Some new results related to Lorentz $G\Gamma$-spaces and interpolation

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The presentation is based:

Ahmed, Irshaad; Fiorenza, Alberto; Formica, Maria Rosaria; Gogatishvili, Amiran; Rakotoson, Jean Michel; Some new results related to Lorentz $G\Gamma$-spaces and interpolation. J. Math. Anal. Appl. **483** (2020), no. 2, 123623.

The original question comes from an unpublished manuscript by H. Brezis.
let f be given in $L^1(\Omega, \mathrm{dist}\,(x, \partial\Omega))$ (Ω bounded smooth open set of $\mathbb{R}^n$), then H. Brezis shows the existence and uniqueness of a function $u \in L^1(\Omega)$ satisfying

$$|u|_{L^1(\Omega)} \leqslant c|f|_{L^1(\Omega, \mathrm{dist}\,(x, \partial\Omega))}$$

with

$$GD(\Omega) = \begin{cases} -\displaystyle\int_\Omega u\Delta\varphi\,dx = \int_\Omega f\varphi\,dx, \quad \forall\varphi \in C_0^2(\overline{\Omega}), \\ \text{with } C_0^2(\overline{\Omega}) = \Big\{\varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \partial\Omega\Big\}. \end{cases}$$

Therefore, the question of the integrability of the generalized derivative of u arises in a natural way and

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Therefore, the question of the integrability of the generalized derivative of u arises in a natural way and It was raised already in the note by H. Brezis and developed in:

J.I. Díaz, J.M. Rakotoson, On the differentiability of the very weak solution with right-hand side data integrable with respect to the distance to the boundary *J. Functional Analysis* **257** (2009) 807-831.
J.M. Rakotoson, New hardy inequalities and behaviour of linear elliptic equations *J. Functional Analysis* **263** 9 (2012) 2893-2920.
J.M. Rakotoson, A sufficient condition for a blow-up on the space. Absolutely conditions functions for the very weak solution *AMO* **73**(2016) 153-163.

More generally, the question of the regularity of u is arised, according to f.
In the papers:
J. I. Díaz, D. Gómez-Castro, J. M. Rakotoson, R. Temam, Linear diffusion with singular absorption potential and/or unbounded convective flow: the weighted space approach, *Discrete and Continuous Dynamical Systems* **38**, 2, (2018) 509-546
A. Fiorenza, M.R. Formica, J.M. Rakotoson, Pointwise estimates for $G\Gamma$-functions and applications, *Differential Integral Equations* **30**, 11-12 (2017) 809-824.
following result have shown:

Theorem (1)

Let Ω *be a bounded open set of class* C^2 *of* $\mathbb{R}^n$, $|\Omega| = 1$ *and* $\alpha \geqslant \frac{1}{n'}$ *where* $n' = \frac{n}{n-1}$, $f \in L^1(\Omega;\delta)$, *with* $\delta(x) = \operatorname{dist}(x;\partial\Omega)$.
Consider $u \in L^{n',\infty}(\Omega)$, *the very weak solution* (v.w.s.) *of*

$$-\int_\Omega u\Delta\varphi dx = \int_\Omega f\varphi dx \qquad \forall\,\varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \partial\Omega. \tag{1}$$

Then,

1. *if* $f \in L^1\Big(\Omega;\delta(1+|\operatorname{Log}\delta|)^\alpha\Big)$ *and* $\alpha > \frac{1}{n'}$:

$$u \in L^{(n',n\alpha-n+1}(\Omega) = G\Gamma(n',1;w_\alpha),\ w_\alpha(t) = t^{-1}(1-\operatorname{Log}t)^{\alpha-1-\frac{1}{n'}}$$

   and

$$||u||_{G\Gamma(n',1;w_\alpha)} \leqslant K_0|f|_{L^1(\Omega;\delta(1+|\operatorname{Log}\delta|)^\alpha)} \tag{2}$$

2. *if* $f \in L^1\Big(\Omega;\delta\big(1+|\operatorname{Log}\delta|\big)^{\frac{1}{n'}}\Big)$ *then* $u \in L^{n'}(\Omega)$ *and*

$$||u||_{L^{n'}} \leqslant K_1|f|_{L^1\left(\Omega;\delta(1+|\operatorname{Log}\delta|)^{1/n'}\right)} \tag{3}$$

