Faculty of Mathematics and Information Sciences,
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Maximal operator in function spaces with nonstandard growth

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Plan of presentation

1. Maximal operator in Musielak–Orlicz–Sobolev spaces
   - Musielak–Orlicz–Sobolev spaces
   - Boundedness of the maximal operator
   - Continuity of the maximal operator
2. Maximal operator in Hölder spaces with variable exponent
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   - Boudedness of the maximal operator
   - Discontinuity of a maximal operator in Lipschitz space

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$$\frac{\varphi(x,s)}{s^p} \le a \frac{\varphi(x,t)}{t^p} \quad \text{for almost all } x \in A \text{ and for all } 0 < s < t,$$

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Furthermore, we denote

$$(\text{aInc}) = \bigcup_{p \in (1,\infty)} (\text{aInc})_p, \; (\text{aDec}) = \bigcup_{p \in (1,\infty)} (\text{aDec})_p.$$

Example

Let $p\colon A \to [1, \infty)$. We define $\varphi\colon A \times [0, \infty) \to [0, \infty)$ as

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We say that $\varphi : A \times [0,\infty) \to [0,\infty]$ is a Φ-prefunction if $\varphi(x,0) = 0$, $\varphi(x,\cdot)$ is increasing, $\lim_{t\to 0^+} \varphi(x,t) = 0$, $\lim_{t\to\infty} \varphi(x,t) = \infty$ for almost every $x \in A$ and the map $x \mapsto \varphi(x,|f(x)|)$ is measurable for $f \in L^0(A)$.

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Example: $\varphi \in \Phi_{\mathrm{s}}(A)$

Musielak–Orlicz spaces

For $\varphi \in \Phi_w(A)$ and $f \in L^0(A)$ we define

$$\rho_\varphi(f) = \int_A \varphi(x, |f(x)|)\, dx,$$

$$\|f\|_{\varphi,A} = \inf\left\{\lambda > 0 \,:\, \rho_\varphi\left(\frac{f}{\lambda}\right) \le 1\right\},$$

and a set

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It is known, that if $\varphi, \psi \in \Phi_w(A)$ and $\varphi \simeq \psi$, then $L^\varphi(A) = L^\psi(A)$ and corresponding quasi-norms are equivalent.

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Equivalence of Φ-functions

Let $\varphi, \psi : A \times [0, \infty) \to [0, \infty]$. We say that φ and ψ are equivalent ($\varphi \simeq \psi$) if there exists $L \ge 1$ such that the inequalities $\psi(x, t/L) \le \varphi(x, t) \le \psi(x, Lt)$ are satisfied for almost every $x \in A$ and for all $t \in [0, \infty)$.

(A0) property

Let $A \subset \mathbb{R}^n$ be a measurable set. We say that $\varphi \in \Phi_{\mathrm{w}}(A)$ satisfies (A0) if there exists a constant $\beta \in (0,1]$ such that $\beta \leq \varphi^{-1}(x,1) \leq 1/\beta$ for almost everyl $x \in A$, where φ^{-1} is left-inverse of φ.

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For $\psi(x,t) = t^p + a(x)t^q$, where $1 \le p < q < \infty$ and a is measurable it can be shown that

$$\psi \text{ satisfies (A0)} \Leftrightarrow a \in L^\infty(A).$$

(A1) and (A2) properties

Let $\Omega \subset \mathbb{R}^n$ be an open set and $\varphi \in \Phi_w(\Omega)$, then

1. φ satisfies (A1) if there exists $\beta \in (0,1)$ such that for every ball B such that $|B| \le 1$ the following inequality holds

$$\beta\varphi^{-1}(x,t) \le \varphi^{-1}(y,t)$$

for every $t \in [1, 1/|B|]$ and for almost every $x, y \in B \cap \Omega$.

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2. φ satisfies (A2) if for all $s > 0$ there exist $\beta \in (0,1]$ and $h \in L^1(\Omega) \cap L^\infty(\Omega)$ such that the following inequality holds

$$\beta\varphi^{-1}(x,t) \leq \varphi^{-1}(y,t)$$

   for almost every $x, y \in \Omega$ and for all $t \in [h(x) + h(y), s]$.

Example

Proposition

The Φ-function $\varphi(x,t) = t^{p(x)}$ satisifes (A1), iff $\frac{1}{p} \in C^{\log}$, i.e., there exists C such that for every distinct $x, y \in \Omega$

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leq \frac{C}{\log(e + 1/|x-y|)},$$

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The Φ-function $\varphi(x,t) = t^{p(x)}$ satisfies (A2), if $\frac{1}{p}$ satisfies log-Hölder decay condition, i.e., there exist C, p_∞ such that

$$\left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| \leq \frac{C}{\log(e + |x|)}.$$

Musielak–Orlicz–Sobolev spaces

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$$\|u\|_{k,\varphi,\Omega} := \sum_{|\alpha|\le 1} \|D_\alpha u\|_{\varphi,\Omega}.$$

Again, we will write simply $\|u\|_{1,\varphi}$ if $\Omega = \mathbb{R}^n$.

