

# Density of smooth functions in Musielak-Orlicz spaces

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# Modular density of smooth functions

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In Musielak-Orlicz-Sobolev spaces the norm is defined by the means of functional

$$\xi \mapsto \int_Z M(z, \xi) dz$$

with typical choice of  $Z = \Omega$  or  $Z = \Omega \times [0, T]$ .

Smooth functions are not dense in norm (in general) in the spaces, but in modular topology and only the growth of  $M$  with respect to  $\xi$  is well balanced with small perturbations of  $z$ -variable.

*Y. Ahmida, I. C., P. Gwiazda, A. Yousfi,*  
Gossez's approximation theorems in Musielak-Orlicz-Sobolev spaces,  
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there will appear also references to several papers joint with Gwiazda & Zatorska-Goldstein with applications of similar density results in existence to renormalized solutions to elliptic and parabolic problems with merely integrable data

# Examples

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- $\xi \mapsto \int_{\Omega} |\xi|^p dx$  (Lebesgue/Sobolev spaces)
- $\xi \mapsto \int_{\Omega} |\xi|^{p(x)} dx$  (variable exponent Lebesgue/Sobolev spaces)
- $\xi \mapsto \int_{\Omega} |\xi|^p + a(x)|\xi|^q dx$ ,  $\xi \mapsto \int_{\Omega} |\xi|^p + a(x)|\xi|^p \log(1 + |\xi|) dx$  (double-phase spaces)
- $\xi \mapsto \int_{\Omega} B(|\xi|) dx$  (Orlicz/Orlicz-Sobolev spaces; also  $L \log L$ ,  $\exp L$ )
- $\xi \mapsto \int_{\Omega} \Phi(\xi) dx$  (anisotropic Orlicz-Sobolev spaces)
- $\xi \mapsto \int_{\Omega} M(x, |\xi|) dx$ ,
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C., A pocket guide to nonlinear differential equations  
in Musielak-Orlicz spaces, Nonl. Analysis 2018.

## (Generalized) $N$ -function $M : \Omega \times [0, \infty) \rightarrow \mathbb{R}$

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1.  $M$  is a Carathéodory function, such that  $M(x, 0) = 0$ ,  
 $s \neq 0 \Rightarrow \inf_{x \in \Omega} M(x, s) > 0$  and  $\sup_{x \in \Omega} M(x, s) < \infty$ ,
2.  $s \mapsto M(x, s)$  is a convex function,
3.  $\lim_{s \rightarrow 0} \frac{M(x, s)}{s} = 0$  for a.e.  $x \in \Omega$ ,
4.  $\lim_{s \rightarrow \infty} \frac{M(x, s)}{s} = \infty$  for a.e.  $x \in \Omega$ .

### Examples of $M(x, s)$

$|s|^p$ ,  $\omega(x)|s|^p$ ,  $|s|^{p(x)}$ ,  $|s|^q + a(x)|s|^p$ , more phases...

Orlicz:  $B(|s|)$ ,  $\omega(x)B(|s|)$ ,  $B_0(|s|) + \sum_i \omega_i(x)B_i(|s|)$ ...

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Anisotropy...



## Doubling conditions

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We say that an  $N$ -function  $M : \Omega \times [0, \infty) \rightarrow \mathbb{R}$  satisfies  $\Delta_2$ -**condition** if there exists a constant  $c > 0$  and  $h \in L^1(\Omega)$ ,  $h \geq 0$ , such that

$$M(x, 2s) \leq cM(x, s) + h(x).$$

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- in  $\Delta_2$ :  $M(x, s) = |\xi|^p$ ;  $a(x)|\xi|^p + |\xi|^q$ ;  $|\xi|^{p(x)}$ ;  $|\xi|^p \log^\alpha(1 + |\xi|)$ ;

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- **NOT** in  $\Delta_2$ :
  - (fast)  $M(x, \xi) = a(x) (\exp(|\xi|) - 1 + |\xi|)$ ;
  - (irregularly increasing) even trapped between  $t^p$  and  $t^{p+\varepsilon}$ ,  $1 < p < \infty$ , cf. C.–Giannetti–Zatorska-Goldstein, JMAA 2019.

# Musielak-Orlicz spaces

Suppose  $\Omega \subset \mathbb{R}^N$ .

- $L_M(\Omega)$  is defined as the set of all measurable functions  $u : \Omega \rightarrow \mathbb{R}$  such that  $\int_{\Omega} M(x, |u(x)|/\lambda) dx < +\infty$  for some  $\lambda > 0$ .
- $E_M(\Omega)$  is defined as the set of all measurable functions  $u : \Omega \rightarrow \mathbb{R}$  such that  $\int_{\Omega} M(x, |u(x)|/\lambda) dx < +\infty$  for all  $\lambda > 0$ .