Question 1

The natural question is *how to extend of Theorem 1 for* $\alpha < \frac{1}{n'}$ *and how to improve the estimate when* $\alpha = \frac{1}{n'}$?
Since the solution of (1) satisfies also

$$|u|_{L^{n',\infty}(\Omega)} \leqslant K_1 |f|_{L^1(\Omega;\delta)}, \tag{4}$$

the natural idea to obtain an estimate is to use the real interpolation method of Marcinkiewicz to derive

$$|u|_{(L^{n',\infty},L^{(n')})_{\alpha,1}} \leqslant K_2 |f|_{L^1(\Omega;\delta(1+|\mathrm{Log}\,\delta|)^\alpha)} \qquad \text{for } 0 < \alpha < 1. \tag{5}$$

Question 2

How to characterize the space $\left(L^{n',\infty}(\Omega), L^{(n'}(\Omega)\right)_{\alpha,1}$?

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How to characterize the space $\left(L^{n',\infty}(\Omega), L^{(n'}(\Omega)\right)_{\alpha,1}$?

We still have not an answer to this question. Therefore, we will provide a lower estimate for the norm of u in relation (5), a particular overbound can be obtained from our work made in

A. Fiorenza, M.R. Formica, A. Gogatishvili, T. Kopaliani, J.M. Rakotoson, Characterization of interpolation between Grand, small or classical Lebesgue spaces, *Non Linear Analysis* **177** (2018) 422-453. DOI https://doi.org/10.1016/j.na.2017.09.005 :

Since $L^{n',\infty}(\Omega) \subset L^{n')}(\Omega)$, then we have

$$\left(L^{n',\infty}(\Omega), L^{(n'}(\Omega)\right)_{\alpha,1} \subset \left(L^{n')}(\Omega), L^{(n'}(\Omega)\right)_{\alpha,1}$$

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we have shown the following

Theorem (characterization of the interpolation between Grand and Small Lebesgue space)

$$\left(L^{n')}(\Omega), L^{(n'}(\Omega)\right)_{\alpha,1} = G\Gamma(n'; 1; w_1; w_2)$$

$$\textit{with } w_1(t) = \frac{(1 - \operatorname{Log} t)^{\alpha-1}}{t}, \; w_2(t) = \frac{1}{1 - \operatorname{Log} t}.$$

Theorem

$$\left(L^{n',\infty}(\Omega), L^{(n'}(\Omega)\right)_{\alpha,1} \subset G\Gamma(\infty; 1; v_1; v_2)$$

$$\text{with } v_1(t) = t^{-1}(1 - \operatorname{Log} t)^{\frac{\alpha}{n}-1}, \; v_2(t) = t^{\frac{1}{n'}}.$$

Proposition

Let u be the solution of (1), $0 < \alpha < 1$. Then,

$$\|u\|_{G\Gamma(n',1;w_1,w_2)\cap G\Gamma(\infty,1;v_1,v_2)} \lesssim |f|_{L^1(\Omega;\delta(1+|\mathrm{Log}\,\delta|)^\alpha)},$$

where $w_1(t) = t^{-1}(1-\mathrm{Log}\,t)^{\alpha-1}$, $w_2(t) = (1-\mathrm{Log}\,t)^{-1}$, $v_1(t) = t^{-1}(1-\mathrm{Log}\,t)^{\frac{\alpha}{n}-1}$ *and* $v_2(t) = t^{\frac{1}{n'}}$.

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$$\begin{aligned}
&\|u\|_{G\Gamma(n',1;w_1,w_2)\cap G\Gamma(\infty,1;v_1,v_2)} \\
&\quad = \int_0^1 (1-\mathrm{Log}\,t)^{\alpha-1}\left(\int_0^t \frac{u_*^{n'}(x)dx}{1-\mathrm{Log}\,x}\right)^{\frac{1}{n'}}\frac{dt}{t} \\
&\quad + \int_0^1 (1-\mathrm{Log}\,t)^{\frac{\alpha}{n}-1}\left(\sup_{0<x<t} x^{\frac{1}{n'}}u_*(x)\right)\frac{dt}{t} \\
&\lesssim |f|_{L^1(\Omega;\delta(1+|\mathrm{Log}\,\delta|)^\alpha)}.
\end{aligned}$$