The maximal operator

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For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we define the maximal function $Mf\colon \mathbb{R}^n \to \mathbb{R}$ in a standard way

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f(z)|\,dz.$$

Boundedness of the maximal operator

Theorem

Let $\varphi \in \Phi_w(\mathbb{R}^n)$ satisfies (A0), (A1), (A2), (aInc) and (aDec). If $f \in W^{1,\varphi}(\mathbb{R}^n)$, then $Mf \in W^{1,\varphi}(\mathbb{R}^n)$ and the inequality

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is satisfied for all $i = 1, \dots, n$ and for almost all $x \in \mathbb{R}^n$.

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The sketch of the proof: For $r > 0$ let

$$h_r = \frac{1}{|B(0,r)|}\chi_{B(0,r)}.$$

Then, we have

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This estimate and properties of convolution yields

$$|D_i(|f| * h_r)(x)| = |(D_i|f|) * h_r(x)| \leq MD_i|f|(x)$$

for almost all $x \in \mathbb{R}^n$.

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For $g_k = \max_{1\le m\le k} |f| * h_{r_m}$ we have

$$|D_i g_k(x)| \le \max_{1\le m\le k} |D_i(|f| * h_{r_m})(x)| \le M(D_i|f|)(x) = M(D_i f)(x)$$

for almost all $x \in \mathbb{R}^n$ and all $k \in \mathbb{N}$.

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for almost all $x \in \mathbb{R}^n$ and all $k \in \mathbb{N}$.Therefore, we have

$$\|g_k\|_{1,\varphi} \le \|Mf\|_\varphi + \sum_{i=1}^n \|M(D_i f)\|_\varphi \le C\|f\|_{1,\varphi}.$$

Auxiliary results

For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we define

$$\mathcal{R}f(x) = \left\{ r \geq 0 \, : \, \exists (\{r_k\} \subset (0, \infty)) \, r_k \to r \wedge Mf(x) = \lim_{k \to \infty} \fint_{B(x, r_k)} |f(y)| \, dy \right\}.$$

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(i) For all $x \in \mathbb{R}^n$ the set $\mathcal{R}f(x)$ is nonempty.

Auxiliary results

For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we define

$$\mathcal{R}f(x) = \left\{ r \geq 0 \,:\, \exists(\{r_k\} \subset (0,\infty))\, r_k \to r \wedge Mf(x) = \lim_{k\to\infty} \fint_{B(x,r_k)} |f(y)|\, dy \right\}.$$

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(iii) For almost all $x \in \mathbb{R}^n$ if $0 \in \mathcal{R}f(x)$, then

$$Mf(x) = |f(x)|.$$

Auxiliary results

Proposition

Let $\varphi \in \Phi_{\mathrm{w}}(\mathbb{R}^n)$ satisfies (A0), (aDec)and (aInc). If $f_m, f \in L^{\varphi}(\mathbb{R}^n)$ and $f_m \to f$ in $L^{\varphi}(\mathbb{R}^n)$, then for all $R > 0$ and $\lambda > 0$ the set $\left\{x \in B(0,R) \,:\, \mathcal{R}f_m(x) \notin \mathcal{R}f(x)_{(\lambda)}\right\}$ is measurable and[a]

$$\lim_{m\to\infty} \left|\left\{x \in B(0,R) \,:\, \mathcal{R}f_m(x) \notin \mathcal{R}f(x)_{(\lambda)}\right\}\right| = 0.$$

[a] For nonempty set $A \subset \mathbb{R}^n$ and $\lambda \geq 0$ we denote $A_{(\lambda)} = \{x \in \mathbb{R}^n : \mathrm{dist}(x, A) \leq \lambda\}$.

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Lemma

Let us assume that $\varphi \in \Phi_w(\mathbb{R}^n)$ satisfies (A0), (A1), (A2), (aInc) and (aDec). If $f \in W^{1,\varphi}(\mathbb{R}^n)$, then for almost all $x \in \mathbb{R}^n$ and for all $i = 1, \ldots, n$ we have

$$D_i Mf(x) = \fint_{B(x,r)} D_i|f|(y)\, dy \text{ for all } r \in \mathcal{R}f(x),\ r > 0 \text{ and}$$
$$D_i Mf(x) = D_i|f|(x) \text{ if } 0 \in \mathcal{R}f(x).$$

Continuity of the maximal operator

Theorem

Let us assume that $\varphi \in \Phi_w(\mathbb{R}^n)$ satisfies (A0), (A1), (A2), (aInc) and (aDec), then the maximal operator

$$M: W^{1,\varphi}(\mathbb{R}^n) \to W^{1,\varphi}(\mathbb{R}^n)$$

is continuous.

