A variational approach to doubly nonlinear equations with nonstandard growth

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Overview

The main results are due to a joint work with

- Verena Bögelein (Salzburg)
- Frank Duzaar (Erlangen-Nürnberg)
- Paolo Marcellini (Firenze)

I. Variational formulation of the problem

The model problem

Cauchy-Dirichlet problem:

Find $u\colon \Omega_T \to [0,\infty)$ with

$$\begin{cases} \partial_t u^m - \operatorname{div}\big(|Du|^{p-2}Du\big) = 0 & \text{in } \Omega_T, \\ u = g & \text{on } \partial_{\mathrm{par}}\Omega_T, \end{cases} \tag{1}$$

where here,

- $\Omega \subset \mathbb{R}^n$ is a bounded domain, $T > 0$, $\Omega_T := \Omega \times (0, T)$.
- $m > 0$, $p > 1$.
- $g : \partial_{\mathrm{par}}\Omega_T \to [0,\infty)$ are prescribed boundary values.

This generalizes both the porous medium equation ($p = 2$) and the parabolic p-Laplace equation ($m = 1$).

Variational Formulation I

We test (1) with

$$\varphi = w - u$$

for a comparison map $w\colon \Omega_T \to [0,\infty)$ with $w = u$ on the lateral boundary $\partial\Omega \times (0,T)$. This leads to

$$\underbrace{\iint_{\Omega_T} \partial_t u^m (w-u)\,\mathrm{d}x\mathrm{d}t}_{=:\,\mathrm{I}} + \underbrace{\iint_{\Omega_T} |Du|^{p-2} Du \cdot \big(Dw - Du\big)\,\mathrm{d}x\mathrm{d}t}_{=:\,\mathrm{II}} = 0$$

By the convexity of $\mathbb{R}^n \ni \xi \mapsto \frac{1}{p}|\xi|^p$ we have

$$\tfrac{1}{p}|Du|^p + |Du|^{p-2}Du \cdot \big(Dw - Du\big) \le \tfrac{1}{p}|Dw|^p.$$

Variational Formulation II

$$\underbrace{\iint_{\Omega_T} \partial_t u^m (w-u)\mathrm{d}x\mathrm{d}t}_{=:\,\mathrm{I}} + \underbrace{\iint_{\Omega_T} |Du|^{p-2}Du \cdot \big(Dw - Du\big)\mathrm{d}x\mathrm{d}t}_{=:\,\mathrm{II}} = 0$$

By convexity,

$$\mathrm{II} \le \tfrac{1}{p}\iint_{\Omega_T} |Dw|^p \mathrm{d}x\mathrm{d}t - \tfrac{1}{p}\iint_{\Omega_T} |Du|^p \mathrm{d}x\mathrm{d}t$$

Integration by parts and elementary calculations imply

$$\begin{aligned}\mathrm{I} = &\iint_{\Omega_T} \partial_t w\big(w^m - u^m\big)\mathrm{d}x \\ &+ \int_\Omega \mathfrak{b}[u(0),w(0)]\mathrm{d}x - \int_\Omega \mathfrak{b}[u(T),w(T)]\mathrm{d}x\end{aligned}$$

with $\mathfrak{b}[u,w] := \frac{1}{m+1}w^{m+1} - \Big[\frac{1}{m+1}u^{m+1} + u^m(w-u)\Big] \ge 0$

Variational Formulation III

This leads to the *variational inequality*

$$\iint_{\Omega_T} \tfrac{1}{p}|Du|^p \mathrm{d}x\mathrm{d}t \leq \iint_{\Omega_T} \Big[\tfrac{1}{p}|Dw|^p + \partial_t w\big(w^m - u^m\big)\Big]\mathrm{d}x\mathrm{d}t \\ + \underbrace{\int_\Omega \mathfrak{b}[u(0), w(0)]\mathrm{d}x - \int_\Omega \mathfrak{b}[u(T), w(T)]\mathrm{d}x}_{=:\mathfrak{B}[u(0),w(0)]-\mathfrak{B}[u(T),w(T)]}$$

for any $w\colon \Omega_T \to [0,\infty)$ with $\partial_t w \in L^{\frac{m+1}{m}}(\Omega_T)$ and $w = u$ on $\partial\Omega \times (0,T)$.
In the case $m = 1$ the boundary term simplifies to

$$\mathfrak{b}[u,w] = \tfrac{1}{2}|u-w|^2$$

so that the usual $\mathrm{L}^2(\Omega)$-boundary terms appear in the variational inequality.

