Homogenization of nonlinear elliptic systems in nonreflexive Musielak-Orlicz spaces

Agnieszka Świerczewska-Gwiazda
University of Warsaw
joint work with Piotr Gwiazda (University of Warsaw), Miroslaw Bulicek (Charles University in Prague), Martin Kalousek (Charles University in Prague)

Warsaw, 14th December 2020

What is the goal?

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. A parameter $\varepsilon > 0$ is considered to be small in comparison to the size of the domain Ω. Given $\mathbf{F}$ and a nonlinear operator $\mathbf{A}$ we study elliptic systems

$$\begin{aligned} \operatorname{div} \mathbf{A}\left(\frac{x}{\varepsilon}, \nabla \mathbf{u}^\varepsilon\right) &= \operatorname{div} \mathbf{F} \text{ in } \Omega, \\ \mathbf{u}^\varepsilon &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\mathbf{u}^\varepsilon : \Omega \to \mathbb{R}^N$ is an unknown.

- As the length scale of oscillating coefficients is visibly smaller than the size of the domain, studying such an equation would be to much complex, and thus in the homogenisation process we let $\varepsilon \to 0$ in (1).
- We expect to show that $\mathbf{u}^\varepsilon \to \mathbf{u}$, where the limit $\mathbf{u}$ solves the nonlinear elliptic problem with an operator independent of a spatial variable.

Inhomogeneous problems

- The spatial inhomogeneity in the studies on homogenisation is motivated by the phenomena of creating porous structure under the influence of electric field.
- Formation of such structures in oxides of metals, such as aluminium and titanium, appearing in the process of anodisation is an example of such a phenomena.
- It was observed in experiments that in the growing oxide layer spatially irregular pores are formed. This occurrence is caused due to dependence of oxide conductivity on the electric field.
- A benefit of an anodisation process is that an oxide film increases resistance to corrosion and wear, as well as provides better adhesion for paint primers and glues than bare metal itself.

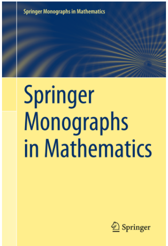
Homogenization of inhomogeneous problems - essential mathematical summary

- In mathematical understanding, homogenization is understood as nothing else than averaging PDEs with oscillating coefficients.
- The dependence of an N-function on spatial variable has a significant impact on the problem as consequently the homogenisation process will change the underlying function spaces and the nonlinear elliptic operator at each step.

Homogenization of elliptic operators with growth in inhomogenous anisotropic Musielak-Orlicz spaces

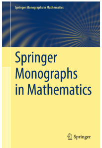
- Existence and homogenization of nonlinear elliptic systems in nonreflexive spaces (joint work with Miroslav Bulícek, Piotr Gwiazda and Martin Kalousek), Nonlinear Anal. (2020),
- Homogenization of nonlinear elliptic systems in nonreflexive Musielak-Orlicz spaces (joint work with Miroslav Bulícek, Piotr Gwiazda and Martin Kalousek) Nonlinearity (2019).

Literature



Iwona Chlebicka, Piotr Gwiazda, Agnieszka Świerczewska-Gwiazda, Aneta Wróblewska-Kamińska
Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces (to appear)

Literature



Part I Overture

1. $N-$functions
2. Musielak-Orlicz spaces

Part II PDEs

1. Weak solutions to elliptic and parabolic problems
2. Renormalized solutions elliptic and parabolic problems
3. Homogenization of elliptic boundary value problems
4. Non-Newtonian fluids

Definition of N-function

Suppose $\Omega \subset \mathbb{R}^N$ is a bounded connected set. A function $M : \Omega \times \mathbb{R}^d \to [0, \infty)$ is called an $\mathcal{N}$-function if it satisfies the following conditions:

