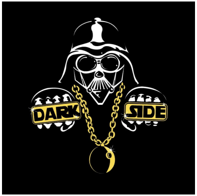
Regularity for non-homogeneous systems

Cristiana De Filippis (University of Turin)
Monday's Nonstandard Seminar - 05/10/2020
cristiana.defilippis@unito.it



$$\mathcal{F}(\mathbf{w},\Omega) := \int_\Omega \varphi(\mathbf{x},|\mathbf{Dw}|)\ \mathbf{dx} \quad \leftsquigarrow \text{ Musielak-Orlicz integral}$$

- P. Harjulehto, P. Hästo, *Lecture Notes in Mathematics* (2019).

$$\mathcal{E}(\mathbf{w},\Omega) := \int_\Omega \mathbf{e}^{\gamma(\mathbf{x})|\mathbf{Dw}|^{\mathbf{p}(\mathbf{x})}}\ \mathbf{dx} \quad \leftsquigarrow \text{ Exponential type integral}$$

- P. Marcellini, *JDE* (1993).

Two different ways of measuring non-uniform ellipticity.

The Double Phase energy

$$\mathcal{P}(\mathbf{w},\Omega) := \int_\Omega \left[|\mathbf{Dw}|^{\mathbf{p}} + \mathbf{a}(\mathbf{x})|\mathbf{Dw}|^{\mathbf{q}}\right] \, \mathbf{dx}$$

- S. M. Kozlov, O. A. Oleinik, V. V. Zhikov, *Springer-Verlag* (1991).
- V. V. Zhikov, *Math. USSR* (1987); *Comp. Med. Hom. Th.* (1995); *Dokl. Ros. Akad. Nauk.* (1995); *Rus. J. Math. Phys.* (1995).

Regularity in the unconstrained case

Theorem (P. Baroni, M. Colombo, G. Mingione, *ARMA* (2015); *Calc. Var. & PDE* (2018))

Let u be a minimizer of the Double Phase energy. Then:

- $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *and* $\frac{q}{p} \leq 1 + \frac{\alpha}{n}$, *then* $u \in C^{1,\beta_0}_{loc}(\Omega, \mathbb{R}^N)$;
- $u \in W^{1,p}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ *and* $q - p \leq \alpha$ ($q - p < \alpha$ *if* $N > 1$), *then* $u \in C^{1,\beta_0}_{loc}(\Omega, \mathbb{R}^N)$;
- $u \in W^{1,p}(\Omega, \mathbb{R}^N) \cap C^{0,\gamma}(\Omega, \mathbb{R}^N)$ *and* $q < p + \frac{\alpha}{1-\gamma}$, *then* $u \in C^{1,\beta_0}_{loc}(\Omega, \mathbb{R}^N)$.

- N. N. Ural'tseva, *Semin. in Mathematics, V. A. Steklov Math. Inst., Leningrad* (1968).

No Sobolev regularity and fractal singular sets

$$\alpha \in (0,1], \ a(\cdot) \in C^{\alpha}(\Omega, [0,\infty))$$

$$1 < p < n < n + \alpha < q$$

- The Double Phase functional admits a minimizer $u \notin W^{1,q}_{loc}(B_1)$. ⇜ L. Esposito, F. Leonetti, G. Mingione, *JDE* (2004).
- For any $\varepsilon > 0$, there exists a Double Phase functional with a minimizer $u \in W^{1,p}(\Omega)$ such that $\dim_{\mathcal{H}}(\Sigma(u)) > n - p - \varepsilon$. The singular set is a fractal of Cantor type. ⇜ I. Fonseca, J. Malý, G. Mingione, *ARMA* (2004).
- The bounds $q \leq p + \alpha$ and $q < p + \frac{\alpha}{1-\gamma}$ are sharp. ⇜ A. Kh. Balci, L. Diening, M. Surnachev, *Calc. Var. & PDE* (2020).

Manifold constrained problems

We consider general functionals of the form

$$\mathbf{W^{1,H(\cdot)}(\Omega,\mathbb{S}^{N-1}) \ni w \mapsto \mathcal{F}(w,\Omega) := \int_\Omega f(x,w,Dw)\ dx},$$

where $f\colon\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}\to\mathbb{R}$ is modelled by

$$w \mapsto \int_\Omega b(x,w)\left[|Dw|^p + a(x)|Dw|^q\right]\ dx.$$

Minima and competitors take values into the sphere $\mathbb{S}^{N-1}$.

$$0 \le a(\cdot) \in C^{0,\alpha}(\Omega), \quad q < p+\alpha, \quad q < N$$

$$H(x,z) := \left[|z|^p + a(x)|z|^q\right], \qquad H^-_{B_r(x_0)}(z) := \left[|z|^p + \left(\inf_{x\in B_r(x_0)} a(x)\right)|z|^q\right]$$

- R. Hardt, D. Kinderlehrer, F.-H. Lin, *Ann. Inst. H. Poincaré Anal. Non Linéaire* (1988).

Partial regularity

Theorem (D., G. Mingione, *J. Geometric Analysis* (2020))

- *There exists* $\delta_g > 0$ *so that*

$$\left(\fint_{B_r} H(x,Du)^{1+\delta_g}\, dx\right)^{\frac{1}{1+\delta_g}} \lesssim \fint_{B_{2r}} H(x,Du)\, dx.$$

- $Du \in C^{0,\beta_0}_{\mathrm{loc}}(\Omega_u, \mathbb{R}^{N\times n})$, $|\Sigma_u| = 0$.
- *When* $p(1+\delta_g) \le n$, *regular set and singular set are respectively characterized by*

$$x_0 \in \Omega_u \iff \left[H^-_{B_r(x_0)}\left(\frac{\varepsilon}{r}\right)\right]^{-1} \fint_{B_r(x_0)} H(x,Du)\, dx < 1;$$

$$\Sigma_u = \left\{x_0 \in \Omega : \limsup_{\varrho\to 0} \left[H^-_{B_\varrho(x_0)}\left(\frac{1}{\varrho}\right)\right]^{-1} \fint_{B_\varrho(x_0)} H(x,Du)\, dx > 0\right\}.$$