The Nonlinearity b

We replace the term u^m by a nonlinearity $b\colon [0,\infty) \to [0,\infty)$

- which is continuous and piecewise C^1 with $b(0) = 0$,
- and satisfies

$$0 < \ell \le \frac{u b'(u)}{b(u)} \le m \qquad (2)$$

whenever $u > 0$, $b(u) > 0$ and $b'(u)$ exists.

Assumption (2) implies the nonstandard growth condition

$$b(1) \min\{u^\ell, u^m\} \le b(u) \le b(1) \max\{u^\ell, u^m\} \qquad \text{for all } u > 0.$$

Orlicz Spaces

The primitive $\phi\colon [0,\infty) \to [0,\infty)$ is defined by

$$\phi(u) := \int_0^u b(s)\mathrm{d}s \quad \forall u \geq 0.$$

Note that ϕ is convex with $\phi(0) = 0$ and that (2) implies that ϕ satisfies the Δ_2- and ∇_2-conditions. We consider the Orlicz-space

$$\mathrm{L}^\phi(\Omega) = \left\{ w\colon \Omega \to \mathbb{R}, \text{ measurable} : \int_\Omega \phi(|w|)\mathrm{d}x < \infty \right\}$$

Assumption (2) implies the nonstandard growth condition

$$\phi(1)\min\{u^{\ell+1}, u^{m+1}\} \leq \phi(u) \leq \phi(1)\max\{u^{\ell+1}, u^{m+1}\}$$

for all $u \geq 0$.

The Integrand f

Instead of the model integrand $\frac{1}{p}|\xi|^p$, we consider an general integrand $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that the following convexity and coercivity conditions hold:

$$\begin{cases} \mathbb{R} \times \mathbb{R}^n \ni (u,\xi) \to f(x,u,\xi) \text{ is convex for a.e. } x \in \Omega, \\ f(x,u,\xi) \geq \nu|\xi|^p \text{ for } (x,u,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n. \end{cases}$$

- **We do not require any growth condition from above.**
- More generally, we can assume

$$f(x,u,\xi) \geq \nu|\xi|^p - g(x)\big(1+|u|\big)$$

for $(x,u,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, with $g \in \mathrm{L}^1(\Omega) \cap \mathrm{L}^{\phi^*}(\Omega)$.

The (time-independent) boundary values

We consider initial values $u_o \colon \Omega \to [0,\infty)$ (which also play the role of time-independent boundary values) that satisfy

$$u_o \in \mathrm{L}^{\phi}(\Omega) \quad \text{and} \quad \int_{\Omega} f(x,u_o,Du_o)\mathrm{d}x < \infty.$$

For the given data, we wish to solve the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t b(u) - \mathrm{div}\, D_\xi f(x,u,Du) = -D_u f(x,u,Du) & \text{in } \Omega_T, \\ u = u_o & \text{on } \partial_{\mathrm{par}}\Omega_T, \end{cases}$$

which generalizes the model case (1) from above.

Variational formulation and integral convexity

Similarly as in the model case, the convexity of f implies

$$
\begin{aligned}
D_\xi f(x,u,Du)\cdot D(w-u) + D_u f(x,u,Du)(w-u) \\
\le f(x,w,Dw) - f(x,u,Du).
\end{aligned}
$$

Actually, what we need in the argument is the **convexity of the integral**

$$
\boldsymbol{F}(u) := \int_\Omega f(x,u,Du)\,\mathrm{d}x
$$

rather than the convexity of the integrand.

Theorem (Bögelein, Dacorogna, Duzaar, Marcellini, S., 2020)

For gradient flows ($b(u)=u$) for functionals with p-growth, **integral convexity is necessary and sufficient** *for existence of variational solutions. (Also in the case of systems.)*

Variational formulation: the time term

For the term involving the time derivative, we calculate (formally)

$$\iint_{\Omega_T} \partial_t b(u)(w-u)\mathrm{d}x$$
$$= \iint_{\Omega_T} \partial_t w\big(b(w)-b(u)\big)\mathrm{d}x$$
$$+ \int_\Omega \mathfrak{b}[u(0),w(0)]\mathrm{d}x - \int_\Omega \mathfrak{b}[u(T),w(T)]\mathrm{d}x$$

with the *boundary term*

$$\mathfrak{b}[u,w] := \phi(w) - \Big[\phi(u) + b(u)(w-u)\Big] \geq 0$$

As mentioned before, $\mathfrak{b}[u,w] = \frac{1}{2}|u-w|^2$ in the case $b(u) = u$.

