Boundary regularity of minimizers of double phase functionals

Atsushi Tachikawa

Monday's Nonstandard Seminar 2020/21
30 November 2020
(Warsaw, On-line seminar)

Department of Mathematics, Faculty of Science and Technology,
Tokyo University of Science
tachikawa_atsushi@ma.noda.tus.ac.jp
a.tachikawa56@gmail.com

Introduction

Nonstandard growth

$\Omega \subset \mathbb{R}^m$, $\partial\Omega$: sufficiently smooth, $u : \Omega \to \mathbb{R}^n$

$$\mathcal{F}(u;\Omega) := \int_\Omega f(x,u,Du)dx,$$

- Growth condition:

$$\lambda|z|^p \leq f(x,u,z) \leq \Lambda(1+|z|)^q \quad (1 \leq p \leq q)$$

$p = q$: Standard growth
$p < q$: Non-standard growth (introduced by P.Marcellini '89)

$*$ Before this talk, in "Non-standard Seminar", in many talks, there have been many good introduction for non-standard growth problems. See, for example, presentation file by C. De Filippis.

$*$ In this talk, we treat only vectorial cases ($n \geq 2$). For scaler-valued case ($n = 1$), see for example presentation files for this seminar by P. Hästö or by P.Harjuleht.

Functionals with variable exponents

V.V. Zhikov '95, '97

$$\int |Du|^{p(x)} dx$$

- Higher integrability of minimizers for continuous $p(\cdot)$.
- Levrentiev phenomenon for discontinuous $p(x)$:
  $m = 2$, $\Omega = B_1 := \{x = (x^1, x^2) \in \mathbb{R}^2 \ ; \ |x| < 1\}$,
  $1 < \alpha < 2 < \beta$,

$$p(x) = \begin{cases} \alpha & \text{for} \quad x^1 x^2 > 0, \\ \beta & \text{for} \quad x^1 x^2 \leq 0. \end{cases}$$

(This functional is also considered as *"double phase"-type.*)

Functional of Double Phase

Functionals of double phase have been introduced in M.Colombo-G.Mingione ('15,'15), P.Baroni-M.Colombo-G.Mingione ('16).

$$\mathcal{H}(u) := \int H(x, Du)dx, \quad H(x,\xi) := |\xi|^p + a(x)|\xi|^q,$$

where $q > p > 1$ and $a(\cdot) \geq 0$ and $a \in C^{0,\alpha}$. Let u be a local minimizer of $\mathcal{H}$ (or more general type of functionals of double phase), then we have the following regularity results:

- $q/p < 1 + (\alpha/m) \Longrightarrow u \in C^{1,\gamma}_{\mathrm{loc}}(\Omega)$. (M.Colombo-G.Mingione ARMA. 215 (2015))
- $u \in L^\infty(\Omega)$, $q/p < \alpha$ ($q/p \leq \alpha$ for $n = 1$) $\Longrightarrow u \in C^{1,\gamma}_{\mathrm{loc}}(\Omega)$. (M.Colombo-G.Mingione ARMA. 218 (2015))
- Manifold constrained, $q/p < \alpha \Longrightarrow$ partial $C^{1,\gamma}$-regularity (C. De Filippis-G.Mingione JGA (2020))

Double phase with variable expponents

$$\mathcal{F}(u;K) := \int_K \left(|Du|^{p(x)} + a(x)|Du|^{q(x)} \right) dx$$

This problem was suggested by Mingione to us (M.A Ragusa and T.) in 2016.

Interior regularity: (Ragusa-T. 2020)
Assume the following conditions on $p(\cdot), q(\cdot)$ and $a(\cdot)$:

- $p(\cdot), q(\cdot) \in C^{0,\sigma}(\Omega) \quad (\sigma \in (0,1))$,
  $q(x) \geq p(x) \geq p_0 > 1$ ($\forall x \in \Omega$),
- $a(\cdot) \in C^{0,\alpha}(\Omega)$, $a(x) \geq 0$, $\alpha \in (0,1]$.
- $\sup\limits_{x \in \Omega} \dfrac{q(x)}{p(x)} < 1 + \dfrac{\beta}{m}, \quad \forall x \in \Omega, \quad \beta = \min\{\alpha, \sigma\}$,

Let u be a local minimizer of $\mathcal{F}$.
$\Longrightarrow u \in C^{1,\gamma}_{\mathrm{loc}}(\Omega)$ for some $\gamma \in (0,1)$.