1. M is a Carathéodory function (i.e. measurable with respect to x and continuous with respect to the second variable);
2. $M(x, 0) = 0$ and $\xi \mapsto M(x, \xi)$ is a convex function for a.a. $x \in \Omega$;
3. $M(x, \xi) = M(x, -\xi)$ for a.a. $x \in \Omega$ and all $\xi \in \mathbb{R}^d$;
4. there exist two convex functions $m_1, m_2 : [0, \infty) \to [0, \infty)$ positive on $(0, \infty)$, such that $m_1(0) = 0 = m_2(0)$ and

$$\lim_{s\to 0^+} \frac{m_1(s)}{s} = 0 = \lim_{s\to 0^+} \frac{m_2(s)}{s} \qquad \text{and} \qquad \lim_{s\to\infty} \frac{m_1(s)}{s} = \infty = \lim_{s\to\infty} \frac{m_2(s)}{s},$$

   and for a.a. $x \in \Omega$

$$m_1(|\xi|) \le M(x, \xi) \le m_2(|\xi|).$$

Some properties of Orlicz spaces

Let $M(x,\xi):\Omega\times\mathbb{R}^d\to[0,\infty)$ be an N-function, then:
The Orlicz class $\mathcal{L}_M(\Omega)$ is the set of all measurable functions $\xi:\Omega\to\mathbb{R}^N$ such that

$$\int_\Omega M(x,\xi)dx<\infty.$$

By $L_M(\Omega)$ we denote the vector valued Orlicz space which is the set of all measurable functions $\xi:\Omega\to\mathbb{R}^n$ which satisfy

$$\int_\Omega M(x,\lambda\xi(x))dx\to 0 \quad \text{as}\, \lambda\to 0.$$

The generalized Orlicz space is a Banach space with respect to the Luxemburg norm

$$\|\xi\|=\inf\left\{\lambda>0: \quad \int_\Omega M\left(x,\frac{\xi}{\lambda}\right)dx\le 1\right\}.$$

Some properties of Orlicz spaces

The space E_M

By $E_M(\Omega)$ we denote the closure of $L^\infty(\Omega)$ in $L_M(\Omega)$

- $(E_M)^* = L_{M^*}$
- If M does not satisfy Δ_2–condition, then $E_M \subsetneq \mathcal{L}_M \subsetneq L_M$
- If M satisfies Δ_2–condition, L_M is separable and $L_M = E_M = \mathcal{L}_M$.

Δ_2–condition

We say that an N–function M satisfies Δ_2–condition if for some constant $C > 0$

$$M(x, 2\xi) \leq CM(x, \xi) \quad \text{for all } \xi \in \mathbb{R}^N.$$

Modular convergence

Definition

A sequence z^j converges modularly to z in $L_M(\Omega)$ if there exists $\lambda > 0$ such that $\int_\Omega M\left((z^j - z)/\lambda\right) dx \to 0$.

Properties

- $z^j \xrightarrow{M} z$ in $L_M(\Omega)$ modularly if and only if $z^j \to z$ in measure and $\exists \lambda > 0$ such that $\{M(\lambda z^j)\}$ is uniformly integrable.
- Orlicz spaces are separable w.r.t. the modular convergence and smooth functions are dense

History

The studies on homogenization of elliptic equations go back to the fundamental lecture of Tartar

L. Tartar, Cours peccot au Collége de France, partially written by F. Murat in Séminaire d'Analyse Fonctionelle et Numérique del Université d'Alger, unpublished., 1979.

and later works

O. A. Oleinik and V. V. Zhikov, On the homogenization of elliptic operators with almost-periodic coefficients, (1985).

Enrique Sánchez-Palencia, Nonhomogeneous media and vibration theory, 1980.

and are of the highest interest among the properties of elliptic systems with periodic structure. The homogenization process was also the starting point for developing the **two-scale convergence technique**, which was introduced in

G. Allaire, Homogenization and two-scale convergence, SIAM J. Math. Anal. (1992)

Homogenization not in a standard L^p-setting

V. V. Zhikov and S. E. Pastukhova. Homogenization of monotone operators under conditionsof coercitivity and growth of variable order. Mat. Zametki, 2011

- Growth prescribed by means of variable exponent $p(x)$, so the corresponding function spaces were varying with respect to $\varepsilon \to 0$ in the homogenization process.
- In $L^{p(x)}$ setting they required that

$$1 < p_{\min} \leq p(x) \leq p_{\max} < \infty,$$

so the corresponding functions spaces were reflexive and separable.

What is the homogenization process?