Variational Solutions

Definition

A non-negative measurable map $u\colon \Omega_T \to [0,\infty)$ in the class

$$u \in \mathrm{C}^0\big([0,T];\mathrm{L}^\phi(\Omega)\big) \cap \mathrm{L}^p\big(0,T;u_o + \mathrm{W}^{1,p}_0(\Omega)\big)$$

is called variational solution if and only if the variational inequality

$$\begin{aligned}
&\mathfrak{B}[u(\tau),w(\tau)] + \iint_{\Omega_\tau} f(x,u,Du)\,\mathrm{d}x\mathrm{d}t \\
&\quad\le \mathfrak{B}[u_o,w(0)] + \iint_{\Omega_\tau} \Big[f(x,w,Dw) + \partial_t w\big(b(w)-b(u)\big)\Big]\mathrm{d}x\mathrm{d}t
\end{aligned}$$

holds true for any $\tau \in [0,T]$, and any $w \in \mathrm{L}^p\big(0,T;u_o + \mathrm{W}^{1,p}_0(\Omega)\big)$ with $\partial_t w \in \mathrm{L}^\phi(\Omega_T)$ and $w(0) \in \mathrm{L}^\phi(\Omega)$.

Here, $\mathfrak{B}[u,w] := \int_\Omega \big[\phi(w) - \phi(u) - b(u)(w-u)\big]\mathrm{d}x$. Here,
$\mathfrak{B}[u,w] := \frac{1}{2}\int_\Omega |w-u|^2\mathrm{d}x$ if $b(u)=u$

II. Existence results

The General Existence Result

Theorem (Bögelein, Duzaar, Marcellini, S., ARMA 2018)

Suppose that the non-linearity b, the integrand f and the initial datum u_o are as before. Then there exists a variational solution

$$u \in \mathrm{C}^0\big([0,T]; \mathrm{L}^\phi(\Omega)\big) \cap \mathrm{L}^p\big(0,T; u_o + \mathrm{W}^{1,p}_0(\Omega)\big)$$

in the sense of the previous definition. The solution satisfies

$$\partial_t \sqrt{\phi(u)} \in \mathrm{L}^2(\Omega_T)$$

and attains the initial datum u_o in the $\mathrm{C}^0 - \mathrm{L}^\phi$-sense.

Weak solutions

Under additional assumptions, the variational solutions constructed in the preceding theorem are distributional solutions of

$$\begin{cases} \partial_t b(u) - \operatorname{div} D_\xi f(x,u,Du) = -D_u f(x,u,Du) & \text{in } \Omega_T, \\ u = u_o & \text{on } \partial_{\mathrm{par}}\Omega_T, \end{cases}$$

The examples include general nonlinearities $b(u)$ with

$$1 \le \ell \le \frac{ub'(u)}{b(u)} \le m$$

and functionals with nonstandard growth such as

- $f(x,\xi) := \alpha(x)|\xi|^p + \beta(x)|\xi|^q$,
  $(1 < p < q \le p+1,\ \alpha(x)+\beta(x) > 0)$;
- $f(\xi) := |\xi|^p \log(1+|\xi|)$;
- $f(\xi) := \mathrm{e}^{\sqrt{1+|\xi|^2}}$.