For a measurable function $f : \Omega \to \mathbb{R}$, we set for $t \geqslant 0$

$$D_f(t) = \text{measure}\left\{x \in \Omega : |f(x)| > t\right\}$$

and f_* the decreasing rearrangement of $|f|$,

$$f_*(s) = \inf\left\{t : D_f(t) \leqslant s\right\} \text{ with } s \in \left(0, |\Omega|\right),\ |\Omega| \text{ is the measure of } \Omega,$$

that we shall assume to be equal to 1 for simplicity.

The Lorentz $G\Gamma$-space is defined as follows :

Definition (of Generalized Gamma space with double weights (Lorentz-$G\Gamma$))

Let w_1, w_2 be two weights on $(0,1)$, $m \in [1,+\infty]$, $1 \leqslant p < +\infty$. We assume the following conditions:

c1) *There exists $K_{12} > 0$ such that $w_2(2t) \leqslant K_{12} w_2(t)$ $\forall\, t \in (0,1/2)$. The space $L^p(0,1;w_2)$ is continuously embedded in $L^1(0,1)$.*

c2) *The function $\int_0^t w_2(\sigma)d\sigma$ belongs to $L^{\frac{m}{p}}(0,1;w_1)$.*

A generalized Gamma space with double weights is the set :

$$G\Gamma(p,m;w_1,w_2) = \Big\{ f : \Omega \to \mathbb{R} \text{ measurable } \int_0^t f_*^p(\sigma) w_2(\sigma) d\sigma \text{ is in } L^{\frac{m}{p}}(0,1;w_1) \Big\}.$$

This space was introduced in: **A. Gogatishvili, M. Krepela, L. Pick, F. Soudsky**, Embeddings of Lorentz-type spaces involving weighted integral means, *J.F.A.* **273**, 9 (2017) 2939-2980.

Property

Let $G\Gamma(p, m; w_1, w_2)$ be a Generalized Gamma space with double weights and let us define for $f \in G\Gamma(p, m; w_1, w_2)$

$$\rho(f) = \left[\int_0^1 w_1(t) \left(\int_0^t f_*^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{m}{p}} dt \right]^{\frac{1}{m}}$$

with the obvious change for $m = +\infty$.
Then,

1. *ρ is a quasinorm.*
2. *$G\Gamma(p, m; w_1, w_2)$ endowed with ρ is a quasi-Banach function space.*
3. *If $w_2 = 1$*
$$G\Gamma(p, m; w_1, 1) = G\Gamma(p, m; w_1).$$

Proposition (1)

Consider the classical Lorentz space $\Lambda^p(w_2)$*. Then it is equal to the set*

$$\left\{ f : \Omega \to \mathbb{R} \text{ measurable : } \left(\int_0^1 f_*^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{1}{p}} = ||f||_{\Lambda^p(w_2)} < +\infty \right\}.$$

If w_1 *and* w_2 *are integrable weights on* $(0,1)$ *and* w_2 *satisfies c1) then*

$$G\Gamma(p, m; w_1, w_2) = \Lambda^p(w_2).$$

Proposition (2)

Assume that $w_1(t) = t^{-1}(1 - \operatorname{Log} t)^{\gamma}$, $w_2(t) = (1 - \operatorname{Log} t)^{\beta}$, $(\gamma, \beta) \in \mathbb{R}^2$, $m \in [1, +\infty[$, $p \in [1, +\infty[$.

1. *If* $\gamma < -1$ *then* $G\Gamma(p, m; w_1, w_2) = \Lambda^p(w_2)$.
2. *If* $\gamma > -1$ *and* $\gamma + \beta \dfrac{m}{p} + 1 \geqslant 0$ *then*

$$G\Gamma(p, m; w_1, w_2) = G\Gamma(p, m; \overline{w_1}, 1), \ \overline{w_1}(t) = t^{-1}(1 - \operatorname{Log} t)^{\gamma + \beta \frac{m}{p}}.$$