Boundary regularity of minimizers

Some known facts and results on the boundary regularity

Roughly speaking, for the case in which we can obtain full-interior regularity, we can get boundary regularity by reflection or other classical methods etc.. (The main result of this talk is in this case.)

On the other hand, even for the case for which we can expect only partial regularity, if so-called the *blow-up method* does work, we can show the regularity near the boundary.

- Harmonic maps between Riem. Mfds.: J.Jost-M.Meier (1983), R.Schoen-K.Uhlenbeck(1983),
- p-growth: F.Duzaar-J.F. Grotowski-M.Kronz (2004),
- $p(x)$-growth: M.A.Ragusa-T. (2016), T.-K.Usuba(2017) ,
- $p(x)$-growth, Mfd. constrained: I.Chlebicka-C.De Filippis-L.Koch (2020)
- Orlicz type: F.Giannetti-A.Passarelli di Napoli-T. (2019)

Functional which we consider in this talk

Writing $|\xi|_g^2 := \left(\delta_{ij} g^{\alpha\beta}(x) \xi_\alpha^i \xi_\beta^j\right)$, $G(x,\xi) := |\xi|_g^{p(x)} + a(x)|\xi|_g^{q(x)}$ for $\xi \in \mathbb{R}^{mn}$, we consider the following functional:

$$\mathcal{G}(u;K) := \int_K G(x, Du)dx, \quad (K \subset \mathbb{R}^m, \text{compact}) \tag{1}$$

We suppose the following conditions:

$$a(x) \geq 0 \ \ \forall x \in \Omega, \quad a(\cdot) \in C^{0,\alpha}(\overline{\Omega}) \ \text{ for some } \alpha \in (0,1]. \tag{2}$$

$$p(x), q(x) \in C^{0,\sigma}(\overline{\Omega}) \quad \text{for some } \ \sigma \in (0,1). \tag{3}$$

Putting

$$\beta := \min\{\alpha, \sigma\} \tag{4}$$

We assume that $p(x)$ and $q(x)$ satisfy

$$q(x) \geq p(x) > 1 \ \text{ on } \overline{\Omega}, \quad \sup_{x\in\Omega}\left(q(x) - p(x)\right) < \beta \tag{5}$$

Conditions on g

$$g^{\alpha\beta}(x) = g^{\beta\alpha}(x) \ \ \forall x \in \Omega, \qquad \alpha, \beta = 1, \dots, n, \tag{6}$$

$$g^{\alpha\beta}(\cdot) \in C^0(\overline{\Omega}), \quad \alpha, \beta = 1, \dots, n. \tag{7}$$

We assume also that for some constants $0 < \lambda_g \le \Lambda_g$, g satisfies

$$\lambda_g |z|^2 \le g^{\alpha\beta}(x) z_\alpha z_\beta, \quad \max_{1 \le \alpha, \beta \le 1} |g^{\alpha\beta}(x)| \le \Lambda_g, \tag{8}$$

for all $(x, z) \in \Omega \times \mathbb{R}^n$.

Let us write the modulus of continuity of g by ω_g, namely ω_g is a increasing continuous function with $\omega_g(0) = 0$ which satisfies

$$\max_{1 \le \alpha, \beta \le m} \left| g^{\alpha\beta}(x) - g^{\alpha\beta}(y) \right| \le \omega_g(|x - y|) \tag{9}$$

for all $x, y \in \Omega$.