Singular set of Double Phase minima: first reduction

Recall that

$$\Sigma_u = \left\{ \mathbf{x_0} \in \Omega : \limsup_{\varrho \to 0} \left[\mathbf{H}^-_{\mathbf{B}_\varrho(\mathbf{x_0})} \left(\frac{1}{\varrho} \right) \right]^{-1} \fint_{\mathbf{B}_\varrho(\mathbf{x_0})} \mathbf{H}(\mathbf{x}, \mathbf{Du}) \, \mathbf{dx} > \mathbf{0} \right\}.$$

Gehring Lemma then renders:

$$\Sigma_u \cap \{x \colon a(x) = 0\} \subset \left\{ x_0 \colon \limsup_{\varrho \to 0} \varrho^{p(1+\delta_g)-n} \int_{B_\varrho(x_0)} H(x, Du)^{1+\delta_g} \, dx > 0 \right\},$$

$$\Sigma_u \cap \{x \colon a(x) > 0\} \subset \left\{ x_0 \colon \limsup_{\varrho \to 0} \varrho^{q(1+\delta_g)-n} \int_{B_\varrho(x_0)} H(x, Du)^{1+\delta_g} \, dx > 0 \right\},$$

then, by Giusti Lemma we can conclude that

$$\mathcal{H}^{\mathbf{n}-\mathbf{p}}(\Sigma_u \cap \{\mathbf{x} \colon \mathbf{a}(\mathbf{x}) = \mathbf{0}\}) = \mathbf{0} \quad \text{and} \quad \mathcal{H}^{\mathbf{n}-\mathbf{q}}(\Sigma_u \cap \{\mathbf{x} \colon \mathbf{a}(\mathbf{x}) > \mathbf{0}\}) = \mathbf{0}.$$

Orlicz-Musielak functions

We consider a function $\Phi: \Omega \times [0, \infty) \to [0, \infty)$ such that

$$x \mapsto \Phi(x, t) \text{ measurable for all } t \geq 0,$$
$$t \mapsto \Phi(x, t) \text{ continuous and non-decreasing.}$$

Moreover, we assume that

$$\Phi(x, 0) = 0 \quad \text{and} \quad \lim_{t \to \infty} \Phi(x, t) = \infty \quad \text{for all} \quad x \in \Omega,$$
$$\Phi(x, t) \lesssim m(x) t^n \quad \text{for all} \quad t \geq 1\,, \quad x \in \Omega, \quad m \in L^1(\Omega)$$

and that there exists $\beta_0 \in (0, 1)$ such that

$$\Phi(x, \beta_0) \leq 1, \quad \Phi\left(x, 1/\beta_0\right) \geq 1,$$
$$t \mapsto \Phi(x, t) \text{ non decreasing for every } x \in \Omega.$$

- J. Musielak, *Lecture Notes in Mathematics, Springer* (1983).

Intrinsic Hausdorff measures - Construction

For any set $E \subset \Omega$ and a ball B of radius $r(B) \in (0, \infty)$, we define the quantity

$$\mathbf{h}_{\Phi}(\mathbf{B}) = \int_{\mathbf{B}} \Phi\left(\mathbf{x}, 1/\mathbf{r}(\mathbf{B})\right)\, \mathbf{dx}\,,$$

and, via Carathéodory's construction, the κ-approximating Hausdorff measures

$$\mathcal{H}_{\Phi,\kappa}(E) = \inf_{\mathcal{C}_E^{\kappa}} \sum_j h_{\Phi}(B_j),$$

$$\mathcal{C}_E^{\kappa} = \left\{ \{B_j\}_{j\in\mathbb{N}} \text{ balls covering } E \text{ such that } r(B_j) \le \kappa \right\}.$$

As $\kappa \mapsto \mathcal{H}_{\Phi,\kappa}(\cdot)$ is non-increasing, there exists the limit

$$\mathcal{H}_{\Phi}(\mathbf{E}) := \lim_{\kappa\to 0} \mathcal{H}_{\Phi,\kappa}(\mathbf{E}) = \sup_{\kappa>\mathbf{0}} \mathcal{H}_{\Phi,\kappa}(\mathbf{E}).$$

- $\mathcal{H}_\Phi(\cdot)$ turns out to be Borel-regular.
- Furthermore, it is convenient to localize the x-dependence of the integrand and define

$$h^+_\Phi(B) := |B| \mathrm{esssup}_{x\in B}\, \Phi\left(x, \frac{1}{r(B)}\right), \quad h^-_\Phi(B) := |B| \mathrm{essinf}_{x\in B}\, \Phi\left(x, \frac{1}{r(B)}\right).$$

- Again, Carathéodory's construction renders

$$\mathcal{H}^\pm_{\Phi,\kappa}(E) = \inf_{\mathcal{C}^\kappa_E} \sum_j h^\pm_\Phi(B_j) \quad \text{and} \quad \mathcal{H}^\pm_\Phi(E) = \lim_{\kappa\to 0} \mathcal{H}^\pm_{\Phi,\kappa}(E).$$

Clearly, $\mathcal{H}^-_\Phi(E) \le \mathcal{H}_\Phi(E) \le \mathcal{H}^+_\Phi(E)$.

- To connect the above measures, we shall also assume that

$$\mathrm{esssup}_{x\in B}\, \Phi(x, \beta t) \lesssim \mathrm{essinf}_{x\in B}\, \Phi(x,t) \quad \text{for all } t \in [1, r(B)^{-1}],$$

therefore $\mathcal{H}^-_\Phi(E) \approx \mathcal{H}^+_\Phi(E) \approx \mathcal{H}_\Phi(E)$.

Some examples

These definitions unify several instances of similar objects, and introduce new ones

- $\Phi(x,t) \equiv t^p$, $p \leq n$, then $\mathcal{H}_\Phi \approx \mathcal{H}^{n-p}$;
- $\Phi(x,t) \equiv t^{p(x)}$, $p(\cdot) \leq n$, and this falls into the realm of variable exponent Hausdorff measures;
- $\Phi(x,t) \equiv \omega(x)t^p$, weighted Hausdorff measures, studied in particular when $\omega(\cdot)$ is a Muckenhoupt weight;
- $\Phi(x,t) = [H(x,t)]^{1+\sigma} \equiv [t^p + a(x)t^q]^{1+\sigma}$ for some $\sigma \geq 0$, $q(1+\sigma) \leq n$.