Hölder spaces with variable exponent

Let (X, d, μ) be a metric measure space equipped with a metric d and the Borel measure μ. We assume that the measure of every open nonempty set is positive and that the measure of every bounded set is finite.

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Let (X, d) be a metric space, by $C(X)$ we denote the space of continuous functions on X such that the norm

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is finite.Moreover, for $\alpha : X \to [0, 1]$ we denote by $C^{0,\alpha(\cdot)}(X)$ the `variable exponent Hölder space`, i.e. the space of $f \in C(X)$ such that

$$\|f\|_{C^{0,\alpha(\cdot)}(X)} := \|f\|_{C(X)} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d^{\alpha(x)}(x,y)} < \infty.$$

Boudedness of the maximal operator

Theorem

Suppose that $0 < \delta \leq 1$, and that (X, d, μ) satisfies the δ-annular property. If $\alpha : X \to [0, \delta]$, then $M : C^{0,\alpha(\cdot)}(X) \to C^{0,\alpha(\cdot)}(X)$ and there exists $C_1 > 0$ such, that for $f \in C^{0,\alpha(\cdot)}(X)$ the following estimate holds

$$\|Mf\|_{C^{0,\alpha(\cdot)}(X)} \leq C_1 \|f\|_{C^{0,\alpha(\cdot)}(X)}.$$

Continuity of the maximal operator

Theorem

Let $\delta \in (0,1]$ and (X,d,μ) satisfies the δ-annular property. If $\alpha : X \to (0,1]$ and $\beta : X \to [0,1]$ satisfy $\sup_{x\in X} \beta(x)/\alpha(x) < 1$, then the operator

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Continuity of the maximal operator

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A sketch of the proof: Let us note since $\beta(\cdot) \le \alpha(\cdot)$, we have $Id : C^{0,\alpha(\cdot)}(X) \to C^{0,\beta(\cdot)}(X)$. Therefore, we get $M : C^{0,\alpha(\cdot)}(X) \to C^{0,\beta(\cdot)}(X)$ is bounded.

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In order to prove the continuity of M we fix $f \in C^{0,\alpha(\cdot)}(X)$ and a sequence $\{f_n\} \subset C^{0,\alpha(\cdot)}(X)$ such that $f_n \to f$ in $C^{0,\alpha(\cdot)}(X)$. It is easy to see that $Mf_n \to Mf$ in $C(X)$.

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$$\sup_{x\neq y} \frac{|Mf_n(x) - Mf(x) - Mf_n(y) + Mf(y)|}{d(x,y)^{\beta(x)}} \to 0.$$

We know that sequence $\{Mf_n\}$ is bounded in $C^{0,\alpha(\cdot)}(X)$, so we can assume that there exists $N \ge 1$ such that $\|Mf_n\|_{C^{0,\alpha(\cdot)}(X)} \le N$ for all n and $\|Mf\|_{C^{0,\alpha(\cdot)}(X)} \le N$.

A sketch of the proof

Therefore, for $x, y \in X$ such that $x \neq y$, we get the following string of inequalities

$$\frac{|Mf_n(x) - Mf(x) - Mf_n(y) + Mf(y)|}{d(x,y)^{\beta(x)}}$$

$$= \left(\frac{|Mf_n(x) - Mf(x) - Mf_n(y) + Mf(y)|}{d(x,y)^{\alpha(x)}}\right)^{\frac{\beta(x)}{\alpha(x)}} |Mf_n(x) - Mf(x) - Mf_n(y) + Mf(y)|^{1-}$$

$$\leq \left(\frac{|Mf_n(x) - Mf_n(y)|}{d(x,y)^{\alpha(x)}} + \frac{|Mf(x) - Mf(y)|}{d(x,y)^{\alpha(x)}}\right)^{\frac{\beta(x)}{\alpha(x)}} \left(2\|Mf_n - Mf\|_{C(X)}\right)^{1-\frac{\beta(x)}{\alpha(x)}}$$

$$\leq (2N)^{\left(\frac{\beta}{\alpha}\right)^+} \left(2\|Mf_n - Mf\|_{C(X)}\right)^{1-\left(\frac{\beta}{\alpha}\right)^+},$$

where $\left(\frac{\beta}{\alpha}\right)^+ = \sup_{x\in X} \frac{\beta(x)}{\alpha(x)}$. Hence,

$$\sup_{x\neq y} \frac{|Mf_n(x) - Mf(x) - Mf_n(y) + Mf(y)|}{d(x,y)^{\beta(x)}} \leq (2N)^{\left(\frac{\beta}{\alpha}\right)^+} \left(2\|Mf_n - Mf\|_{C(X)}\right)^{1-\left(\frac{\beta}{\alpha}\right)^+}.$$

Since the right-hand side of the above inequality goes to 0 when $n \to \infty$, the proof follows.