Extensions

The preceding existence result has been extended in various directions:

- to unbounded domains, in particular to the Cauchy-problem on $\Omega = \mathbb{R}^n$ (Bögelein, Duzaar, Marcellini, S., ARMA 2018);
- to time-dependent boundary values (Bögelein, Duzaar, Marcellini, S., Rend. Lincei 2018);
- to doubly nonlinear systems with $b(u) = u^m$, $m > 1$, and time-dependent boundary values (Schätzler, J. Elliptic Parabol. Equ 2019)

Known Results I

- Grange & Mignot (1972), Alt & Luckhaus (1983): equations/systems of the type

$$\partial_t b(u) - \operatorname{div}\big(\boldsymbol{A}(b(u), Du)\big) = f(b(u))$$

with $u = g$ on $\partial_{\mathrm{par}}\Omega_T$. The coefficients $\boldsymbol{A}$ satisfy

$$\begin{cases} |\boldsymbol{A}(s,\xi)| \le C(1+|\xi|^{p-1}), \\ \big(\boldsymbol{A}(s,\xi) - \boldsymbol{A}(s,\eta)\big)\cdot(\xi-\eta) \ge C_o|\xi-\eta|^p, \\ b \text{ is the continuous gradient of a convex } \mathrm{C}^1\text{-function } \phi. \end{cases}$$

The boundary values satisfy

$$\begin{cases} g \in \mathrm{L}^p\big(0,T;\mathrm{W}^{1,p}(\Omega)\big) \cap \mathrm{L}^\infty(\Omega_T), \\ \partial_t g \in \mathrm{L}^1(0,T;\mathrm{L}^\infty(\Omega)) \end{cases}$$

Proof by Galerkin type method.

Known Results II

- Bernis (1988): Higher order doubly non-linear equations on unbounded domains.
- Ivanov & Mkrtychyan (1992, 1997): Existence of Hölder continuous solutions to equations of the type

  $$\partial_t u - \operatorname{div}\big(u^{m-1}|Du|^{p-2}Du\big) = 0.$$

  Existence via approximation by strictly positive solutions and a-priori Hölder-estimates.
- Akagi & Stefanelli (2011): Equations of the type

  $$b(\partial_t u) - \operatorname{div}\big(|Du|^{p-2}Du\big) = 0$$

  Existence via elliptic regularization.

Known Results III

- Akagi & Stefanelli (2014):

$$\partial_t b(u) - \mathrm{div}\big(\boldsymbol{A}(Du)\big) = f$$
$$\underset{v:=b(u)}{\Longleftrightarrow} \quad -\mathrm{div}\big(\boldsymbol{A}(Db^{-1}(v))\big) = f - \partial_t v$$

  for b and $\boldsymbol{A}$ with polynomial growth.
  Existence for the dual problem via elliptic regularization.
- Ambrosio & Gigli & Savare (2008): Gradient flows in metric spaces.

III. Strategy of the proof

Modified Minimizing Movements

- Fix a step size $h \in (0, 1]$. The goal is to construct approximations u_i of the solution at times $t = ih$, $i \in \mathbb{N}_0$.
- Let $u_0 = u_o$.
- Suppose that for some $i \in \mathbb{N}$ with $ih \leq T$ the non-negative map $0 \leq u_{i-1} \in \mathrm{L}^{\phi}(\Omega) \cap \big(u_o + \mathrm{W}_0^{1,p}(\Omega)\big)$ has been defined.
- In the case $b(u) = u$, we define u_i as the minimizer of

$$\boldsymbol{F}_i[v] := \int_\Omega f(x, v, Dv)\mathrm{d}x + \frac{1}{2h} \int_\Omega \big|u_{i-1} - v\big|^2 \mathrm{d}x.$$

  in the class of functions $0 \leq v \in \mathrm{L}^2(\Omega) \cap \big(u_o + \mathrm{W}_0^{1,p}(\Omega)\big)$.
- In the general case, we define u_i as the minimizer of

$$\boldsymbol{F}_i[v] := \int_\Omega f(x, v, Dv)\mathrm{d}x + \frac{1}{h} \int_\Omega \mathfrak{b}[u_{i-1}, v]\mathrm{d}x.$$

  in the class of functions $0 \leq v \in \mathrm{L}^{\phi}(\Omega) \cap \big(u_o + \mathrm{W}_0^{1,p}(\Omega)\big)$.