Classical references are:

- E. Nieminen, *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes* (1991).
- B. O. Turesson, *Lecture Notes in Math.* (2000).

Intrinsic capacities

For $K \subset \mathbb{R}^n$, the relative $\Phi(\cdot)$-capacity is defined as

$$cap^*_\Phi(K) \equiv cap^*_\Phi(K,\Omega) = \inf_{f \in \mathcal{R}(K)} \int_\Omega \Phi(x, |Df|)\, dx$$

$$\mathcal{R}(K) = \left\{ f \in W^{1,\Phi}(\Omega) \cap C_0(\Omega) \colon \ f \geq 1 \ in \ K \right\}$$

As usual, for open subsets $U \subset \Omega$ and general sets $E \subset \Omega$ we define, in sequence

$$cap_\Phi(U) := \sup_{K \subset U, \ K \text{ compact}} cap^*_\Phi(K), \quad cap_\Phi(E) := \inf_{E \subset \tilde{U} \subset \Omega, \ \tilde{U} \text{ open}} cap_\Phi(\tilde{U})$$

cap_Φ is Choquet

- D. Baruah, P. Harjulehto, P. Hästo, *J. Funct. Spaces* (2018).

Φ-Hausdorff measures and intrinsic capacities

Assuming also that, for $0 < s \leq t$

$$\frac{\Phi(x,s)}{s^p} \lesssim \frac{\Phi(x,t)}{t^p} \quad \text{and} \quad \frac{\Phi(x,t)}{t^q} \lesssim \frac{\Phi(x,s)}{s^q}$$

holds for some $1 < p \leq q < \infty$, then

Proposition (D., G. Mingione, *J. Geometric Analysis (2020)*)

Let $E \subset \mathbb{R}^n$ be such that $\mathcal{H}_\Phi(E) < \infty$, then $cap_\Phi(E) = 0$.

The above result extends the standard one connecting $(n-p)$-Hausdorff measures and relative p-capacities.

- H. Federer, W. Ziemer, *Indiana U. Math. J.* (1972).

Singular set estimates via intrinsic capacities

Theorem (D., G. Mingione, *J. Geometric Analysis* (2020))

Let $u \in W^{1,1}_{\rm loc}(\Omega, \mathbb{S}^{N-1})$ *be a local minimizer and let* $\Omega_u \subset \Omega$ *be its regular set. Assume that*

$$[H(\cdot, Du)]^{1+\delta_g} \in L^1_{loc}(\Omega)\,, \quad q(1+\delta_g) \leq n\,, \quad \delta_g \geq 0\,.$$

Then

$$\mathcal{H}_{H^{1+\delta_g}}(\Sigma_u) = 0$$

and therefore

$$cap_{H^{1+\delta_g}}(\Sigma_u) = 0\,.$$

- I. Chlebicka, C. De Filippis, *Ann. Mat. Pura Appl.* (2020).
- I. Chlebicka, A. Karppinen, *JMAA* (2020).

The $p(x)$-laplacian

$$\mathcal{E}(\mathbf{w},\Omega) := \int_\Omega |\mathbf{Dw}|^{\mathbf{p(x)}}\ \mathbf{dx}$$

- V. V. Zhikov, *Math. USSR*, (1987); *Comp. Med. Hom. Th.*, (1995); *Dokl. Ros. Akad. Nauk.*, (1995); *Rus. J. Math. Phys.*, (1995).

$p(x)$-Laplacian: regularity in the unconstrained case

The energy

$$\mathbf{W}^{1,\mathbf{p}(\cdot)}(\Omega, \mathbb{R}^{\mathbf{N}}) \ni \mathbf{w} \mapsto \min \int_{\Omega} |\mathbf{D}\mathbf{w}|^{\mathbf{p}(\mathbf{x})} \, \mathbf{dx}$$

enters in the modelling of electro-rheological fluids. The regularity of the exponent $p(\cdot)$ crucially influences the regularity of minima.

- $\lim_{\varrho \to 0} \omega(\varrho) \log(\varrho^{-1}) = \lambda > 0 \Rightarrow u \in C^{0,\gamma}_{loc}(\Omega, \mathbb{R}^N)$, E. Acerbi, G. Mingione, *ARMA* (2001);
- $\lim_{\varrho \to 0} \omega(\varrho) \log(\varrho^{-1}) = 0 \Rightarrow u \in C^{0,\gamma}_{loc}(\Omega, \mathbb{R}^N)$ for all $\gamma \in (0,1)$, E. Acerbi, G. Mingione, *ARMA* (2001);
- $p(\cdot)$ Hölder continuous $\Rightarrow u \in C^{1,\gamma}_{loc}(\Omega, \mathbb{R}^N)$, A. Coscia, G. Mingione, *C. R. Acad. Sci. Paris* (1999); E. Acerbi, G. Mingione, *ARMA* (2001).

Non-autonomous functionals: the Lavrentiev phenomenon

$$W^{1,p(\cdot)}(\Omega) \ni w \mapsto \int_{\Omega} |Dw|^{p(x)} \, dx, \quad \Omega \subset \mathbb{R}^n, \; n \geq 2$$

Discontinuous coefficients

Let $n = 2$ and $\Omega = B_1$. If $p(\cdot)$ has a saddle point in zero, then we can find a function $u \in W^{1,p(\cdot)}(B_1)$ which cannot be approximated by smooth functions in any neighborhood of the origin.

Log-continuity

$$\omega(t) := \left[\log\left(\frac{1}{t}\right) \right]^{-s}, \quad \begin{cases} s \geq 1 \Rightarrow \text{smooth maps dense in } W^{1,p(\cdot)}(\Omega), \\ 0 < s < 1 \Rightarrow \text{density may fail.} \end{cases}$$

Sobolev regularity $\Leftrightarrow$ Absence of Lavrentiev phenomenon

- L. Esposito, F. Leonetti, G. Mingione, *JDE* (2004).

