On Musielak-Orlicz-Sobolev spaces and Lavrentiev's phenomenon

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based on joint work with Iwona Chlebicka and Błażej Miasojedow

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On the other hand, there exist a constant $c > 0$ such that if u is a Lipschitz function, then $\mathcal{I}[u] \geq c$.

This means that the minimizer of functional $\mathcal{I}$ cannot be appropriately approximated by Lipschitz functions, and we deal with **Lavrentiev's phenomenon**.

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$$\inf_{u \in u_0 + W(\Omega)} \mathcal{F}[u] < \inf_{u \in u_0 + C_c^\infty(\Omega)} \mathcal{F}[u]\,,$$

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The phenomenon is named after Lavrentiev, who provided the first example of its occurrence and conditions needed for its absence. [Lavrentiev, 1926]

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  Examples of Lavrentiev's phenomenon for wider range of p, q.

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Regularity of minimizers in those special instances were studied by [Zhikov '80-'10], [Belloni & Buttazzo, '92], [Buttazzo & Mizel '95], [Esposito, Leonetti, Mingione '04], [Fonseca, Maly & Mingione '04], [Balci, Diening & Surnachev '20], [Esposito, Leonetti & Petricca '19], [Leonetti & De Filippis '22], [Koch '22], [Bousquet '22], [Baasandorj & Byun '23].

This is the moment where Musielak–Orlicz spaces come in.

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With x-dependence: $M(x, \xi) = |\xi|^{p(x)}$ for $1 < p^- \leq p(\cdot) \in L^\infty$.

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Some things that we lack:

- homogeneity, i.e., dependence on x is important
- monotonicity
  If $\eta \leq \xi$ ($\eta_i \leq \xi_i$ for every i), then not necessarily $M(x,\eta) \leq M(x,\xi)$.

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\mathcal{L}_M &= \{\xi : \varrho_M(\xi) < \infty\},\\
L_M &= \{\xi : \exists_{\lambda>0} : \varrho_M(\xi/\lambda) < \infty\},\\
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But we have that $E_M = \mathcal{L}_M = L_M$ if

$$M(x, 2\xi) \le h(x) + cM(x, \xi) \text{ for some } h \in L^1(\Omega) \quad (M \in \Delta_2)$$

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We always have that $E_M \subseteq \mathcal{L}_M \subseteq L_M$.
But we have that $E_M = \mathcal{L}_M = L_M$ if

$$M(x, 2\xi) \leq h(x) + cM(x, \xi) \text{ for some } h \in L^1(\Omega) \quad (M \in \Delta_2)$$

Two types of convergence:

- In L_M: $\varrho_M((\xi - \xi_n)/\lambda) \to 0$ for **some** λ – **modular convergence**;
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Function spaces

Having an N-function M, we may define a modular

$$\varrho_M(\xi) = \int_\Omega M(x, \xi(x))\,dx\,.$$

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If $M \in \Delta_2$, they both coincide with $\varrho_M(\xi - \xi_n) \to 0$.

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Given the definitions of L_M and E_M, one can define

$$V_0L_M = \{u \in W_0^{1,1}(\Omega) : \nabla u \in L_M\}$$
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We want to know under what conditions on M we have the density in V_0L_M and V_0E_M.

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- Absence of Lavrentiev's gap;
- Existence results in the theory of PDEs with non–standard growth.

Absence of Lavrentiev's phenomenon

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For simplicity, let us assume that $M \in \Delta_2$. Then, if smooth functions are dense in $V_0L_M = V_0E_M$, it means that for any $u \in V_0L_M$, there exists a sequence $(u_\delta)_\delta \subset C_c^\infty(\Omega)$, such that

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Note: The above holds true provided that we have density in $V_0 E_M$, regardless Δ_2 condition.

Nice references

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[I. Chlebicka, P. Gwiazda, A. Świerczewska-Gwiazda, and A. Wróblewska-Kamińska. Partial differential equations in anisotropic Musielak-Orlicz spaces. Springer Monographs in Mathematics. Springer, ©2021.]

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Now we shall talk about the results concerning density in Musielak–Orlicz–Sobolev spaces.

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Theorem (Gossez, 1982
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Let $M(\xi)$ be an N-function, Ω – Lipschitz domain. Then for any $u \in V_0L_M$ (V_0E_M), the exists a sequence $(u)_\delta \subseteq C_c^\infty(\Omega)$ approximating u in $V_0L_M(V_0E_M)$.