  Minimizers exist by the Direct Method of the Calculus of

The Euler operator for the time term

For a test function $\psi \in C_0^\infty(\Omega)$, we consider variations $u_i + s\psi$, $s \in (-\varepsilon, \varepsilon)$, of the minimizers u_i and calculate

$$\frac{d}{ds}\Big|_{s=0}\left(\frac{1}{h}\int_\Omega \mathfrak{b}[u_{i-1}, u_i + s\psi]\mathrm{d}x\right)$$

$$= \frac{1}{h}\int_\Omega \frac{\partial}{\partial s}\Big|_{s=0}\Big[\phi(u_i + s\psi) - \phi(u_{i-1}) - b(u_{i-1})(u_i + s\psi - u_{i-1})\Big]\,\mathrm{d}x$$

$$= \frac{1}{h}\int_\Omega \Big[\phi'(u_i)\psi - b(u_{i-1})\psi\Big]\,\mathrm{d}x$$

$$= \int_\Omega \frac{b(u_i) - b(u_{i-1})}{h}\,\psi\,\mathrm{d}x$$

Energy Estimates I

Observe that u_{i-1} is an admissible competitor for u_i, and therefore $\boldsymbol{F}_i[u_i] \le \boldsymbol{F}_i[u_{i-1}]$. This can be iterated and leads to

$$\underbrace{\int_\Omega f(x,u_k,Du_k)\mathrm{d}x}_{\ge \nu |Du_k|^p} + \frac{1}{h}\sum_{i=1}^{k}\int_\Omega \mathfrak{b}[u_{i-1},u_i]\mathrm{d}x \le \underbrace{\int_\Omega f(x,u_o,Du_o)\mathrm{d}x}_{=:M<\infty},$$

whenever $k \in \mathbb{N}$ with $kh \le T$.

Monotonicity

For the boundary term $\mathfrak{b}[u, w]$, we have the following bounds:

Lemma

For any $u, w \geq 0$, we have

$$\begin{aligned} \mathfrak{b}[u, w] &\leq \big(b(u) - b(w)\big)(u - w) \\ &\leq C\Big|\sqrt{\phi(u)} - \sqrt{\phi(w)}\Big|^2 \\ &\leq C^2 \mathfrak{b}[u, w], \end{aligned}$$

with a constant $C = C(\ell, m) \geq 1$.

The proof relies on the assumption

$$\ell \leq \frac{ub'(u)}{b(u)} \leq m.$$

Energy Estimates II

From the previous energy estimate we immediately obtain:

$$\begin{aligned} \tfrac{1}{h}\sum_{i=1}^{k}\int_\Omega\Big|\sqrt{\phi(u_i)}-\sqrt{\phi(u_{i-1})}\Big|^2\mathrm{d}x \\ \leq \tfrac{C}{h}\sum_{i=1}^{k}\int_\Omega \mathfrak{b}[u_{i-1},u_i]\mathrm{d}x \leq C(\ell,m)M \end{aligned} \tag{3}$$

and

$$\int_\Omega |Du_k|^p \mathrm{d}x \leq \tfrac{M}{\nu}, \tag{4}$$

whenever $k\in\mathbb{N}$ with $kh\leq T$. Furthermore,

$$\begin{aligned} \int_\Omega \phi(u_k)\mathrm{d}x \leq 2k\sum_{i=1}^{k}\int_\Omega\Big|\sqrt{\phi(u_i)}-\sqrt{\phi(u_{i-1})}\Big|^2\mathrm{d}x + 2\int_\Omega \phi(u_o)\mathrm{d}x \\ \leq 2TM + 2\int_\Omega \phi(u_o)\mathrm{d}x. \end{aligned} \tag{5}$$

Construction of the Limit Map I

From now on we consider only such $h \in (0,1]$ such that $h_k = T/k$ with $k \in \mathbb{N}$. From the construction before we obtain minimizers $u_i^{(k)}$ with $i \in \{0,1,\dots k\}$.

We define $u^{(k)}\colon \Omega \times (-h_k, T] \to [0,\infty)$ by

$$u^{(k)}(\cdot,t) := u_i^{(k)} \text{ for } t \in ((i-1)h_k, ih_k],\ i \in \{0,\dots,k\}.$$

The preceding estimates for $u_i^{(k)}$ imply the uniform energy bound

$$\sup_{t\in[0,T]} \int_\Omega \Big[\phi\big(u^{(k)}(t)\big) + \big|Du^{(k)}(t))\big|^p\Big]\mathrm{d}x + \iint_{\Omega_T} \Big|\partial_t^{(-h_k)}\sqrt{\phi(u^{(k)})}\Big|^2 \mathrm{d}x\,\mathrm{d}t \le C, \tag{6}$$

where the constant C is independent of k.