Partial regularity

We consider a minimizer $u \in W^{1,p(\cdot)}(\Omega, \mathcal{M})$ of the functional:

$$\mathbf{W^{1,p(\cdot)}}(\Omega, \mathcal{M}) \ni \mathbf{w} \mapsto \int_\Omega \mathbf{k(x)|Dw|^{p(x)}\ dx},$$

where $0 < k(\cdot) \in C^{0,\nu}(\Omega)$ and $1 < p(\cdot) \in C^{0,\alpha}(\Omega)$. Set

$$\gamma_1 := \inf_{x \in \Omega} p(x) \quad \text{and} \quad \gamma_2 := \sup_{x \in \Omega} p(x).$$

The manifold $\mathcal{M} \subset \mathbb{R}^N$ is isometrically embedded in $\mathbb{R}^N$ and it is $[\gamma_2] - 1$-connected, in the sense that

$$\Pi_0(\mathcal{M}) = \cdots = \Pi_{[\gamma_2]-1}(\mathcal{M}) = \mathbf{0}.$$

- R. Hardt, F.-H. Lin, *CPAM* (1987).

Partial regularity

Theorem (D., Calc. Var. & PDE (2019))

- *There exists $\delta_g > 0$ so that for all $\delta \in (0, \delta_g)$:*

$$\left(\fint_{B_{\varrho/2}(x_0)} |Du|^{(1+\delta)p(x)} \ dx\right)^{\frac{1}{1+\delta}} \lesssim \fint_{B_\varrho(x_0)} (1+|Du|^2)^{p(x)/2} \ dx.$$

- $Du \in C^{0,\beta_0}_{\mathrm{loc}}(\Omega_u, \mathbb{R}^{N\times n})$, $\mathcal{H}^{n-\gamma_1}(\Sigma_u) = 0$.
- *Regular set and singular set are respectively characterized by*

$$\left((2r)^{p_2(2r,x_0)-n} \int_{B_{2r}(x_0)} (1+|Du|^2)^{p_2(2r,x_0)/2} \ dx\right)^{\frac{1}{p_2(2r,x_0)}} < \varepsilon;$$

$$x_0 \in \Omega\colon \ \limsup_{\varrho\to 0} \left(\varrho^{p_2(\varrho,x_0)-n} \int_{B_\varrho(x_0)} |Du|^{p_2(\varrho,x_0)} \ dx\right)^{\frac{1}{p_2(\varrho,x_0)}} > 0.$$

- M. A. Ragusa, A. Tachikawa, A. Takabayashi, *Trans. AMS* (2013).

Partial boundary regularity

Consider the Dirichlet problem

$$\mathbf{g} + \left(\mathbf{W_0^{1,p(\cdot)}}(\Omega, \mathbb{R}^\mathbf{N}) \cap \mathbf{W^{1,p(\cdot)}}(\Omega, \mathcal{M})\right) \ni \mathbf{w} \mapsto \min \int_\Omega \mathbf{k(x)|Dw|^{p(x)}} \ \mathbf{dx}$$

with $g \in W^{1,q}(\bar{\Omega}, \mathcal{M})$ with $q > \max\{n, \gamma_2\}$.

Theorem (I. Chlebicka, D., L. Koch, *Preprint* (2020))

There exists a relatively (to $\bar{\Omega}$) open set $\Omega_u \subset \bar{\Omega}$ so that $u \in C^{0,1-\frac{n}{q}}_{loc}(\Omega_0, \mathcal{M})$ and $\mathcal{H}^{n-\gamma_1}(\Sigma_u) = 0$.

- M. A. Ragusa, A. Tachikawa, *Annales IHP - AN* (2016).

$p(x)$-harmonic maps: dimension reduction

Let $(k_j)_{j\in\mathbb{N}}, (p_j)_{j\in\mathbb{N}}$ be two sequences of α-Hölder continuous functions, $\alpha \in (0,1]$, satisfying

$$\begin{cases} \sup_{j\in\mathbb{N}}[k_j]_{0,\alpha} < c_k \\ \lambda \le k_j(x) \le \Lambda \\ \|k_j - k\|_{L^\infty(B_1)} \to 0, \ k(\cdot) \in C^{0,\alpha}(B_1) \end{cases} , \quad \begin{cases} \sup_{j\in\mathbb{N}}[p_j]_{0,\alpha} < c_p \\ p_j(x) \ge \gamma_1 > 1 \\ \|p_j - p_0\|_{L^\infty(B_1)} \to 0 \end{cases}$$

respectively. Here, $p_0 \ge \gamma_1 > 1$ is a constant. For each $j \in \mathbb{N}$, let $u_j \in W^{1,p_j(\cdot)}(B_1,\mathcal{M})$ be a constrained local minimizer of

$$\mathcal{E}_j(w, B_1) := \int_{B_1} k_j(x)|Dw|^{p_j(x)}\, dx.$$

Lemma (Compactness of minimizers)

Then, there exists a subsequence such that

$$u_j \rightharpoonup v \quad \text{weakly in } W^{1,(1+\tilde{\sigma})p_0}(B_r, \mathcal{M}), \quad \tilde{\sigma} > 0,$$

for any $r \in (0,1)$ and v is a constrained local minimizer of the functional

$$\mathcal{E}_0(w, B_1) := \int_{B_1} k(x)|Dw|^{p_0} \, dx.$$

Moreover, $\mathcal{E}_j(u_j, B_r) \to \mathcal{E}_0(v, B_r)$ for all $r \in (0,1)$. Finally, if x_j is a singular point of u_j and $x_j \to \bar{x}$, then $\bar{x}$ is a singular point for v.