Both are equipped with the Luxemburg norm

$$\|u\|_{L_M(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} M\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

\*Then  $L_M(\Omega)$  is a Banach space and  $E_M(\Omega)$  is its closed subset.

\*\*  $E_M(\Omega)$  coincides with norm closure of bounded functions, provided that  $\forall c > 0 \int_{\Omega} M(x, c) dx < \infty$ .

Thus here  $E_M(\Omega) = \overline{L^{\infty}(\Omega)}^{\|\cdot\|_{L_M(\Omega)}}$ , but  $E_M(\Omega) \neq L_M(\Omega)$ .

# Musielak-Orlicz-Sobolev spaces

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For a positive integer  $m$ , we define the Musielak-Orlicz-Sobolev spaces  $W^m L_M(\Omega)$  and  $W^m E_M(\Omega)$  as follows

$$W^m L_M(\Omega) = \{u \in L_M(\Omega) : D^\alpha u \in L_M(\Omega), |\alpha| \leq m\},$$

$$W^m E_M(\Omega) = \{u \in E_M(\Omega) : D^\alpha u \in E_M(\Omega), |\alpha| \leq m\},$$

where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ ,  $|\alpha| = |\alpha_1| + |\alpha_2| + \dots + |\alpha_N|$  and  $D^\alpha$  denote the distributional derivatives.

$W^m L_M(\Omega)$  and  $W^m E_M(\Omega)$  are endowed with the Luxemburg norm

$$\|u\|_{W^m L_M(\Omega)} = \inf \left\{ \lambda > 0 : \sum_{|\alpha| \leq m} \int_{\Omega} M(x, \frac{1}{\lambda} |D^\alpha u|) dx \leq 1 \right\}.$$

$(W^m L_M(\Omega), \|u\|_{W^m L_M(\Omega)})$  is a Banach space.

## Fenchel–Young conjugate function $M^*$

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Fenchel–Young conjugate function (complementary function, the Legendre transform) to a function  $M : \Omega \times [0, \infty) \rightarrow \mathbb{R}$  is defined by

$$M^*(x, s) = \sup_{r \in \mathbb{R}} (r \cdot s - M(x, r)), \quad s \geq 0, \quad x \in \Omega.$$

For  $M(x, s) = \frac{1}{p}|s|^p$ , then  $M^*(x, s) = \frac{1}{p'}|s|^{p'}$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$ .

Fenchel–Young's inequality

$$|\eta \cdot \xi| \leq M(x, |\eta|) + M^*(x, |\xi|),$$

Hölder's inequality  $\|\eta \cdot \xi\|_{L^1} \leq 2\|\eta\|_{L_M} \cdot \|\xi\|_{L_{M^*}}$  but in general

$$(L_M)' \neq L_{M^*}$$

» spaces  $(L_M)'$  and  $L_{M^*}$  are associate, but not dual «

# Doubling condition vs. properties of the spaces

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## Separability

$$M \in \Delta_2^\infty \quad \implies \quad E_M(\Omega) = L_M(\Omega)$$

## Reflexivity

$$M, M^* \in \Delta_2^\infty \quad \Longleftrightarrow \quad (L_M(\Omega; \mathbb{R}^N))' = L_{M^*}(\Omega; \mathbb{R}^N)$$

and  $L_M$  is **reflexive and separable**.

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# Modular density 1/3

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## Modular density - Orlicz case (Gossez, Studia Math.'82)

In Orlicz spaces  $C_c^\infty$  is dense in  $W_0^1 L_B$  modularly.

## Modular convergence

- $u_k \xrightarrow[k \rightarrow \infty]{M} u$   $((u_k)_k$  converges modularly to  $u$  in  $L_M$ ) if

$$\exists \lambda > 0 \quad \int_{\Omega} M\left(x, \frac{|u_k - u|}{\lambda}\right) dx \xrightarrow[k \rightarrow \infty]{} 0 .$$

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- $u_k \xrightarrow[k \rightarrow \infty]{mod} u$   $((u_k)_k$  converges modularly to  $u$  in  $W^m L_M$ )

$$\text{if} \quad \forall_{|\alpha| \leq m} \quad D^\alpha u_k \xrightarrow[k \rightarrow \infty]{M} D^\alpha u .$$

## Modular density 2/3

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### Modular density - Orlicz case (Gossez, Studia Math.'82)

In Orlicz spaces  $C_c^\infty$  is dense in  $W_0^1 L_B$  with respect to sequential modular closure, not the norm one.