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Let us use approximation by convolution with shrinking

$$S_\delta(\nabla u)(x) = \int_{B_\delta(0)} \rho_\delta(y)\nabla u((x-y)/\kappa_\delta)\, dy\,.$$

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Vitali Converging Theorem actually ends the proof. □

Difference in general case

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By using Jensen's inequality

$$M(x, S_\delta(\nabla u)(x)) = M\left(x, \int_{B_\delta(0)} \rho_\delta(y) \nabla u((x-y)/\kappa_\delta)\right) dy$$

$$\leq \int_{B_\delta(0)} \rho_\delta(y) M(x, \nabla u((x-y)/\kappa_\delta))\, dy \neq S_\delta\left(M(\cdot, \nabla u(\cdot))\right)(x)\,.$$

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Isotropic case

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Isotropic condition ($\mathsf{B}_0^{\mathsf{iso}}$)**.** For some constants $C, c > 0$:

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Density in $V_0 E_M$ requires a bit stronger condition, which is described in our paper.

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Infimum of convex 1-dimensional functions is **almost convex**, *actually*

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Note: The above fact is not true in higher dimensions.

Theorem

If isotropic $M(x, |\xi|)$ *satisfies* ($\mathrm{B}_0^{\mathrm{iso}}$)*, then any function* $u \in V_0 L_M$ *may be approximated by functions from* $C_c^\infty(\Omega)$*.*

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$$M(x, \tfrac{1}{\lambda}|S_\delta(\nabla u)(x)|) \lesssim M^-\left(\tfrac{1}{\lambda}\,|S_\delta(\nabla u)(x)|\right)$$
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Again, Vitali Converging Theorem ends the proof. □

Let us see how to generalize this result.

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If $u \in C^{0,\gamma}(\Omega)$, $\gamma \in (0,1]$, *then*

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Note that for $\gamma = 1$ the condition ($\mathsf{B}_\gamma^{\mathrm{iso}}$) is always satisfied.

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Some conditions using this object were considered in

- [P. Gwiazda, I. Skrzypczak, and A. Zatorska-Goldstein, 2018]
- [I. Chlebicka, P. Gwiazda, A. Świerczewska-Gwiazda, and A. Wróblewska-Kamińska, 2021].

As $\left(M_\delta^-\right)^{**}$ is a very complicated object, the conditions are hard to verify.

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Theorem (P. Hästö, 2022)

The following conditions are equivalent.

$$i) \quad \exists_{C>0} \quad M(x,\xi) \le \left(M_\delta^-\right)^{**}(C\xi) + 1 \quad \textit{whenever} \quad \left(M_\delta^-\right)^{**}(C\xi) \le \frac{1}{|B_\delta|};$$

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Using this theorem, we formulated the general condition embracing $(\mathrm{B}_\gamma^{\mathrm{iso}})$ and $(\mathrm{B}_\gamma^{\mathrm{ort}})$.
[B, I. Chlebicka, B. Miasojedow, arXiv:2210.15217]

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The proof of this fact uses Theorem of P. Hästö and some tricks to keep full anisotropy of the space.

Absence of Lavrentiev's phenomenon

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Let the functional $\mathcal{F}$ *be given by*

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where Ω *is Lipschitz domain.*

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$$\mathcal{F}[u] = \int_\Omega F(x, u(x), \nabla u(x))\, dx\,, \text{ where } \quad F \approx M\,,$$

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- $M(x,\xi) = |\xi|^p + a(x)|\xi|^q$, if $a \in C^{0,\alpha}$, $q \le p + \frac{\alpha}{1-\gamma}$;
- $M(x,\xi) = \sum_{i=1}^n |\xi_i|^{p_i} + a_i(x)|\xi_i|^{q_i}$, if $a_i \in C^{0,\alpha_i}$, $q_i \le p_i + \frac{\alpha_i}{1-\gamma}$;
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Let us again consider the double phase functional

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We have answers and we will publish them soon:

B, I. Chlebicka, F. De Filippis, B. Miasojedow, *Absence and presence of Lavrentiev's phenomenon in double phase functionals for every choice of exponents*.

The end

Thank you for your attention!