$p(x)$-harmonic maps: dimension reduction

Lemma (Monotonicity formula)

Let $k(\cdot) \in C^{0,\alpha}(\Omega)$, $\alpha \in (0,1]$ be such that $k(0) = 1$, $p(\cdot) \in Lip(\Omega)$ and $n > \gamma_2 \geq p(x) \geq 2$ for all $x \in \Omega$. If $u \in W^{1,p(\cdot)}(\Omega, \mathcal{M})$ is a constrained local minimizer of $\mathcal{E}(\cdot)$ on B_1, then for any $\gamma \in (0,1)$ there exist a positive constant c and $T \in (0,1)$ such that for all $0 < r < R < T$, we have

$$\int_{\partial B_1} |u(Rx) - u(rx)|^{p_2(r)} \, d\mathcal{H}^{n-1}(x)$$

$$\leq c r^{p_2(r) - p_2(R)} \left(\log \frac{R}{r} \right)^{p_2(r)-1} \left((\Phi(R) - \Phi(r)) + (R^\gamma - r^\gamma) \right),$$

where

$$\Phi(t) := t^{p_2(t)-n} \exp(At^\alpha) \int_{B_t} k(x) |Du|^{p_2(t)} \, dx.$$

- A. Tachikawa, *CalcVar & PDE* (2014).

Dimension reduction

Theorem (D., *Calc. Var. & PDE* (2019))

Let $u \in W^{1,p(\cdot)}(\Omega, \mathcal{M})$ be a constrained local minimizer of energy

$$W^{1,p(\cdot)}(\Omega, \mathcal{M}) \ni w \mapsto \mathcal{E}(w, \Omega) := \int_\Omega |Dw|^{p(x)} \, dx,$$

where $p(\cdot) \in Lip(\Omega)$ and $\gamma_1 \geq 2$. Then,

- *if $n \leq [\gamma_1] + 1$, then u can have only isolated singularities;*
- *if $n > [\gamma_1] + 1$, then the Hausdorff dimension of the singular set is at the most $n - [\gamma_1] - 1$.*

Full boundary regularity

Theorem (I. Chlebicka. D., L. Koch, *Preprint* (2020))

Let $u \in W^{1,p(\cdot)}(\Omega, \mathcal{M})$ *be a solution of the Dirichlet problem*

$$g + \left(W^{1,p(\cdot)}(\Omega, \mathcal{M}) \cap W_0^{1,p(\cdot)}(\Omega, \mathbb{R}^N) \right) \ni w \mapsto \min \int_\Omega |Dw|^{p(x)} \, dx,$$

with $2 \le p(\cdot) \in \mathrm{Lip}(\Omega)$ *and* $g \in W^{1,q}(\bar{B}_1^+, \mathcal{M})$, $q > \max\{\gamma_2, n\}$ *satisfying the smallness condition*

$$[g]_{0,1-\frac{n}{q}} < \Upsilon,$$

for suitable $\Upsilon \in (0,1]$. *Then,* $\Sigma_u \Subset \Omega$.

- R. Schoen, K. Uhlenbeck, *J. Differential Geometry* (1982); *J. Differential Geometry* (1983).

Everywhere discontinuous solutions

The presence of the singular set is in general unavoidable.
Sphere-valued harmonic maps satisfy in a suitably weak sense

$$-\Delta u = |Du|^2 u.$$

- J. Eels, J. H. Sampson, *Amer. J. Math.* (1963). ⇜ Regularity when target manifold is compact and with non-positive curvature.
- C. B. Morrey, *Annals of Math.* (1948). ⇜ Regularity in two space dimensions.
- S. Hildebrand, H. Kaul, K. O. Widman, *Acta Math.* (1970). ⇜ Regularity under suitable restrictions on the image of solutions.
- T. Riviére, *D. Phil. Thesis* (1993); *Acta Math.* (1995). ⇜ Regularity for maps with values in the two-dimensional toruus of revolution; everywhere discontinuous $\mathbb{S}^2$-valued harmonic maps.

Ural'tseva conjecture

Consider the elliptic system

$$-\operatorname{div} a(x, Du) = f \quad \text{in} \quad \Omega \subset \mathbb{R}^n, \quad n \geq 2.$$

What are the optimal conditions on f assuring that solutions are regular?

The optimal conditions on f do not depend from the structure of the nonlinear tensor $\Omega \times \mathbb{R}^{N\times n} \ni (x, z) \mapsto a(x, z)$.

- The regularity of the partial map $x \mapsto a(x, z)$ subtly influences the regularity of solutions.
- The optimal conditions to be assumed on the forcing term f do not depend on the structure of $a(x, z)$.
- Both conditions we'll find are sharp.

Sobolev-Morrey embedding theorem

Given any $w \in W^{1,p}_{loc}(\mathbb{R}^n, \mathbb{R}^N)$, then

- $p \in [1, n) \Rightarrow w \in L^q_{loc}(\mathbb{R}^n, \mathbb{R}^N)$ for all $q \in \left[p, \frac{np}{n-p}\right]$;
- $p = n \Rightarrow w \in L^q_{loc}(\mathbb{R}^n, \mathbb{R}^N)$ for all $q \in [p, \infty)$;
- $p > n \Rightarrow w \in C^{0,\lambda}_{loc}(\mathbb{R}^n, \mathbb{R}^N)$ with $\lambda := 1 - \frac{n}{p}$.

A sufficiently high level of integrability of the gradient of a function results in a boost of regularity for the function itself.

Lorentz spaces

Look at the Lorentz space

$$\mathbf{L(n,1)(\mathbb{R}^n,\mathbb{R}^k)} := \left\{ \mathbf{w}\colon \mathbb{R}^n \to \mathbb{R}^k\colon \int_0^\infty |\{\mathbf{x}\colon |\mathbf{w}(\mathbf{x})| > \mathbf{t}\}|^{\frac{1}{n}} \, \mathbf{dt} < \infty \right\}.$$

It is well-known that

$$\mathbf{L^{n+\varepsilon}} \hookrightarrow \mathbf{L(n,1)} \hookrightarrow \mathbf{L^n} \quad \text{for all} \quad \varepsilon > \mathbf{0}.$$

Theorem (E. M. Stein, *Annals of Math.* (1981))

If $w \in W(1;n,1)_{loc}(\mathbb{R}^n,\mathbb{R}^N)$, then

- $w \in C_{loc}(\mathbb{R}^n,\mathbb{R}^N)$;
- w *is differentiable a.e. in the sense that*

$$w(x+h) - w(x) - h \cdot Dw(x) = o(|h|) \quad \text{as} \quad |h| \to 0.$$

A (linear) PDE interpretation of Stein's theorem

Combining the implication

$$\mathbf{Dw} \in \mathbf{L(n,1)_{loc}}(\mathbb{R}^n, \mathbb{R}^{N\times n}) \Rightarrow \mathbf{w} \in \mathbf{C_{loc}}(\mathbb{R}^n, \mathbb{R}^N)$$

with standard Calderón-Zygmund theory there holds that

$$-\Delta \mathbf{u} \in \mathbf{L(n,1)} \Rightarrow \mathbf{Du} \quad \text{is continuous.}$$

- A. Cianchi, *J. Geom. Analysis* (1993).