The closures coincide (only) in doubling case (i.e. reflexive).

### Modular density - Musielak-Orlicz case (Ahmida, Chlebicka, Gwiazda, Youssfi, JFA'2018)

In Musielak-Orlicz spaces to get modular density one needs to exclude Lavrentiev's phenomenon by imposing a condition reflecting log-Hölder continuity of the variable exponent, as well as sharp closeness of powers in double-phase spaces.

We need to control modulus of continuity of function  $M(x, s)$ : balance its asymptotic behaviour for small perturbations of spacial variable  $x$  and big values of  $s$ .

# Lavrentiev's phenomenon 1/2

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Consider variational functional

$$u \mapsto \mathcal{F}[u] = \int_{\Omega} F(x, u, Du) \, dx$$

with  $F$  whose growth is governed by inhomogeneous function  $M$

$$\frac{1}{c}M(x, s) \leq F(x, r, s) \leq cM(x, s).$$

Then always

$$\inf_{u \in u_0 + W_0^1 L_M(\Omega)} \mathcal{F}[u] \leq \inf_{u \in u_0 + C_c^\infty(\Omega)} \mathcal{F}[u]$$

We deal with Lavrentiev's phenomenon if the inequality is strict.

# Lavrentiev's phenomenon 2/2

## Examples of spaces with functions that cannot be approximated

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### Variable exponent spaces

$$W^{1,p(\cdot)} = \{f \in W_{loc}^{1,1} : |Df|^{p(x)} \in L^1\}$$

when the exponent  $p$  is **not** log-Hölder continuous

$$(p \in \mathcal{P}_{\log} \text{ if } |p(x) - p(y)| \leq -c / \log |x - y|)$$

the condition is essentially sharp

### Double phase spaces

$$\{f \in W_{loc}^{1,1} : |Df|^p + a(x)|Df|^q \in L^1\}$$

with  $a : \Omega \rightarrow [0, \infty)$ ,  $a \in C^{0,\alpha}$ ,

when powers **do not** satisfy  $q/p \leq 1 + \alpha/n$

range is sharp due to Colombo & Mingione, ARMA 2015

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We say that an  $N$ -function  $M : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$  satisfies  $\Delta_2$ -**condition** if there exists a constant  $c > 0$  and  $h \in L^1(\Omega)$ ,  $h \geq 0$ , such that

$$M(x, 2\xi) \leq cM(x, \xi) + h(x).$$

If  $M, M^* \in \Delta_2^\infty$ , then  $(L_M(\Omega; \mathbb{R}^N))^* = L_{M^*}(\Omega; \mathbb{R}^N)$   
and  $L_M$  is **reflexive and separable**.

### Example-reminder

- in  $\Delta_2$ :  $M(x, \xi) = |\xi|^p$ ;  $a(x)|\xi|^p + |\xi|^q$ ;  $|\xi|^{p(x)}$ ;  $|\xi|^p \log^\alpha(1 + |\xi|)$ ;
- **NOT** in  $\Delta_2$ : 'fast growth' like exponential or 'irregular growth'.

# Balance condition in the general setting

Ahmida, Chlebicka, Gwiazda, Youssfi, JFA 2018

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For small perturbations of  $x$  and sufficiently large  $s \in \mathbb{R}$  we have

$$\frac{M(x, s)}{M(y, s)} \leq \Theta(|x - y|, s)$$

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**(M)** with  $\limsup_{\delta \rightarrow 0} \Theta(\delta, \delta^{-N}) < \infty$

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**(M)<sub>p</sub>** if  $M(x, s) \geq cs^p$ , then we need  $\limsup_{\delta \rightarrow 0} \Theta(\delta, \delta^{-N/p}) < \infty$



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\* fully anisotropic case involves  $(\inf_{x \in B_\delta} M(x, \xi))^{**}$ .

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where  $\sigma(\tau) = -c/\log \tau$ .

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Then  $\limsup_{\varepsilon \rightarrow 0^+} \Theta(\varepsilon, c\varepsilon^{-N/p}) < \infty$  forces sharp condition  $q/p \leq 1 + \alpha/N$ .
4. [Weighted Orlicz]  $M(x, s) = \sum_{i=1}^k k_i(x)M_i(s) + M_0(x, s)$ ,  
 $k_i : \overline{\Omega} \rightarrow (0, +\infty)$ , there exists a nondecreasing function  $\Theta_i$  satisfying  $k_i(x) \leq \Theta_i(|x - y|)k_i(y)$  with  $\limsup_{\varepsilon \rightarrow 0^+} \Theta_i(\varepsilon) < \infty$ ,  
whereas  $M_0(x, s)$  satisfies is  $\Theta$ -regular with  $\Theta_0$ .  
Then, we can take  $\varphi(\tau, s) = \sum_{i=1}^k \Theta_i(\tau) + \Theta_0(\tau, s)$ .