Lemma

Assume that $w_1(t) = t^{-1}(1 - \mathrm{Log}\, t)^{\gamma}$, $w_2(t) = (1 - \mathrm{Log}\, t)^{\beta}$, $(\gamma, \beta) \in \mathbb{R}^2$, $m \in [1, \infty[$, $p \in [1, \infty[$. *If* $\gamma > -1$ *and* $\gamma + \beta \frac{m}{p} + 1 < 0$, *then*

$$\|f\|^m_{G\Gamma(p,m;w_1,w_2)} \approx \int_0^1 (1 - \mathrm{Log}\, t)^{\gamma + \beta \frac{m}{p}} \left(\int_t^1 f_*(x)^p dx \right)^{m/p} \frac{dt}{t}.$$

Lemma

Assume that $w_1(t) = t^{-1}(1 - \mathrm{Log}\, t)^\gamma$, $w_2(t) = (1 - \mathrm{Log}\, t)^\beta$, $(\gamma, \beta) \in \mathbb{R}^2$, $m \in [1, \infty[$, $p \in [1, \infty[$. *If* $\gamma > -1$ *and* $\gamma + \beta \frac{m}{p} + 1 < 0$, *then*

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$$\|f\|_{G\Gamma(p,\infty;w_1,w_2)} \approx \sup_{0<t<1} (1 - \mathrm{Log}\, t)^{\gamma + \frac{\beta}{p}} \left(\int_t^1 f_*(x)^p dx \right)^{1/p}.$$

Definition (of the small Lebesgue space)

The small Lebesgue space associated to the parameters $p \in]1, +\infty[$ *and* $\theta > 0$ *is the set*

$$L^{(p,\theta}(\Omega) = \Big\{ f : \Omega \to \mathbb{R} \textit{ measurable :}$$

$$||f||_{(p,\theta} = \int_0^1 (1 - \operatorname{Log} t)^{-\frac{\theta}{p}+\theta-1} \left(\int_0^t f_*^p(\sigma) d\sigma \right)^{1/p} \frac{dt}{t} < +\infty \Big\}.$$

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Definition (of the grand Lebesgue space)

The grand Lebesgue space is the associate space of the small Lebesgue space, with the parameters $p \in]1, +\infty[$ *and* $\theta > 0$ *is the set*

$$L^{p),\theta}(\Omega) = \Big\{ f : \Omega \to \mathbb{R} \text{ measurable :}$$

$$||f||_{p),\theta} = \sup_{0<t<1} (1 - \operatorname{Log} t)^{-\frac{\theta}{p}} \left(\int_t^1 f_*^p(\sigma) d\sigma \right)^{1/p} \frac{dt}{t} < +\infty \Big\}.$$

Corollary

Assume that $w_1(t) = t^{-1}(1 - \mathrm{Log}\, t)^{\gamma}$, $w_2(t) = (1 - \mathrm{Log}\, t)^{\beta}$, $(\gamma, \beta) \in \mathbb{R}^2$, $p \in]1, +\infty[$. *If* $\gamma + 1 + \frac{\beta}{p} > 0$ *and* , $\gamma > -1$, *then*

$$G\Gamma(p, 1; w_1, w_2) = L^{(p,\theta}, \ \theta = p' \left(\gamma + 1 + \frac{\beta}{p}\right).$$

$$L^{(p,1}(\Omega) = L^{(p}(\Omega).$$

Corollary

Assume that $w_1(t) = (1 - \mathrm{Log}\, t)^\gamma$, $w_2(t) = (1 - \mathrm{Log}\, t)^\beta$, $(\gamma, \beta) \in \mathbb{R}^2$, $p \in]1, +\infty[$. *If* $\gamma + \dfrac{\beta}{p} < 0$ *and* $\gamma > 0$, *then*

$$G\Gamma(p, \infty; w_1, w_2) = L^{p),\theta}, \ \theta = -p\left(\gamma + \frac{\beta}{p}\right).$$

$$L^{p),1}(\Omega) = L^{p)}(\Omega).$$

We recall also the following definition of interpolation spaces.
Let $(X_0, ||\cdot||_0)$, $(X_1, ||\cdot||_1)$ two Banach spaces contained continuously in a Hausdorff topological vector space (that is (X_0, X_1) is a compatible couple).
For $g \in X_0 + X_1$, $t > 0$ one defines the so called K functional $K(g, t; X_0, X_1) \doteq K(g, t)$ by setting