Main Theorem

Theorem 1 (T. to appear in JMAA)

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with the boundary $\partial\Omega$ of class $C^{1,1}$. Suppose that $a(x)$, $p(x)$ and $q(x)$ satisfy (2), (3), (5) , and that g satisfies (6), (7), (8). Let h be a function in the class $W^{1,s}(\Omega;\mathbb{R}^n)$ for some $s > \max\left\{\sup_{x\in\Omega} q(x),\ m \cdot \sup_{x\in\Omega} q(x)/p(x)\right\}$. Then a minimizer $u \in W^{1,1}(\Omega;\mathbb{R}^n)$ of $\mathcal{G}(\,\cdot\,;\Omega)$ with the boundary condition $u = h$ on $\partial\Omega$ is in the class $C^{0,\gamma_0}(\overline{\Omega};\mathbb{R}^n) \cap C^{0,\gamma}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ for any $\gamma_0 \in \left(0,\ 1 - \dfrac{m}{s}\sup_{\Omega}\dfrac{q(x)}{p(x)}\right)$ and $\gamma \in (0,1)$.

When g is Hölder continuous, then $Du \in C^{0,\zeta}_{\mathrm{loc}}(\Omega;\mathbb{R}^{mn})$ for some $\zeta \in (0,1)$.

Outline of the Proof of Hölder continuity

Roughly speaking, the proof is divided to the following steps:

Step 1 Fix $x_0 \in \partial\Omega$. By flatten $\partial\Omega$ near x_0 by a suitable transformation Ψ s.t. $\Psi(x_0) = 0$, $\Psi(\partial\Omega \cap B_\delta(x_0)) \subset \{x; x^m = 0\}$ and $g(x_0) \mapsto I_m$(identity matrix). (As in the paper by F.Duzaar-J.F.Grotowsky-M.Kronz ('04))

Step 2 For the transformed functional $\bar{\mathcal{G}}$, we consider *semi-frozen functional* of type

$$\mathcal{H}(u) := \int (|Du|^p + b(x)|Du|^q)dx$$

Step 3 By a reflexion argument we prove Hölder regularity on the flat part of the boundary for weak solutions w of the Euler-Lagrange eq. for $\mathcal{H}$.

Step 4 By estimating the difference between minimizers v of $\mathcal{H}$ and u of $\bar{\mathcal{G}}$, we get Morrey type estimate for Du.

Step 1: Notaion, and change of Coordinates

We use the following notation:
for $x_0 = (x_0^1, \cdots, x_0^{m-1}, 0) \in \mathbb{R}^m$ and $M > 0$,

$$B_M^+(x_0) := \{x \in \mathbb{R}^m \; ; \; |x - x_0| < M, \; x^m > 0\}, \quad B_M^+ := B_M^+(0),$$

$$\Gamma := \{x \in \mathbb{R}^m \; ; \; x^m = 0\}, \;\; \Gamma_M(x_0) := B_M(x_0) \cap \Gamma, \;\; \Gamma_M := \Gamma_M(0).$$

Choosing a suitable diffeomorphism which flatten locally the boundary, and a suitable linear transoformation, we can find the coordinate transformation Ψ with desired properties.

Step 2: Transformed functional and semi-frozen one

By Ψ, locally the functional $\mathcal{G}$ is transformed to the following type:

$$\overline{\mathcal{G}}(u;K) := \int_K \overline{G}(x, Du)dx, \quad \overline{G}(x,\xi) = |\xi|_g^{p(x)} + a(x)|\xi|_g^{q(x)}.$$

with $g(0) = I_n$. We consider the above functional on B_M^+ with the boundary condition on Γ_M.

For $x_0 \in \Gamma_M$ and sufficiently small $R > 0$ with $B_R^+(x_0) \subset B_M^+$, we put

$$p_2 = p_2(x_0, R) := \sup_{B_R^+(x_0)} p(x), \quad q_2 = q_2(x_0, R) := \sup_{B_R^+(x_0)} q(x),$$

We consider a *semi-frozen functional* for $\overline{\mathcal{G}}$:

$$\mathcal{H}_R(u) := \int_{B_R^+(0)} H_R(x, Du)dx, \quad H_R(x,\xi) := |\xi|^{p_2} + a(x)^{\frac{q_2}{q(x)}}|\xi|^{q_2}.$$

Let us put $a(x)^{q_2/q(x)} = b(x)$.

A remark on the exponents of H_R

• When we use, for example, Hölder or reverse Hölder inequalities, we must treat $a(x)$ and $|Du|^{q(x)}$ together. So, the second term of the semi-frozen functional should be $\left(a(x)^{1/q(x)}|Dv|\right)^{q_2}$.