# Key ideas of the proof

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## Construction of approximation

- split domain into starshaped regions
- shrink support of an approximated function
- mollify (re-scaled convolution)
- glue domains

We need: 
$$\int_{\Omega} M(x, \frac{1}{\lambda} u_{\varepsilon}) dx \leq C \int_{\Omega} M(x, \frac{1}{\lambda} u) dx.$$

For this, using (M) when  $x, y$  are close, we control

$$M(x, \frac{1}{\lambda} |u_{\varepsilon}(x)|) = \frac{M(x, \frac{1}{\lambda} |u_{\varepsilon}(x)|)}{M(y, \frac{1}{\lambda} |u_{\varepsilon}(x)|)} \frac{M(y, \frac{1}{\lambda} |u_{\varepsilon}(x)|)}{(M_{x,\varepsilon})^{**}(\frac{1}{\lambda} |u_{\varepsilon}(x)|)} (M_{x,\varepsilon})^{**}(\frac{1}{\lambda} |u_{\varepsilon}(x)|).$$

# References for approximation

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## Modular approximation in Orlicz spaces

- isotropic: Gossez, Studia Math.'82
- anisotropic: Alberico, C., Cianchi, Zatorska-Goldstein, arXiv

## Modular approximation in Musielak-Orlicz spaces

- isotropic: Ahmida, C., Gwiazda, Youssfi, JFA 2018
- anisotropic: C. (Skrzypczak), Gwiazda, Zatorska-Goldstein, 2xJDE 2018  
» covering sharply  $p(x)$ -case, but not double-phase (but existence does!)
- anisotropic: C., Gwiazda, Świerczewska-Gwiazda, Wróblewska-Kamińska, book » covering sharply  $p(x)$ -case and double-phase
- anisotropic parabolic with  $M(t, x, \xi)$ : C-G-ZG (AIHP 2019 & JDE 2019)



# Applications to existence and regularity

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Alberico, C., Cianchi, Zatorska-Goldstein, *Fully anisotropic elliptic problems with minimally integrable data*, arXiv 2019.

$$-\operatorname{div} a(x, \nabla u) = \mu$$

with fully anisotropic operator and measure data.

We prove the existence and regularity in anisotropic Marcinkiewicz scale of *approximable solutions* in particular to

$$\begin{cases} -\sum_{i=1}^n (b_i(x) |u_{x_i}|^{p_i-2} u_{x_i})_{x_i} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $u_{x_i}$  denotes partial derivative with respect to the variable  $x_i$ , the functions  $b_i \in L^\infty(\Omega)$  are such that  $b_i(x) \geq 1$ , and  $p_i > 1$ .

(but also for problems with slower growth like  $L \log L$  etc.)

# Applications to existence

see <https://www.mimuw.edu.pl/~ichlebicka/>

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- Renormalized solutions to elliptic problem  $-\operatorname{div} a(x, \nabla u) = f \in L^1$   
Gwiazda, Skrzypczak, Zatorska-Goldstein, JDE 2018
- Renormalized solutions to parabolic problem with  $M = M(x, \nabla u)$   
C., Gwiazda, Zatorska-Goldstein, JDE 2018
- Weak solutions to parabolic problem with  $M = M(t, x, \nabla u)$   
C., Gwiazda, Zatorska-Goldstein, Ann.I.H.Poincaré 2019
- Renormalized solutions to parabolic problem with  $M = M(t, x, \nabla u)$   
C., Gwiazda, Zatorska-Goldstein, JDE 2019

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- Renormalized solutions to elliptic problem  $-\operatorname{div} a(x, \nabla u) = f \in L^1$   
Gwiazda, Skrzypczak, Zatorska-Goldstein, JDE 2018
- Renormalized solutions to parabolic problem with  $M = M(x, \nabla u)$   
C., Gwiazda, Zatorska-Goldstein, JDE 2018
- Weak solutions to parabolic problem with  $M = M(t, x, \nabla u)$   
C., Gwiazda, Zatorska-Goldstein, Ann.I.H.Poincaré 2019
- Renormalized solutions to parabolic problem with  $M = M(t, x, \nabla u)$   
C., Gwiazda, Zatorska-Goldstein, JDE 2019

book: *Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces*  
C., Gwiazda, Świerczewska-Gwiazda, Wróblewska-Kamińska (soon)