Construction of the Limit Map II

From (6) we conclude that

$$(u^{(k)})_{k\in\mathbb{N}} \text{ is uniformly bounded in } \mathrm{L}^\infty\big(0,T;\mathrm{W}^{1,p}(\Omega)\big).$$

Therefore, we find a subsequence (still denoted by k) and

$$u \in \mathrm{L}^\infty\big(0,T;u_o + \mathrm{W}^{1,p}_0(\Omega)\big)$$

such that

$$u^{(k)} \overset{*}{\rightharpoondown} u \quad \text{weakly}^* \text{ in } \mathrm{L}^\infty\big(0,T;\mathrm{W}^{1,p}(\Omega)\big).$$

Construction of the Limit Map III: Compactness

Compactness lemma

The energy estimate

$$\max_{i\in\{0,1,\ldots k\}} \int_\Omega \phi(u_i^{(k)})\mathrm{d}x + \sup_{i\in\{0,1,\ldots k\}} \int_\Omega |Du_i^{(k)}|^p \mathrm{d}x \le C, \qquad (7)$$

and the continuity estimate

$$\frac{1}{h}\sum_{i=1}^{k} \int_\Omega \left| \sqrt{\phi(u_i^{(k)})} - \sqrt{\phi(u_{i-1}^{(k)})} \right|^2 \mathrm{d}x \le C(\ell, m)M, \qquad (8)$$

imply, after passing to a subsequence, that

$$\begin{cases} \sqrt{\phi(u^{(k)})} \to \sqrt{\phi(u)} & \text{strongly in } L^1(\Omega_T), \\ u^{(k)} \to u & \text{a.e. in } \Omega_T. \end{cases}$$

Compactness lemma

$$\max_{i\in\{0,1,\dots k\}} \int_\Omega \phi(u_i^{(k)})\mathrm{d}x + \sup_{i\in\{0,1,\dots k\}} \int_\Omega \big|Du_i^{(k)}\big|^p \mathrm{d}x \le C, \qquad (7)$$

and $$\frac{1}{h}\sum_{i=1}^{k} \int_\Omega \left|\sqrt{\phi(u_i^{(k)})} - \sqrt{\phi(u_{i-1}^{(k)})}\right|^2 \mathrm{d}x \le C(\ell, m)M, \qquad (8)$$

$$\Longrightarrow \begin{cases} \sqrt{\phi(u^{(k)})} \to \sqrt{\phi(u)} & \text{strongly in } L^1(\Omega_T), \\ u^{(k)} \to u & \text{a.e. in } \Omega_T. \end{cases}$$

- This lemma can be interpreted as a Jacques Simon type compactness result adapted to doubly nonlinear equations.
- For the proof, we relied on techniques by Alt & Luckhaus.

Construction of the Limit Map IV: Time Derivative

From (6), we recall the uniform estimate

$$\iint_{\Omega_T} \left|\partial_t^{(-h_k)} \sqrt{\phi(u^{(k)})}\right|^2 \mathrm{d}x\, \mathrm{d}t \le CM$$

Extract a further subsequence such that

$$\partial_t^{(-h_k)} \sqrt{\phi(u^{(k)})} \rightharpoonup w \quad \text{weakly in } \mathrm{L}^2(\Omega_T).$$

Since $\sqrt{\phi(u^{(k)})} \to \sqrt{\phi(u)}$ strongly in $\mathrm{L}^1(\Omega_T)$, we have

$$w = \partial_t \sqrt{\phi(u)}.$$

We deduce $\partial_t \sqrt{\phi(u)} \in L^2(\Omega_T)$, with the estimate

$$\iint_{\Omega_T} \left|\partial_t \sqrt{\phi(u)}\right|^2 \mathrm{d}x\mathrm{d}t \le C \underbrace{\int_\Omega f(x, u_o, Du_o)\mathrm{d}x}_{\equiv M}.$$

Variational Inequality for the Limit Map I

Let

$$\boldsymbol{F}^k[v] := \iint_{\Omega_T} \Big(f(x,v,Dv) + \tfrac{1}{h_k}\mathfrak{b}\big[u^{(k)}(t-h_k), v(t)\big]\Big)\mathrm{d}x\mathrm{d}t.$$

