The self-improving property of higher integrability in the obstacle problem for the porous medium equation

Yumi Cho

Seoul National University

(This is a joint work with Christoph Scheven)

Monday's Nonstandarad Seminar 2020/21
February 08, 2021

Our problem

Let $\Omega_T := \Omega \times (0,T)$ for a bounded open domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and $T > 0$. The signed porous medium equation (PME):

$$\partial_t u - \Delta\big(|u|^{m-1}u\big) = 0 \quad \text{in } \Omega_T.$$

- $m > 1$: degenerate case (slow diffusion)
  In this case, disturbances propagate with finite speed and solutions might vanish outside of a compact subset of the spatial domain.
- $m < 1$: singular case (fast diffusion)
  In this case, solutions exhibit infinite propagation speed and we may observe extinction in finite time.

Introduce a vector valued function $\mathbf{A}(x,t,u,\xi) : \Omega_T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which is measurable in (x,t) and continuous in (u,ξ). In addition, $\mathbf{A}$ satisfies the following ellipticity and growth conditions with some constants $0 < \nu \leq L < \infty$:

$$\begin{cases} \mathbf{A}(x,t,u,\xi) \cdot \xi \geq \nu |\xi|^2, \\ |\mathbf{A}(x,t,u,\xi)| \leq L|\xi|, \end{cases} \quad \text{for a.e. } (x,t) \in \Omega_T \text{ and all } (u,\xi) \in \mathbb{R} \times \mathbb{R}^n.$$

We can consider a generalization of the signed PME,

$$\partial_t u - \operatorname{div} \mathbf{A}(x,t,u,D\boldsymbol{u}^m) = 0 \quad \text{in } \Omega_T,$$

where $\boldsymbol{u}^m := |u|^{m-1}u$.

Our problem

Let $\Omega_T := \Omega \times (0, T)$ for a bounded open domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and $T > 0$. The signed porous medium equation (PME):

$$\partial_t u - \Delta\big(|u|^{m-1}u\big) = 0 \quad \text{in } \Omega_T.$$

- $m > 1$: degenerate case (slow diffusion)
  In this case, disturbances propagate with finite speed and solutions might vanish outside of a compact subset of the spatial domain.
- $m < 1$: singular case (fast diffusion)
  In this case, solutions exhibit infinite propagation speed and we may observe extinction in finite time.

Introduce a vector valued function $\mathbf{A}(x, t, u, \xi) : \Omega_T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which is measurable in (x, t) and continuous in (u, ξ). In addition, $\mathbf{A}$ satisfies the following ellipticity and growth conditions with some constants $0 < \nu \leq L < \infty$:

$$\begin{cases} \mathbf{A}(x,t,u,\xi) \cdot \xi \geq \nu |\xi|^2, \\ |\mathbf{A}(x,t,u,\xi)| \leq L|\xi|, \end{cases} \quad \text{for a.e. } (x,t) \in \Omega_T \text{ and all } (u,\xi) \in \mathbb{R} \times \mathbb{R}^n.$$

We can consider a generalization of the signed PME,

$$\partial_t u - \operatorname{div} \mathbf{A}(x, t, u, D\boldsymbol{u}^m) = 0 \quad \text{in } \Omega_T,$$

where $\boldsymbol{u}^m := |u|^{m-1}u$.

Our problem

We now consider the obstacle problem related to the equation

$$\partial_t u - \operatorname{div} \mathbf{A}(x,t,u,D\boldsymbol{u}^m) = g - \operatorname{div} F \quad \text{in } \Omega_T, \tag{1}$$

with an obstacle constraint given by the condition $u \geq \psi$ a.e. in Ω_T.

- We restrict our attention to the case $m > 1$.
- We consider inhomogeneities

$$F \in L^2(\Omega_T, \mathbb{R}^n) \qquad \text{and} \qquad g \in L^2(\Omega_T, \mathbb{R}),$$

  and an obstacle function $\psi : \Omega_T \to \mathbb{R}$ with

$$\boldsymbol{\psi}^m \in L^2(0,T;W^{1,2}(\Omega)) \qquad \text{and} \qquad \partial_t \boldsymbol{\psi}^m \in L^{\frac{2m}{2m-1}}(\Omega_T).$$

- We define the function classes

$$K_\psi := \big\{ w \in C^0\big([0,T]; L^{m+1}(\Omega)\big) : \boldsymbol{w}^m \in L^2\big(0,T;W^{1,2}(\Omega)\big),\ w \geq \psi \text{ a.e. in } \Omega_T \big\}$$

  and

$$K'_\psi := \big\{ w \in K_\psi : \partial_t \boldsymbol{w}^m \in L^{\frac{2m}{2m-1}}(\Omega_T) \big\}.$$