# Parabolic existence and uniqueness 1/2

Chlebicka, Gwiazda, Zatorska-Goldstein

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$$\partial_t u - \operatorname{div} A(t, x, \nabla u) = f(x, t) \in L^\infty/L^1$$

(A1)  $A$  is a Carathéodory function

(A2) “Growth of  $A(t, x, \xi)$  is governed by  $M(t, x, \xi)$ ”

(A3)  $(A(t, x, \xi) - A(t, x, \eta)) \cdot (\xi - \eta) \geq 0$

# Parabolic existence and uniqueness 2/2

Chlebicka, Gwiazda, Zatorska-Goldstein

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## Very weak solutions

Suppose  $[0, T]$  is a finite interval,  $\Omega$  is a bounded Lipschitz domain in  $\mathbb{R}^N$ ,  $N > 1$ ,  $f \in L^1(\Omega_T)$ ,  $u_0 \in L^1(\Omega)$ . Let an  $N$ -function  $M$  satisfy assumption (M) and function  $A$  satisfy assumptions (A1)-(A3). Then there exists **unique** renormalized solution to the problem

$$\begin{cases} \partial_t u - \operatorname{div} A(t, x, \nabla u) = f(t, x) & \text{in } \Omega_T, \\ u(t, x) = 0 & \text{on } \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega. \end{cases}$$

# Parabolic existence and uniqueness 2/2

Chlebicka, Gwiazda, Zatorska-Goldstein

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## Remark

Skip balance condition (M) in reflexive spaces if  $M = M(x, \nabla u)$ .

# Main examples not covered before

Chlebicka, Gwiazda, Zatorska-Goldstein, JDE 2018, AIHP 2019, JDE 2019

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I. Evolutionary equations with growth below a power ( $s^{1+\nu}$ )

$$\begin{cases} \partial_t u - \operatorname{div} \left( a(t, x) \frac{\log^\alpha(e + |\nabla u|)}{|\nabla u|} \nabla u \right) = f(t, x) \in L^1(\Omega_T), \\ u(t, x) = 0 \quad \text{on } \partial\Omega, \\ u(0, x) = u_0(x) \in L^1(\Omega) \end{cases}$$

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II. Evolutionary  $p(t, x)$ -Laplace equation ( $1 \ll p \ll \infty$ )

$$\begin{cases} \partial_t u - \operatorname{div} \left( a(t, x) |\nabla u|^{p(t, x)-2} \nabla u \right) = f(t, x) \in L^1(\Omega_T), \\ u(t, x) = 0 \text{ on } \partial\Omega, \\ u(0, x) = u_0(x) \in L^1(\Omega) \end{cases}$$

Exponent  $p$  can vary around 2 – we only need to have it:

$1 \ll p(\cdot, \cdot) \ll \infty$  and log-Hölder continuous in  $\Omega_T$



# Main examples not covered before

Chlebicka, Gwiazda, Zatorska-Goldstein, JDE 2018, AIHP 2019, JDE 2019

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## III. Evolutionary double-phase problem

$$\begin{cases} \partial_t u - \operatorname{div} \left( (|\nabla u|^{p-2} + a(t, x) |\nabla u|^{q-2}) \nabla u \right) = f(t, x) \in L^1(\Omega_T), \\ u(t, x) = 0 \quad \text{on } \partial\Omega, \\ u(0, x) = u_0(x) \in L^1(\Omega) \end{cases}$$

We cover it provided  $a \in C^{0,\alpha}(\Omega_T)$  and  $q/p \leq 1 + \alpha/n$   
(range is sharp for excluding Lavrentiev's phenomenon)

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Chlebicka, Gwiazda, Zatorska-Goldstein, JDE 2018, AIHP 2019, JDE 2019

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## IV. Anisotropic problems

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Chlebicka, Gwiazda, Zatorska-Goldstein, JDE 2018, AIHP 2019, JDE 2019

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## IV. Anisotropic problems

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# Summary

## Modular density of smooth functions

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In Musielak-Orlicz-Sobolev spaces the norm is defined by the means of functional

$$\xi \mapsto \int_Z M(z, \xi) dz$$

with typical choice of  $Z = \Omega$  or  $Z = \Omega \times [0, T]$ .

Smooth functions are not dense in norm (in general) in the spaces, but in modular topology and only the growth of  $M$  with respect to  $\xi$  is well balanced with small perturbations of  $z$ -variable.

*Y. Ahmida, I. Chlebicka, P. Gwiazda, A. Youssfi,*  
Gossez's approximation theorems in Musielak-Orlicz-Sobolev spaces,  
J. Functional Analysis 275 (9) (2018), 2538-2571.

see <https://www.mimuw.edu.pl/~ichlebicka/publications>

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