Orlicz spaces

Consider the Orlicz space

$$\mathbf{L^n}(\mathrm{LogL})^{\alpha}(\mathbb{R}^n,\mathbb{R}^k) := \left\{ \mathbf{w} \in \mathbf{L^n}(\mathbb{R}^{\mathbf{n}},\mathbb{R}^{\mathbf{k}}) \colon \int_{\mathbb{R}^{\mathbf{n}}} |\mathbf{Dw}|^{\mathbf{n}} \log^{\alpha}(\mathbf{e} + |\mathbf{Dw}|) \ \mathbf{dx} < \infty \right\}.$$

If $w \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^k)$ is so that

$$\int_{\Omega} |\mathbf{Dw}|^{\mathbf{n}} \log^{\alpha}(\mathbf{e} + |\mathbf{Dw}|) \ \mathbf{dx} < \infty \quad \text{for some} \quad \alpha > \mathbf{n} - \mathbf{1}$$

for all open subset $\Omega \subset \mathbb{R}^n$, then $\mathbf{u} \in \mathbf{C}_{\mathrm{loc}}(\mathbb{R}^{\mathbf{n}},\mathbb{R}^{\mathbf{k}})$.

- J. Kauhaunen, P. Koskela, J. Malý, *Manuscripta Math.* (1999).

Nonlinear Stein's theorem

Theorem (T. Kuusi, G. Mingione, *Calc. Var. & PDE* (2014))

Consider the p-laplacean system

$$-\operatorname{div}\left(\gamma(x)|Du|^{p-2}Du\right) = f \quad \textit{in} \ \ \Omega,$$

where

$$f \in L(n,1)_{loc}(\Omega,\mathbb{R}^N) \quad \textit{and} \quad \gamma\colon \Omega \to (0,\Gamma] \ \ \textit{Dini-continuous.}$$

Then $Du \in C_{loc}(\Omega,\mathbb{R}^{N\times n})$.

The condition $\mathbf{f \in L(n,1)_{loc}(\Omega,\mathbb{R}^N)}$ does not depend on p.

- P. Daskalopoulos, T. Kuusi, G. Mingione, *Comm. PDE* (2014) ⇜ Fully nonlinear elliptic equations.
- S. Sil, *Calc. Var. & PDE* (2019) ⇜ Systems of differential forms.
- A. Banerjee, I. Munive, *CCM* (2020) ⇜ Normalized p-Laplacean.

Uniform ellipticity

For the autonomous system

$$-\operatorname{div} \tilde{a}(|Du|)Du = f \quad \text{in} \quad \Omega,$$

the **uniform ellipticity** condition reads as

$$-1 < i_a \leq \frac{\tilde{a}(t)'t}{\tilde{a}(t)} \leq s_a < \infty \quad \text{for all} \quad t > 0$$
$$\tilde{a} \in C^1_{loc}((0,\infty),[0,\infty)).$$

Set $A(t) := \int_0^t \tilde{a}(s)s \ ds$. The uniform ellipticity condition yields that

$$\mathbf{A(2t) \lesssim A(t)} \quad \text{and} \quad \mathbf{t^{i_a+2} \lesssim A(t) \lesssim t^{s_a+2}}$$

for $t > 0$ large enough.

- P. Baroni, *Calc. Var. & PDE* (2015);
- A. Cianchi, V. G. Maz'ya, *ARMA* (2014); *JEMS* (2014).

Non-uniform ellipticity

Functionals of the type

$$w \mapsto \int_\Omega \left[\exp\left(\exp\left(\cdots \exp\left(|Dw|^p\right)\right)\right) - f \cdot w \right] \, dx$$

or

$$w \mapsto \int_\Omega \left[(\max\{|Dw| - T, 0\})^p + \sum_{i=1}^{n} |D_i w|^{q_i} - f \cdot w \right] \, dx$$

are excluded by the uniform ellipticity condition.

- A. Cellina, *ESAIM Control Optim. Calc. Var.* (2016);
- A. Cellina, V. Staicu, *Calc. Var. & PDE* (2018);
- L. C. Evans, *Calc. Var. & PDE* (2003).

Their common feature is that the **ellipticity ratio** may blow up:

$$1 \leq \mathcal{R}(z) := \frac{\text{highest eigenvalue of } \partial_z^2 A(|z|)}{\text{lowest eigenvalue of } \partial_z^2 A(|z|)} \to_{|z|\to\infty} \infty.$$

- L. Beck, G. Mingione, *CPAM* (2020).

Non-autonomous functionals

We consider non-autonomous functionals

$$w \mapsto \int_{\Omega} \left[F(x, Dw) - f \cdot w \right] \, dx.$$

The ellipticity ratio associated to $F(x, z)$ is defined as

$$\mathbf{1} \le \mathcal{R}(\mathbf{x}, \mathbf{z}) := \frac{\text{highest eigenvalue of } \partial_{\mathbf{z}}^{2} \mathbf{F}(\mathbf{x}, \mathbf{z})}{\text{lowest eigenvalue of } \partial_{\mathbf{z}}^{2} \mathbf{F}(\mathbf{x}, \mathbf{z})}.$$

- $\sup_{x \in B} \mathcal{R}(x, z) \to_{|z| \to \infty} \infty$ for at least one ball $B \Subset \Omega$ $\leadsto$ Non-uniform ellipticity.