Then, $u^{(k)}$ minimizes $\boldsymbol{F}^k$, i.e.

$$\boldsymbol{F}^k[u^{(k)}] \le \boldsymbol{F}^k[v]$$

for any $0 \le v \in \mathrm{L}^\phi(\Omega_T) \cap \mathrm{L}^p\big(0,T; u_o + \mathrm{W}^{1,p}_0(\Omega)\big)$. We test the minimality with the admissible comparison map

$$w_s := u^{(k)} + s\big(v - u^{(k)}\big), \quad s \in (0,1),$$

and use the **convexity** of $\int_\Omega f(x,v,Dv)$. Letting $s \downarrow 0$, we get

$$\iint_{\Omega_T} f\big(x, u^{(k)}, Du^{(k)}\big)\mathrm{d}x\mathrm{d}t \\ \le \iint_{\Omega_T} \Big(f\big(x,v,Dv\big) + \Delta_{-h_k} b\big(u^{(k)}\big)\big(v - u^{(k)}\big)\Big)\mathrm{d}x\mathrm{d}t$$

Variational Inequality for the Limit Map II

By lower semicontinuity

$$\iint_{\Omega_T} f\big(x,u,Du\big)\mathrm{d}x\mathrm{d}t \le \liminf_{k\to\infty} \iint_{\Omega_T} f\big(x,u^{(k)},Du^{(k)}\big)\mathrm{d}x\mathrm{d}t.$$

Formally, in the limit $k \to \infty$ we have

$$\iint_{\Omega_T} \underbrace{\Delta_{-h_k} b\big(u^{(k)}\big)}_{\to \partial_t b(u)} \big(\underbrace{v - u^{(k)}}_{\to v-u}\big)\mathrm{d}x\mathrm{d}t \to \iint_{\Omega_T} \partial_t b(u)\big(v-u\big)\mathrm{d}x\mathrm{d}t$$

$$= \iint_{\Omega_\tau} \partial_t v\big(b(v) - b(u)\big)\mathrm{d}x\mathrm{d}t + \mathfrak{B}[u_o, v(0)] - \mathfrak{B}[u(T), v(T)].$$

This formal argument can be made rigorous by a discrete integration by parts formula.

The Initial Datum I

Testing the variational equation with u_o on $\Omega_\tau = \Omega \times (0,\tau)$ for a (small) time $\tau > 0$, we obtain

$$\mathfrak{B}[u(\tau), u_o] + \iint_{\Omega_\tau} \underbrace{f(x,u,Du)}_{\geq 0} \mathrm{d}x\mathrm{d}t \leq \tau \underbrace{\int_\Omega f(x,u_o,Du_o)\mathrm{d}x}_{=:M<\infty}.$$

Letting $\tau \downarrow 0$, we deduce

$$\lim_{\tau\downarrow 0} \mathfrak{B}[u(\tau), u_o] = 0.$$

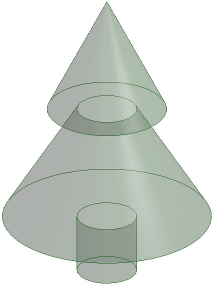
The Initial Datum II

Now with the Cauchy-Schwarz inequality and the monotonicity lemma

$$
\begin{aligned}
&\int_\Omega \big|\phi(u(\tau)) - \phi(u_o)\big| \mathrm{d}x \\
&\quad \le C \bigg[\int_\Omega \Big| \sqrt{\phi(u(\tau))} - \sqrt{\phi(u_o)} \Big|^2 \mathrm{d}x \bigg]^{\frac{1}{2}} \bigg[\int_\Omega \big[\phi(u(\tau)) + \phi(u_o)\big] \mathrm{d}x \bigg]^{\frac{1}{2}} \\
&\quad \le C \underbrace{\sqrt{\mathfrak{B}[u(\tau), u_o]}}_{\longrightarrow 0} \bigg[\int_\Omega \big[\phi(u(\tau)) + \phi(u_o)\big] \mathrm{d}x \bigg]^{\frac{1}{2}}.
\end{aligned}
$$

This implies that u attains the initial data in the sense

$$u(\tau) \to u_o \text{ in } L^\phi(\Omega) \text{ as } \tau \downarrow 0.$$



Merry Christmas!