$$K(g, t) = \inf_{g=g_0+g_1} \left(||g_0||_0 + t||g_1||_1\right). \tag{6}$$

For $0 \leqslant \theta \leqslant 1$, $1 \leqslant p \leqslant +\infty$, $\alpha \in \mathbb{R}$ we shall consider

$$(X_0, X_1)_{\theta,p;\alpha} = \left\{ g \in X_0 + X_1,\ ||g||_{\theta,p;\alpha} = ||t^{-\theta-\frac{1}{p}}\left(1 - \operatorname{Log} t\right)^{\alpha} K(g, t)||_{L^p(0,1)} \text{ is finite} \right\}.$$

Here $||\cdot||_{L^p(0,1)}$ denotes the norm in a Lebesgue space $L^p(0,1)$, $0 < p \leqslant +\infty$.

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Here $||\cdot||_{L^p(0,1)}$ denotes the norm in a Lebesgue space $L^p(0,1)$, $0 < p \leqslant +\infty$.

Our definition of the interpolation space is different from the usual one since we restrict the norms on the interval $(0, 1)$.
If we consider ordered couple, i.e. $X_1 \hookrightarrow X_0$ and $\alpha = 0$,

$$(X_0, X_1)_{\theta,p;0} = (X_0, X_1)_{\theta,p}$$

is the interpolation space as it is defined by J. Peetre.

Theorem

Let $\varphi(t) = e^{1-\frac{1}{t^{p'}}}$, $0 < t \leqslant 1$. *Then*

$$K(f,t;L^p,L^{(p}) \approx t\int_{\varphi(t)}^{1}(1-\operatorname{Log}\sigma)^{-\frac{1}{p}}\left(\int_0^{\sigma} f_*^p(x)dx\right)^{\frac{1}{p}}\frac{d\sigma}{\sigma} \doteq K^2(t)$$

for all $f \in L^p + L^{(p}$.

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for all $f \in L^p + L^{(p}$.

Corollary

One has, for $r \in [1,+\infty[,\ \ 0 < \theta < 1,$

$$||f||^r_{(L^p,L^{(p)})_{\theta,r}} \approx \int_0^1 (1-\operatorname{Log}x)^{\frac{\theta r}{p'}}\left(\int_0^x f_*^p(s)ds\right)^{\frac{r}{p}}\frac{dx}{x(1-\operatorname{Log}x)}.$$

Interpolation between grand and classical Lebesgue spaces in the critical case

Lemma

Let $1 < p < \infty$, *and let* $f \in L^{p)}$. *Then, for all* $0 < t < 1$,

$$K(f, t; L^{p)}, L^p) \approx \sup_{0<s<\varphi(t)} (1 - \operatorname{Log} s)^{-1/p} \left(\int_s^1 f_*(x)^p dx \right)^{1/p},$$

where $\varphi(t) = e^{1-\frac{1}{t^p}}$.

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where $\varphi(t) = e^{1-\frac{1}{t^p}}$*.*

Theorem

Let $1 < p < \infty$*,* $0 < \theta < 1$*, and* $1 \leqslant r < \infty$*. Then*

$$(L^{p)}, L^p)_{\theta,r} = G\Gamma(p, r; w_1, w_2),$$

where $w_1(t) = t^{-1}(1 - \operatorname{Log} t)^{r\theta/p-1}$ *and* $w_2(t) = (1 - \operatorname{Log} t)^{-1}$*.*

Interpolation between grand Lebesgue spaces in the critical case

Lemma

Let $1 < p < \infty$ *and* $0 < \beta < \alpha$. *Let* $f \in L^{p),\alpha}$. *Then, for all* $0 < t < 1$,

$$K(f, t; L^{p),\alpha}, L^{p),\beta}) \approx \sup_{0<s<\varphi(t)} (1 - \operatorname{Log} s)^{-\frac{\alpha}{p}} \left(\int_s^{\varphi(t)} f_*(x)^p dx \right)^{1/p} + t \sup_{\varphi(t)<s<1} (1 - \operatorname{Log} s)^{-\frac{\beta}{p}} \left(\int_s^1 f_*(x)^p dx \right)^{1/p},$$

where $\varphi(t) = e^{1 - t^{\frac{p}{\beta - \alpha}}}$.