Is $\mathbf{1} \le \sup_{\mathbf{x} \in \mathbf{B}} \mathcal{R}(\mathbf{x}, \mathbf{z}) \le \mathbf{M}$ *for all balls* $B \Subset \Omega$ *enough to assure* ***uniform ellipticity****?* ***No!*** *We need to take into account also of the interaction between the coefficient term and the underlying energy.*

The general problem

As we are studying general vectorial functionals, we assume **radial** structure:

$$F(x,z) = \tilde{F}(x,|z|).$$

- Control of eigenvalues. $\leftsquigarrow$ Description of Ellipticity.

$$\mathbf{g_1(x,|z|)\mathbb{I} \lesssim \partial_{zz}F(x,z) \lesssim g_2(x,|z|)\mathbb{I}.}$$

- Controlled non-uniform ellipticity. $\leftsquigarrow$ Growth of the ellipticity ratio.

$$\mathbf{\frac{g_2(x,|z|)}{g_1(x,|z|)} \lesssim K\left(\int_0^{|z|} g_1(x,s)s\,ds\right).}$$

for a suitable increasing function $K(\cdot)$ which is of **power-type**.

The variational setting - Sobolev coefficients

- Controlled differentiability. $\leadsto$ Sobolev coefficients.

$$|\partial_{xz}F(x,z)| \le h(x)g_3(x,|z|), \quad 0 \le h(\cdot) \in L^d(\Omega), \quad d > n;$$

$$|\partial_x g_1(x,t)| \le h(x)g_1(x,t)K_1\left(\int_0^{|z|} g_1(x,s)s \ ds\right);$$

$$g_3(x,t)\sqrt{t^2+\mu^2} \le K_2\left(\int_0^{|z|} g_1(x,s)s \ ds\right);$$

$$\frac{[g_3(x,t)]^2}{g_1(x,t)} \le K_3\left(\int_0^{|z|} g_1(x,s)s \ ds\right).$$

Functions K_i, $i \in \{1,2,3\}$ are suitable increasing functions of **power-type**.

Theorem (D., G. Mingione, *Preprint* (2020))

Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ *be a minimizer of the above variational problem. If*

$$f \in X_{\mathrm{loc}}(\Omega,\mathbb{R}^N) := \begin{cases} L(n,1)_{\mathrm{loc}}(\Omega,\mathbb{R}^N) & \text{if } n \geq 3 \\ L^2_{\mathrm{loc}}(\mathrm{LogL})^{\alpha}(\Omega,\mathbb{R}^N),\ \alpha > 2 & \text{if } n = 2, \end{cases}$$

then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$.

We now consider the assumptions

$$\begin{cases} \nu(\mu^2 + |z|^2)^{\frac{p-2}{2}}|\xi|^2 \leq \left\langle \partial_{zz} F(x,z)\xi, \xi \right\rangle & \mu \geq 0 \\ |\partial_{zz} F(x,z)| \leq L(\mu^2 + |z|^2)^{\frac{q-2}{2}} & 1 < p < q \\ |\partial_{xz} F(x,z)| \leq h(x)(1 + |z|^2)^{\frac{q-1}{2}} & \text{for } h(\cdot) \in L^d \quad d > n. \end{cases}$$

$$\mathbf{g_1(t) \sim t^{p-2}} \qquad \mathbf{g_2(t) \sim t^{q-2}} \qquad \mathbf{g_3(t) := t^{q-1}}$$

- M. Eleuteri, P. Marcellini, E. Mascolo, *AMPA* (2016).
- P. Marcellini, *ARMA* (1989); *JDE* (1991).

The (p,q)-growth case

Theorem (D., G. Mingione, *Preprint* (2020))

If u is a local minimizer of the functional

$$v \mapsto \int_\Omega \left[F(x,Dv) - f\cdot v\right]\, dx$$

and assume that, when $n>2$,

$$\frac{q}{p} < 1 + \min\left\{\frac{1}{n} - \frac{1}{d}, \frac{4(p-1)}{p(n-2)}\right\}, \qquad f \in L(n,1)\ ,$$

or for $n=2$

$$\frac{q}{p} < 1 + \min\left\{\frac{1}{n} - \frac{1}{d}, \frac{2(p-1)}{\vartheta p}\right\}, \qquad f \in L^2(\mathrm{LogL})^\alpha, \quad \alpha > 2.$$

Then Du is locally bounded in Ω. When $f\equiv 0$, the bound reduces to

$$\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{d}.$$

(p,q)-growth vs structure

$$\mathbf{W^{1,1}_{loc}(\Omega,\mathbb{R}^N) \ni v \mapsto \int_\Omega \left[|Dw|^p + a(x)|Dw|^q - f\cdot w\right] \ dx \qquad p>1}$$

$$\mathbf{g_1(x,t) \sim g_2(x,t) \sim t^{p-2} + a(x)t^{q-2} \qquad g_3(x,t) = t^{q-1}}$$

In this case we have

$$\frac{\mathbf{g_2(x,t)}}{\mathbf{g_1(x,t)}} \sim \text{const} \qquad \text{for all balls } \mathbf{B} \Subset \Omega.$$

The unbalanced growth will be visible when relating to the energy those terms involving g_3, g_3^2/g_1 or $\partial_x g_1$. Severe loss of information if we set

$$\mathbf{g_1(t) \sim t^{p-2} \qquad g_2(t) \sim t^{q-2}}.$$

$$\mathbf{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^{\mathbf{N}}) \ni \mathbf{w} \mapsto \int_{\Omega} \left[\gamma(\mathbf{x})|\mathbf{Dw}|^{\mathbf{p}} \log(\mathbf{1}+|\mathbf{Dw}|) - \mathbf{f}\cdot\mathbf{w}\right] \ \mathbf{dx} \qquad \mathbf{p}>\mathbf{1}$$

$$\mathbf{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^{\mathbf{N}}) \ni \mathbf{w} \mapsto \int_{\Omega} \gamma(\mathbf{x})|\mathbf{Dw}|^{\mathbf{p}(\mathbf{x})} - \mathbf{f}\cdot\mathbf{w} \ \mathbf{dx} \qquad \inf_{x\in\Omega} p(x)>1.$$

In general, we can treat integrands with Orlicz-Musielak structure

$$\mathbf{W}^{1,\varphi(\cdot)}_{\mathrm{loc}}(\Omega,\mathbb{R}^{\mathbf{N}}) \ni \mathbf{w} \mapsto \int_{\Omega} \varphi(\mathbf{x},|\mathbf{Dw}|) \ \mathbf{dx}.$$

- S.-S. Byun, J. Oh, *Anal. PDE* (2020).
- C. De Filippis, J. Oh, *JDE* (2019).
- P. Hästö, J. Ok, *JEMS*, to appear.
- M. A. Ragusa, A. Tachikawa, *Adv. Nonlinear Anal.* (2020).