The homogenization process is letting $\varepsilon \to 0$ in

$$\operatorname{div} \mathbf{A}\left(\frac{x}{\varepsilon}, \nabla \mathbf{u}^\varepsilon\right) = \operatorname{div} \mathbf{F} \quad \text{in } \Omega,$$
$$\mathbf{u}^\varepsilon = 0 \quad \text{on } \partial\Omega.$$

One expects that $\mathbf{u}^\varepsilon \to \mathbf{u}$, where $\mathbf{u}$ is a solution to the following nonlinear elliptic problem with the nonlinear operator **independent** of a spatial variable, i.e.,

$$\operatorname{div} \hat{\mathbf{A}}(\nabla \mathbf{u}) = \operatorname{div} \mathbf{F} \text{ in } \Omega,$$
$$\mathbf{u} = 0 \quad \text{on } \partial\Omega.$$

where $\hat{\mathbf{A}}(\boldsymbol{\xi}) := \int_Y \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}(y))\,\mathrm{d}y,$, $Y := (0,1)^d$. The function $\mathbf{w}_{\boldsymbol{\xi}} : \mathbb{R}^d \to \mathbb{R}^N$ is the solution of the cell problem, i.e., $\mathbf{w}_{\boldsymbol{\xi}}$ is Y-periodic and solves in the sense of distributions

$$\operatorname{div} \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}(y)) = 0 \text{ in } Y.$$

Assumptions for A

We first formulate certain minimal assumptions on the operator $\mathbf{A}$, that will be used in what follows:

1. $\mathbf{A}$ is a Carathéodory mapping, i.e., $\mathbf{A}(\cdot, \boldsymbol{\xi})$ is measurable for any $\boldsymbol{\xi} \in \mathbb{R}^{d\times N}$ and $\mathbf{A}(y, \cdot)$ is continuous for a.a. $y \in \mathbb{R}^d$,
2. $\mathbf{A}$ is $Y-$periodic, i.e., periodic in each argument $y_i, i = 1, \dots, d$ with the period 1,
3. There exists an N–function $M : \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0, \infty)$ and a constant $c > 0$ such that for a.a. $y \in Y$ and all $\boldsymbol{\xi} \in \mathbb{R}^{d\times N}$ there holds
$$\mathbf{A}(y, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq c(M(y, \boldsymbol{\xi}) + M^*(y, \mathbf{A}(y, \boldsymbol{\xi}))),$$
4. For all $\xi, \zeta \in \mathbb{R}^{d\times N}$ such that $\xi \neq \zeta$ and a.a. $y \in Y$, we have
$$(\mathbf{A}(y, \boldsymbol{\xi}) - \mathbf{A}(y, \zeta)) \cdot (\boldsymbol{\xi} - \zeta) \geq 0.$$

Assumptions on the modular function

(M1) M or conjugate M^* satisfy Δ_2 condition.
or

(M2) M is log-Hölder continuous, that is there exist constants $a > 0$ and $b \geq 1$, such that for all $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$ and all $\xi \in \mathbb{R}^N$ we have

$$\frac{M(x,\xi)}{M(y,\xi)} \leq \max\left\{ |\xi|^{-\frac{a}{\log|x-y|}}, b^{-\frac{a}{\log|x-y|}} \right\}.$$

Assume that there exist constants $a_2, c_1, c_2 > 0$ and $b_2 \geq 1$, such that for all $x \in Q_j^\delta$ and all $\xi \in \mathbb{R}^N$ we have

$$\frac{M(x,\xi)}{(M_j^\delta(\xi))^{**}} \leq c_1 \max\left\{ |\xi|^{-\frac{a_2}{\log(c_2\delta)}}, b_2^{-\frac{a_2}{\log(c_2\delta)}} \right\},$$

where $\delta < \delta_0 = \frac{1}{2c_2}$ and $M_j^\delta(\xi) := \inf_{x \in \widetilde{Q}_j^\delta \cap \Omega} M(x,\xi)$.

Homogenization - tools

The method of **periodic unfolding** is one of the tools used in homogenisation problems, having its origins in L^p setting. It basically relies on two ideas:

1. Firstly one doubles the dimension by introducing the unfolding operator S_ε.
2. The second equally important element of the periodic unfolding method is separating the characteristic scales, which means that every function is decomposed in two parts.

Unfolding operator

1. This step allows one to use standard weak or strong convergence result in L^p instead of the tools of *two-scale convergence*:
   Associate to a function in $L^p(\Omega)$ a function $v(S_\varepsilon)$, which is an element of $L^p(\Omega \times Y)$ and it appears that two-scale convergence of a sequence in L^p, is equivalent to the weak convergence in $L^p(\Omega \times Y)$ of the unfolded sequence.