Discontinuity of a maximal operator in Lipschitz space

Theorem

There exist $f, f_n \in C^{0,1}(\mathbb{R})$ such that $f_n \to f$ in $C^{0,1}(\mathbb{R})$ and $Mf_n \not\to Mf$ in $C^{0,1}(\mathbb{R})$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and 2−periodic function such that $f(x) = |x|$ for $x \in [-1, 1]$.

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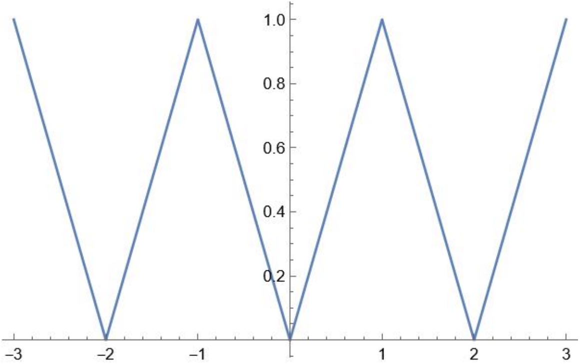
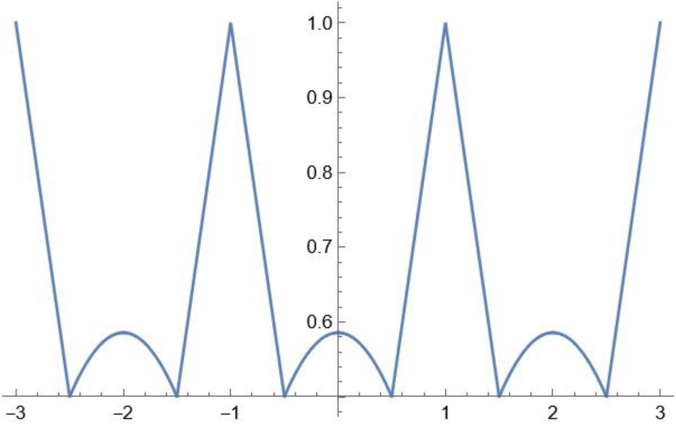


Figure: A graph of the funtion f

A discontinuity of a maximal operator in Lipschitz space

$$Mf(x) = \begin{cases} 2 - \sqrt{x^2+2}, & \text{for } 0 \le x \le \frac{1}{2} \\ x, & \text{for } \frac{1}{2} < x \le 1. \end{cases}$$



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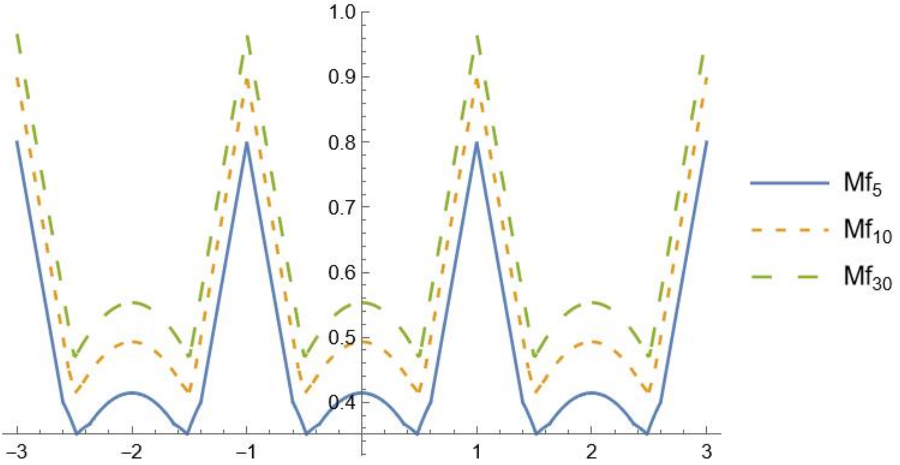

Figure: A graph of few of functions Mf_n

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$$\begin{aligned} \frac{|Mf_n(\frac{1}{2}) - Mf(\frac{1}{2}) - Mf_n(d_n) + Mf(d_n)|}{|\frac{1}{2} - d_n|} &= \frac{Mf(d_n) - Mf(\frac{1}{2})}{\frac{1}{2} - d_n} + \frac{Mf_n(\frac{1}{2}) - Mf_n(d_n)}{\frac{1}{2} - d_n} \\ &\geq \frac{Mf(d_n) - Mf(\frac{1}{2})}{\frac{1}{2} - d_n} \geq \frac{1}{6}. \end{aligned}$$

Hence, it is not true that $Mf_n \to Mf$ in $C^{0,1}(\mathbb{R})$.

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Thank you for your attention!