L^∞-bounds for structures

Theorem (D., G. Mingione, *Preprint* (2020))

If u is a local minimizer of the functional

$$v \mapsto \int_\Omega \left[|Dw|^p + a(x)|Dw|^q - f \cdot v \right] \, dx$$

and assume that, when $n > 2$,

$$\frac{q}{p} \leq 1 + \frac{1}{n} - \frac{1}{d}, \qquad f \in L(n,1) \, ,$$

or for $n = 2$

$$\frac{q}{p} \leq 1 + \frac{1}{n} - \frac{1}{d} \quad \text{and} \quad q < p^2, \qquad f \in L^2(\mathrm{LogL})^\alpha, \quad \alpha > 2.$$

Then Du is locally bounded in Ω. *When* $f \equiv 0$, *in both the cases we have*

$$\frac{q}{p} \leq 1 + \frac{1}{n} - \frac{1}{d}.$$

Fast growth conditions

Estimates apply to

$$\mathbf{w} \mapsto \int_\Omega \left[\gamma(\mathbf{x}) \exp(|\mathbf{Dw}|^{\mathbf{p}(\mathbf{x})}) - \mathbf{f}\cdot\mathbf{w}\right] \mathbf{dx}\,, \qquad \mathbf{p}(\cdot) > \mathbf{1},\ \gamma(\cdot) > \mathbf{0},$$

and, more in general, to

$$\mathbf{w} \mapsto \int_\Omega \gamma_1(\mathbf{x}) \left[\exp(\exp(\ldots\exp(\gamma_2(\mathbf{x})|\mathbf{Dw}|^{\mathbf{p}(\mathbf{x})})\ldots)) - \mathbf{f}\cdot\mathbf{w}\right] \mathbf{dx}$$

under the assumptions

$$D\gamma, D\gamma_1, D\gamma_2, Dp \in L^{n+\varepsilon};$$

$$f \in L(n,1) \ \text{ if } \ n > 2 \quad \text{or} \quad f \in L^2(\mathrm{LogL})^\alpha,\ \alpha > 2 \ \text{ if } \ n = 2.$$

$$\frac{g_2(x,|z|)}{g_1(x,|z|)} \sim |z|^{p(x)}$$

$$\int_0^{|z|} g_1(x,s)s \ ds \sim \exp(\exp(\ldots\exp(|Dw|^{p(x)})\ldots))$$

- E. Mascolo, A. P. Migliorini, *ESAIM COCV* (2003).

Back to the uniformly elliptic case

Theorem (D., G. Mingione, *Preprint* (2020))

Let $u\colon\Omega(\subset\mathbb{R}^n)\to\mathbb{R}^N\in W^{1,p}(\Omega;\mathbb{R}^N)$ *be a solution to the p-Laplacean system*

$$-\operatorname{div}\left(\gamma(x)|Du|^{p-2}Du\right)=f\ ,\qquad p>1$$

with $0<\nu\le\gamma(x)\le L$. *Assume that*

$$D\gamma, f\in L(n,1)\quad \text{if}\quad n>2;$$
$$D\gamma, f\in L^2(LogL)^{\alpha},\ \alpha>2\quad \text{if}\quad n=2.$$

Then Du is locally bounded.

No Dini continuity of the coefficient $\gamma(\cdot)$ is assumed here.

- J. Kauhanen, P. Koskela, J. Malý, *Manuscripta Math.* (1999).
- T. Kuusi, G. Mingione, *Calc. Var. & PDE* (2014).
- E. M. Stein, *Ann. of Math.* (1981).

Applications to obstacle problems

$$\mathcal{K}_{\psi}(\Omega) \ni w \mapsto \min \int_{\Omega} F(x, Dw) \ dx$$

$$\mathcal{K}_{\psi}(\Omega) := \left\{ w \in W^{1,1}_{\mathrm{loc}}(\Omega) \colon w(x) \geq \psi(x) \text{ in } \Omega \right\}$$

After approximation and linearization, obstacle problems can be rearranged in form

$$W^{1,1}_{\mathrm{loc}}(\Omega) \ni w \mapsto \int_{\Omega} \left[F(x, Dw) - f \cdot w \right] \ dx,$$

for some f depending on $D\psi$ and $D^2\psi$.

- M. Fuchs, *Nonlinear Anal.* (1990).
- M. Fuchs, G. Mingione, *Manuscripta Math.* (2000).

Non-autonomous obstacle problems with fast exponential growth can be treated under the sharp assumptions on the obstacle:

$$\psi \in W_{\mathrm{loc}}(2; n, 1)(\Omega) \quad \text{if} \quad n \geq 3$$

$$\psi \in W_{\mathrm{loc}}(2; L^2(\mathrm{LogL})^{\alpha})(\Omega), \text{ with } \alpha > 2 \quad \text{if} \quad n = 2.$$

References

- K. Adimurthi, A. Banerjee, *Preprint* (2020).
- P. Baroni, *JDE* (2013).
- M. Carozza, J. Kristensen, A. Passarelli di Napoli, *Annales IHP - AN* (2011); *Annali SNS* (2014).
- C. De Filippis, *Calc. Var. & PDE* (2020).
- L. Diening, B. Stroffolini, A. Verde, *Manuscripta Math.* (2009); *JDE* (2012).
- F. Giannetti, A. Passarelli di Napoli, C. Scheven, *Proc. R. Soc. Edinb. A* (2020).
- J. Hirsch, M. Schäffner, *Comm. Cont. Math.* (2020).
- A. Karppinen, M. Lee, *Preprint* (2020).
- T. Kuusi, G. Mingione, *J. Funct. Anal.* (2012); *JEMS* (2018).
- G. Mingione, *Annali SNS* (2007); *JEMS* (2011).