Two-scale convergence

A sequence of functions v^ε in $L^p(\Omega)$, $p \in (1, \infty)$, is said to two-scale converge to a limit $v^0(x, y) \in L^p(\Omega \times Y)$ if for any function $\varphi(x, y) \in C_c^\infty(\Omega, C_{per}^\infty(Y))$ it holds

$$\lim_{\varepsilon \to 0} \int_\Omega v^\varepsilon(x)\varphi(x, \frac{x}{\varepsilon})dx = \int_\Omega \int_Y v^0(x, y)\varphi(x, y)dxdy.$$

How it works in non-reflexive spaces?

- The current setting of Musielak-Orlicz spaces, by the reason of their non-reflexivity, only provides conclusions on the weak* compactness of bounded sets.
- For our considerations it is more clearly to set a condition on convergence of unfolded sequence $v^\varepsilon \circ S_\varepsilon$ as a definition of two-scale convergence, underlining additionally what type of convergence precisely we have in mind.

How it works in non-reflexive spaces?

Definition

We say that a sequence of functions $\{v^\varepsilon\} \subset L_M(\mathbb{R}^d)$

(i) converges to v^0 weakly* two–scale in $L_M(\mathbb{R}^d \times Y)$ if $v^\varepsilon \circ S_\varepsilon$ converges to v^0 weakly* in $L_M(\mathbb{R}^N \times Y)$,

(ii) converges to v^0 strongly two–scale in $E_M(\mathbb{R}^d \times Y)$ if $v^\varepsilon \circ S_\varepsilon$ converges to v^0 strongly in $E_M(\mathbb{R}^N \times Y)$.

The key result is an introduction of a technique, which is not based on the Hemholtz-like decomposition, but rather deals with the combination of two–scale limit and the modular convergence.

Function S_ε

we define functions $n : \mathbb{R} \to \mathbb{Z}$

$$n(t) := \max\{n \in \mathbb{Z} : n \leq t\} \ \forall t \in \mathbb{R},$$

and

$$[x] := (n(x_1), \dots, n(x_d)), \ \forall x \in \mathbb{R}^N.$$

Set $r(x) := x - [x]$. Then obviously for any $x \in \mathbb{R}^N$, $\varepsilon > 0$, we have a two–scale decomposition

$$x = \varepsilon \left(\left[\frac{x}{\varepsilon} \right] + r\left(\frac{x}{\varepsilon} \right) \right),$$

where r is the reminder function. Then we define for any $\varepsilon > 0$ a two–scale composition function $S_\varepsilon : \mathbb{R}^N \times Y \to \mathbb{R}^N$ as

$$S_\varepsilon(x, y) := \varepsilon \left(\left[\frac{x}{\varepsilon} \right] + y \right).$$

It follows immediately that

$$S_\varepsilon(x, y) \to x \text{ uniformly in } \mathbb{R}^N \times Y \text{ as } \varepsilon \to 0$$

since $S_\varepsilon(x, y) = x + \varepsilon \left(y - r\left(\frac{x}{\varepsilon} \right) \right)$.

Main steps of the proof

1. Properties of cell problem
2. Existence of solutions for a fixed ε
3. Uniform (in ε) estimates
4. Limit passage to the homogenized problem

What is the homogenization process?

The homogenization process is letting $\varepsilon \to 0$ in

$$\operatorname{div} \mathbf{A}\left(\frac{x}{\varepsilon}, \nabla \mathbf{u}^\varepsilon\right) = \operatorname{div} \mathbf{F} \quad \text{in } \Omega,$$
$$\mathbf{u}^\varepsilon = 0 \qquad \text{on } \partial\Omega.$$

One expects that $\mathbf{u}^\varepsilon \to \mathbf{u}$, where $\mathbf{u}$ is a solution to the following nonlinear elliptic problem with the nonlinear operator **independent** of a spatial variable, i.e.,

$$\operatorname{div} \hat{\mathbf{A}}(\nabla \mathbf{u}) = \operatorname{div} \mathbf{F} \text{ in } \Omega,$$
$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega.$$

where $\hat{\mathbf{A}}(\boldsymbol{\xi}) := \int_Y \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}(y))\, \mathrm{d}y,$, $Y := (0,1)^d$. The function $\mathbf{w}_{\boldsymbol{\xi}} : \mathbb{R}^d \to \mathbb{R}^N$ is the solution of the cell problem, i.e., $\mathbf{w}_{\boldsymbol{\xi}}$ is Y-periodic and solves in the sense of distributions

$$\operatorname{div} \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}(y)) = 0 \text{ in } Y.$$

Existence of weak solutions - important question

Already a well–understood problem in Musielak-Orlicz space

$$-\mathrm{div} A(x, \nabla u) = f,$$

with assumptions: A has growth conditions prescribed by an $N-$function.