It is easy to see that

$$a(x)^{\frac{q_2}{q(x)}} \in C^{0,\beta}, \quad \beta := \min\{\alpha, \sigma\}.$$

• Since we are assuming

$$(*) \qquad\qquad q(x) < p(x) + \beta,$$

we can choose $M > 0$ sufficiently small so that

$$q_2(0, M) < p_1(0, M) + \beta, \quad p_1 := \inf_{B_M} p(x).$$

This enables us to apply the previous results by Colombo-Mingione to $\mathcal{H}_R$. This is the reason why we assume $(*)$.

Proposition 1

Assume that $v \in W^{1,1} \cap L^\infty(B_R^+, \mathbb{R}^n)$ *with* $\mathcal{H}_R(v) < \infty$ *satisfies*

$$(**) \begin{cases} \displaystyle\int_{B_R^+} \left(p_2|Dv|^{p_2-2} + b(x)q_2|Dv|^{q_2-2}\right) \langle Dv, D\varphi\rangle dx = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\varphi \in W_0^{1,q_2}(B_R^+; \mathbb{R}^n), \\ v = h \qquad \text{on } \Gamma_R, \end{cases}$$

where $h \in W^{1,s}(B_R^+; \mathbb{R}^n), (s > q_2)$. *Then* ${}^\forall\mu \in (0, n]$, ${}^\exists C$ *s.t.*

$$\int_{B_r^+} H_R(x, Dv)dx$$
$$\lesssim \left[\left(\frac{r}{R}\right)^{m-\mu} \int_{B_R^+} H_R(x, Dv)dx + r^{m(1-\frac{q_2}{s})}\left(\int_{B_R^+}(1 + |Dh|^s)dx\right)^{\frac{q_2}{s}}\right]$$

$\forall r \in (0, R)$.

Sketch of Proof of Prop. 1

Let w be a minimizer of $\mathcal{H}_R$ with $w - (v - h) = 0$ on ∂B_R. Then w satisfies $(**)$ with $h = 0$. Let $\bar{w}$ and $\bar{b}$ be odd and even extensions of w and b respectively to B_R. For a test function φ, let us put $\bar{\varphi}(x^1, ..., x^{m-1}, x^m) = \varphi(x^1, ..., x^{m-1}, -x^m)$. Then we have

$$\int_{B_R} \left(p_2|D\bar{w}|^{p_2-2} + \bar{b}(x)q_2|D\bar{w}|^{q_2-2}\right)\langle D\bar{w}, D\varphi\rangle dx$$

$$= \int_{B_R^+} \left(p_2|Dw|^{p_2-2} + b(x)q_2|Dw|^{q_2-2}\right)\langle Dw, D(\varphi - \bar{\varphi})\rangle dx = 0,$$

since $\varphi - \bar{\varphi} = 0$ on Γ_R and $\varphi = \bar{\varphi} = 0$ on $\partial B_R^+ \setminus \Gamma_R$.

By the convexity of the functional the weak solution w minimizes $\bar{\mathcal{H}}_R(w, B_R) := \int_{B_R}(|Dw|^{p_2} + \bar{b}(x)|Dw|^{q_2})dx$, and therefore w satisfies for every $\mu \in (0, n)$.

$$\int_{B_r^+} H_R(x, Dw)dx \lesssim \left(\frac{r}{R}\right)^{m-\mu} \int_{B_R^+} H_R(x, Dw)dx.$$

(by C.De Filippis-G.Mingione '20, M.Colombo-G.Mingione '15)

Now, mentioning that $w - (v - h) \in W_0^{1.1}(B_R^+)$ and that w, v both minimize $\mathcal{H}_R$, we can estimate the difference between $\int H_R(x, Dv)dx$ and $\int H_R(x, Dw)dx$ to obtain the desired estimate:

$$\begin{aligned}&\int_{B_r^+} H_R(x, Dv)dx \\ \lesssim\ &\left[\left(\frac{r}{R}\right)^{m-\mu} \int_{B_R^+} H_R(x, Dv)dx + r^{m(1-\frac{q_2}{s})}\left(\int_{B_R^+}(1 + |Dh|^s)dx\right)^{\frac{q_2}{s}}\right].\end{aligned}$$

