frac{\gamma}{p}u}$ and $w = h - 1$. Then

$$(w, \frac{\gamma}{p} h \nabla u) \in QH_0^1(v; \Omega)$$

is a weak (sub)solution to

$$\begin{cases} -v^{-1}\operatorname{Div}(Q\nabla w) = \frac{\gamma}{p} hf & x \in \Omega, \\ w = 0 & x \in \partial\Omega. \end{cases}$$

Step 2: Auxiliary Dirichlet problem

Let $h = e^{\frac{\gamma u}{p}}$ and $w = h - 1$. Then

$$(w, \frac{\gamma}{p} h \nabla u) \in QH_0^1(v;\Omega)$$

is a weak (sub)solution to

$$\begin{cases} -v^{-1}\,\mathrm{Div}(Q\nabla w) = \frac{\gamma}{p} hf & x \in \Omega, \\ w = 0 & x \in \partial\Omega. \end{cases}$$

Step 3: Moser iteration

Assume $\|f\|_{L^{\sigma'}(v;\Omega)} = 1$.

Repeat Moser iteration and induction argument on auxiliary solution w to conclude

$$\|w\|_{L^\infty(v;\Omega)} = e^{c\|u\|_{L^\infty(v;\Omega)}} \leqslant \tau_0\big(1 + \|f\|_{L^A(v;\Omega)}\big).$$

Hence,

$$\|u\|_{L^\infty(v;\Omega)} \leqslant C\log\big(1 + \|f\|_{L^A(v;\Omega)}\big) + C.$$

Step 3: Moser iteration

Assume $\|f\|_{L^{\sigma'}(v;\Omega)} = 1$.

Repeat Moser iteration and induction argument on auxiliary solution w to conclude

$$\|w\|_{L^\infty(v;\Omega)} = e^{c\|u\|_{L^\infty(v;\Omega)}} \leqslant \tau_0\big(1 + \|f\|_{L^A(v;\Omega)}\big).$$

Hence,

$$\|u\|_{L^\infty(v;\Omega)} \leqslant C\log\big(1 + \|f\|_{L^A(v;\Omega)}\big) + C.$$

Additional Results

Rodney & MacDonald (undergraduate honors thesis) have extended this result to the most general degenerate linear elliptic operator with first order and zero order terms.

Future work

Possible generalizations:

- Prove sharp result (Cianchi 1999)
- Extend to parabolic equations (Xu to appear)
- Extend to other function spaces: $L^{p(\cdot)}(v;\Omega)$, grand Lebesgue spaces
- Extend to nonlinear elliptic equations (SR, DCU, KM)

Future work

Possible generalizations:

- Prove sharp result (Cianchi 1999)
- Extend to parabolic equations (Xu to appear)
- Extend to other function spaces: $L^{p(\cdot)}(v;\Omega)$, grand Lebesgue spaces
- Extend to nonlinear elliptic equations (SR, DCU, KM)

Future work

Possible generalizations:

- Prove sharp result (Cianchi 1999)
- Extend to parabolic equations (Xu to appear)
- Extend to other function spaces: $L^{p(\cdot)}(v;\Omega)$, grand Lebesgue spaces
- Extend to nonlinear elliptic equations (SR, DCU, KM)

Future work

Possible generalizations:

- Prove sharp result (Cianchi 1999)
- Extend to parabolic equations (Xu to appear)
- Extend to other function spaces: $L^{p(\cdot)}(v;\Omega)$, grand Lebesgue spaces
- Extend to nonlinear elliptic equations (SR, DCU, KM)

Future work

Possible generalizations:

- Prove sharp result (Cianchi 1999)
- Extend to parabolic equations (Xu to appear)
- Extend to other function spaces: $L^{p(\cdot)}(v;\Omega)$, grand Lebesgue spaces
- Extend to nonlinear elliptic equations (SR, DCU, KM)

Dziękuję Ci!

Thank You!

Roll Tide!