BUT:

- For the problem

$$\operatorname{div} \mathbf{A}(y, \xi + \nabla \mathbf{w}_{\xi}(y)) = 0 \text{ in } Y.$$

we need to solve it in a periodic cell

- For the problem

$$\begin{aligned} \operatorname{div} \hat{\mathbf{A}}(\nabla \mathbf{u}) &= \operatorname{div} \mathbf{F} \text{ in } \Omega, \\ \mathbf{u} &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

the growth conditions of $\hat{\mathbf{A}}$ need to be prescribed

Assumptions on the modular function - once more

(M1) M or conjugate M^* satisfy Δ_2 condition.
or

(M2) M is log-Hölder continuous, that is there exist constants $a > 0$ and $b \geq 1$, such that for all $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$ and all $\xi \in \mathbb{R}^N$ we have

$$\frac{M(x,\xi)}{M(y,\xi)} \leq \max\left\{|\xi|^{-\frac{a}{\log|x-y|}}, b^{-\frac{a}{\log|x-y|}}\right\}.$$

Assume that there exist constants $a_2, c_1, c_2 > 0$ and $b_2 \geq 1$, such that for all $x \in Q_j^\delta$ and all $\xi \in \mathbb{R}^N$ we have

$$\frac{M(x,\xi)}{(M_j^\delta(\xi))^{**}} \leq c_1 \max\left\{|\xi|^{-\frac{a_2}{\log(c_2\delta)}}, b_2^{-\frac{a_2}{\log(c_2\delta)}}\right\},$$

where $\delta < \delta_0 = \frac{1}{2c_2}$ and $M_j^\delta(\xi) := \inf_{x \in \widetilde{Q}_j^\delta \cap \Omega} M(x,\xi)$.

For (M2) - Approximation properties

Density

Let Ω be a Lipschitz domain in $\mathbb{R}^N$ and an N-function M satisfy condition (M2). Then for any φ such that $\nabla\varphi \in L_M(\Omega)$ there exists a sequence of compactly supported smooth functions $\{\varphi_\delta\}_{\delta>0}$ such that $\nabla\varphi_\delta \xrightarrow{M} \nabla\varphi$.

(M1) = two different cases

(M1)-1 A case when M^* satisfies Δ_2 condition wouldn't surprise you much.

(M1)-2 M satisfies Δ_2 condition: A naive approach to show existence in this case would be to follow the lines of the proof of (M1)-1. *This however breaks down:*

- We only know that $\mathbf{A}(\cdot, \nabla \mathbf{u}) \in L_{M^*}(\Omega)$, and it is not the predual space to $L_M(\Omega)$ anymore.
- Therefore we cannot use the arguments of weak-* convergence in $L_M(\Omega)$.

Different approach for (M1)-2

For that reason the strategy is different here:

- First a weak solution of the dual problem will be found.
- Then we deduce the existence of a weak solution to the original problem.
- One needs to specify how the *dual problem* is understood. For construction we use an inverse operator to $\mathbf{A}$, which we shall denote as $\mathbf{B}$, i.e.

$$\mathbf{A}(x, B(\xi)) = \xi$$

- Here we need that $\mathbf{A}$ is strictly monotone, therefore the inverse operator exists and is also strictly monotone.
- We could also work in the language of maximal monotone graphs to avoid assumption on strict monotonicity.

Very few details

Instead of the growth conditions on **A**

$$\mathbf{A}(y,\xi)\cdot\xi \geq c(M(y,\xi) + M^*(y,\mathbf{A}(y,\xi))),$$

we work with growth conditions of **B**

$$\mathbf{B}(x,\zeta)\cdot\zeta \geq c\,(M(x,\mathbf{B}(x,\zeta)) + M^*(x,\zeta))\,.$$

Thank you for your attention