□

Let v be the minimizer of $\mathcal{H}_R$ with the boundary condition $v = u$ on ∂B_R^+. Here, mention that by virtue of the maximum principle due to F.Leonetti-F.Siepe ('05), u is bounded, and therefore v is also bounded. So, for v we can use Prop. 1 to see that

$$\int_{B_r^+} H_R(x, Dv)dx \lesssim \left(\frac{r}{R}\right)^{m-\mu} \int_{B_R^+} H_R(x, Du)dx + R^{m(1-\frac{q_2(R)}{s})} \int_{B_R^+} (1 + |Dh|^s)dx. \quad (10)$$

For $t > 1$, put

$$V_t(\xi) := \begin{cases} |\xi|^{(t-2)/2}\xi & (\xi \neq 0), \\ 0 & (\xi = 0). \end{cases}$$

Then $|\xi|^t \leq 2(|\eta|^t + |V_t(\xi) - V_t(\eta)|^2$ holds. By this inequality, we have

$$\begin{aligned} \int_{B_r^+} H_r(x, Du)dx &\leq \int_{B_r^+} (2 + H_R(x, Du))\, dx \\ \lesssim\ & r^m|B_1| + \int_{B_r^+} \big(H_R(x, Dv)\big)dx \\ &+ \int_{B_r^+} \left[|V_{p_2}(Du) - V_{p_2}(Dv)|^2 + b_R(x)|V_{q_2}(Du) - V_{q_2}(Dv)|^2\right] dx \\ =:& |B_1|r^m + I + II. \end{aligned}$$

I can be estimate by (10).

For II, using the fact that v is a weak solution of E-L eq. of $\mathcal{H}_R$ and that $u - v = 0$ on ∂B_R^+ (so $u - v$ is an admissible test function for the E-L eq. of H_R), we see that

$$II \lesssim \ [\mathcal{H}_R(u) - \mathcal{H}_R(v)].$$

Now, putting

$$\mathcal{H}_{R,g}(w) := \int_{B_R^+} H_{R,g}(x, Dw)dx, \ \ H_{R,g}(x, \xi) := |\xi|_g^{p_2(R)} + b(x)|\xi|_g^{q_2(R)},$$

and using the minimality of u, we can estimate II as follows.

$$
\begin{aligned}
II =&[\mathcal{H}_R(u) - \mathcal{H}_{R,g}(u) + \mathcal{H}_{R,g}(u) - \overline{\mathcal{G}}(u; B_R^+) \\
&+ \overline{\mathcal{G}}(u; B_R^+) - \overline{\mathcal{G}}(v; B_R^+) \\
&+ \overline{\mathcal{G}}(v; B_R^+) - \mathcal{H}_{R,g}(v) + \mathcal{H}_{R,g}(v) - \mathcal{H}_R(v)]
\end{aligned}
$$

- $|(\mathrm{black}) - (\mathrm{blue})|$: The difference is "$|\cdot|$ and $|\cdot|_g$". So, we can estimate them using the continuity of g
- green part : Nonpositive by virtue of the minimality of u.
- magenta parts : The exponents are different. So, we can estimate them using the following estimate: for $\varepsilon > 0$ there exists a constant $c(\varepsilon) > 0$ s.t.

$$
|t^\tau - t^\rho| \le c(\varepsilon)|\tau - \rho|(1 + t^{\tau+\varepsilon})
$$

holds for any $t > 0$ and $1 < \rho < \tau$ (see A.Coscia-G.Mingione ('99)).

Reverse Hölder inequality (or Gehring-type inequality)

For $y \in B_M^+$ and $R \in (0, (M - |y|)/2)$, let us put $\Omega_R(y) := B_R(y) \cap \Omega$. Then for a minimizer u for $\overline{\mathcal{G}}$ with $u = h$ on Γ_M we have the following estimate: for sufficiently small $\delta > 0$,

$$\left(\fint_{\Omega_R(y)} (\overline{G}(x, Du))^{1+\delta} dx \right)^{\frac{1}{1+\delta}}$$

$$\lesssim \fint_{\Omega_{2R}(y)} \overline{G}(x, Du) + \left(\fint_{\Omega_{2R}(y)} (\overline{G}(x, Dh))^{1+\delta} dx \right)^{\frac{1}{1+\delta}} + 1.$$

When $B_{2R}(y) \subset B_M$, the above estimate holds without the second term in the right-hand side.