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where $\varphi(t) = e^{1-t^{\frac{p}{\beta-\alpha}}}$.

Theorem

Let $1 < p < \infty$, $0 < \beta < \alpha$, $0 < \theta < 1$, *and* $1 \leqslant r < \infty$. *Then*

$$(L^{p),\alpha}, L^{p),\beta})_{\theta,r} = G\Gamma(p, r; w_1, w_2),$$

where $w_1(t) = t^{-1}(1 - \operatorname{Log} t)^{\frac{r\theta}{p}(\alpha-\beta)-1}$ *and* $w_2(t) = (1 - \operatorname{Log} t)^{-\alpha}$.

The K-functional for the couple $(L^{p,\infty}, L^p)$, $1 < p < +\infty$

Theorem

For a measurable set $E \subset [0,1]$, we denote $|E|_\nu = \int_E \frac{dx}{x}$ and for $f \in L^{p,\infty} + L^p$, $1 < p < +\infty$, we define

$$K_p(f,t) = t \sup \left\{ \left(\int_E f_*^p(\sigma) d\sigma \right)^{\frac{1}{p}} : |E|_\nu = t^{-p} \right\} \qquad t \in]0,1].$$

Then

$$K(f,t; L^{p,\infty}, L^p) \approx K_p(f,t)$$

and

$$K_p(f,t) = t \left[\int_0^{t^{-p}} \psi_{*,\nu}(x)^p dx \right]^{\frac{1}{p}}$$

where $\psi(s) = s^{\frac{1}{s}} f_*(s)$, $\psi_{*,\nu}$ its decreasing rearrangement with respect to the measure ν.

Lemma

Let $1 < p < \infty$. *Then for any* $f \in L^{p,\infty}$ *and all* $0 < t < 1$,

$$\sup_{0<s<t} s^{\frac{1}{p}} f_*(s) \lesssim K(\rho(t), f; L^{p,\infty}, L^{(p}),$$

where $\rho(t) = (1 - \mathrm{Log}\, t)^{-1+\frac{1}{p}}$.

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where $\rho(t) = (1 - \operatorname{Log} t)^{-1+\frac{1}{p}}$.

Theorem

Let $1 < p < \infty$, $0 < \theta < 1$, *and* $1 \leqslant r < \infty$. *Then, for any* $f \in (L^{p,\infty}, L^{(p)})_{\theta,r}$, *one has*

$$||f||_{G\Gamma(\infty,r;v_1,v_2)} \lesssim ||f||_{(L^{p,\infty},L^{(p)})_{\theta,r}},$$

where $v_1(t) = t^{-1}(1 - \operatorname{Log} t)^{r\theta(1-1/p)-1}$ *and* $v_2(t) = t^{1/p}$.

Theorem

Let $1 < p < \infty$, $0 < \theta < 1$, *and* $1 \leqslant r < \infty$. *Then*

$$\|f\|_{G\Gamma(p,r;w_1,w_2)} \lesssim \|f\|_{(L^{p,\infty},L^{(p})_{\theta,r}},$$

where $w_1(t) = t^{-1}(1 - \operatorname{Log} t)^{r\theta-1}$ *and* $w_2(t) = (1 - \operatorname{Log} t)^{-1}$.

Combination these two theorem we have

Theorem

Let $1 < p < \infty$, $0 < \theta < 1$ *and* $1 \leqslant r < \infty$. *Then*

$$\|f\|_{G\Gamma(p,r;w_1,w_2)\cap G\Gamma(\infty,r;v_1,v_2)} \lesssim \|f\|_{(L^{p,\infty},L^{(p})_{\theta,r}},$$

where $w_1(t) = t^{-1}(1-\mathrm{Log}\, t)^{r\theta-1}$, $w_2(t) = (1-\mathrm{Log}\, t)^{-1}$,
$v_1(t) = t^{-1}(1-\mathrm{Log}\, t)^{r\theta(1-1/p)-1}$ *and* $v_2(t) = t^{1/p}$.

THANK YOU FOR ATANTION!!