Using also the reverse Hölder inequality (with boundary data) etc..., for sufficiently small $\delta > 0$, we have

$$II \lesssim R^{m+\sigma} + \left(\omega_g(R) + R^{\sigma-\delta n}\right) \int_{B_{2R}^+} H_{2R}(x, Du)dx + R^{\sigma} \int_{B_R^+} H_{2R}(x, Dh)dx$$

where ω_g is the modulus of continuity of g and δ comes from the reverse Hölder inequality. We can take $\delta > 0$ sufficiently small.

Combining the estimate for I and II, we get

$$\int_{B_r^+} \mathcal{H}_R(x, Du)dx \lesssim \left[\left(\frac{r}{R}\right)^{m-\mu} + \omega_g(R) + R^{\sigma-\delta n}\right] \int_{B_{2R}^+} H_{2R}(x, Du)dx + R^{m(1-\frac{q_2(2R)}{s})} \left(1 + \int_{B_{2R}^+} |Dh|^s dx\right) \tag{11}$$

Hölder continuity (interior)

For a interior point $y \in \Omega$, proceeding without considering boundary data, we can show the following estimate:

$$\int_{B_r(y)} H_r(x, Du)dx \lesssim \left[\left(\frac{r}{R}\right)^{m-\mu} + \omega_g(2R) + R^{\sigma - n\delta}\right] \int_{B_{2R}(y)} H_R(x, Du)dx + CR^m. \tag{12}$$

From the above estimate, mentioning that we can take $\mu \in (0, 1)$ arbitrarily, we obtain for any $\lambda \in (0, m)$

$$\int_{B_r(y)} H_r(x, Du)dx \leq Cr^{m-\lambda}.$$

by the following (next page) lemma. Now, by Morrey's theorem, we see that $u \in C^{0,\gamma}_{\mathrm{loc}}(B^+_M)$. for any $\gamma \in (0, 1)$.

Lemma 2

Let A, B, α be positive constants and $\beta \in (0, \alpha)$. There exists a positive constant $\varepsilon_0 = \varepsilon_0(\alpha, \beta, A)$ with the following property: if a non-negative and nondecreasing function Φ defined on $[0, R_0]$ for some $R_0 > 0$ satisfies

$$\Phi(r) \leq A\left[\left(\frac{r}{R}\right)^{\alpha} + \varepsilon_0\right]\Phi(R) + BR^{\beta},$$

all $0 < r < R_0$. Then, for some constant $C = C(\alpha, \beta, A)$

$$\Phi(r) \leq C(\alpha, \beta, \gamma, A)\left[\left(\frac{r}{R}\right)^{\beta}\Phi(R) + Br^{\beta}\right]$$

holds for any $r \in (0, R_0]$.

Hölder continuity (up to the boundary)

Fix $x_0 \in B_M^+$ and choose $R > 0$ sufficiently small. For every $y \in B_R(x_0) \cap B_M^+$, using (11) and (12), we can show

$$r^{m - m\frac{q_2(x_0,R)}{s}} \int_{B_r(x_0)\cap\Omega} H_r(x, Du)dx \le C$$

for some constant $C > 0$.

Thus we obtain

$$\int_{B_r(x_0)\cap\Omega} |Du|^{p_1(x_0,R)} dx \le C r^{m - m\frac{q_2(x_0,R)}{s}}.$$

Now, for any $\gamma_0 \in (0, 1 - (m/s)\sup_\Omega(q(x)/p(x)))$, by virtue of the continuity of the exponents, we can choose $R > 0$ sufficiently small so that

$$\gamma_0 \leq 1 - \frac{mq_2(x_0, R)}{sp_1(x_0, R)}$$

Then we get

$$\int_{B_r(x_0)\cap\Omega} |Du|^{p_1(x_0,R)} dx \leq Cr^{m - p_1(x_0,R) + p_1(x_0,R)\gamma_0}.$$

By Morrey's theorem, we have $u \in C^{0,\gamma_0}(B_M^+)$. □

Thank you for your attention!