Definition

We say that a function $u \in K_\psi$ is a local weak solution of the obstacle problem related to the equation (1) if the variational inequality

$$\langle\!\langle \partial_t u, \alpha\eta(\boldsymbol{w}^m - \boldsymbol{u}^m)\rangle\!\rangle + \iint_{\Omega_T} \alpha \mathbf{A}(x,t,u,D\boldsymbol{u}^m) \cdot D\big(\eta(\boldsymbol{w}^m - \boldsymbol{u}^m)\big)\,\mathrm{d}x\mathrm{d}t \geq \iint_{\Omega_T} \alpha\big(F \cdot D\big(\eta(\boldsymbol{w}^m - \boldsymbol{u}^m)\big) + \eta g(\boldsymbol{w}^m - \boldsymbol{u}^m)\big)\,\mathrm{d}x\mathrm{d}t$$

holds true for all comparison maps $w \in K'_\psi$, any cut-off function $\alpha \in W_0^{1,\infty}([0,T], \mathbb{R}_{\geq 0})$ in time, and any cut-off function $\eta \in W_0^{1,\infty}(\Omega, \mathbb{R}_{\geq 0})$ in space. Here, the term containing the time derivative is defined by

$$\langle\!\langle \partial_t u, \alpha\eta(\boldsymbol{w}^m - \boldsymbol{u}^m)\rangle\!\rangle := \iint_{\Omega_T} \eta\big\{\alpha'\big(\tfrac{1}{m+1}|u|^{m+1} - u\boldsymbol{w}^m\big) - \alpha u \partial_t \boldsymbol{w}^m\big\}\,\mathrm{d}x\mathrm{d}t.$$

(Note: A corresponding existence result was showed by Bögelein, Lukkari, and Scheven in Math. Ann, 2015.)

Our goal

$|\partial_t \boldsymbol{\psi}^m|^{\frac{m}{2m-1}},\ |D\boldsymbol{\psi}^m|,\ |F|,\ |g| \in L^{2+\gamma}_{\mathrm{loc}}(\Omega_T)$ for some $\gamma > 0$

$\implies$ The gradient $D\boldsymbol{u}^m$ of a local weak solution is more integrable than assumed in the definition. More precisely, $|D\boldsymbol{u}^m| \in L^{2+\sigma_1}_{\mathrm{loc}}(\Omega_T)$ for some $\sigma_1 > 0$.

- Elliptic p-Laplace problems $(1 < p < \infty)$: Meyers and Elcrat (1975)
  Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution of the equation

  $$\mathrm{div}(|Du|^{p-2}Du) = 0 \quad \text{in } \Omega.$$

  $\Longrightarrow$ There exists $\varepsilon > 0$ such that

  $$\fint_{B_r} |Du|^{p+\varepsilon} \mathrm{d}x \le c \left(\fint_{B_{2r}} |Du|^p \mathrm{d}x \right)^{\frac{p+\varepsilon}{p}}$$

  for any ball $B_{2r} \subset \Omega$.

(Idea of the proof)

1. An energy estimate and Sobolev-Poincaré inequality:

$$\fint_{B_r} |Du|^p \mathrm{d}x \le \frac{c}{r^p} \fint_{B_{2r}} |u - (u)_{B_{2r}}|^p \mathrm{d}x$$
$$\le c\left(\fint_{B_{2r}} |Du|^{p_*} \mathrm{d}x\right)^{\frac{p}{p_*}}.$$

2. A reverse Hölder inequality:

$$\fint_{B_r} |Du|^p \mathrm{d}x \le c\left(\fint_{B_{2r}} |Du|^{p_*} \mathrm{d}x\right)^{\frac{p}{p_*}}$$

   for any ball $B_{2r} \subset \Omega$.

3. Gehring's lemma:
   There exists $\varepsilon > 0$ such that

$$\fint_{B_r} |Du|^{p+\varepsilon} \mathrm{d}x \le c\left(\fint_{B_{2r}} |Du|^p \mathrm{d}x\right)^{\frac{p+\varepsilon}{p}}.$$

Parabolic problems with p-Laplacian type

- Parabolic systems
  1. $p = 2$: Giaquinta and Struwe (Math. Z., 1982).
  2. $p > \frac{2n}{n+2}$: Kinnunen and Lewis (Duke Math. J., 2000).
     – an intrinsic scaling method
- Global higher integrability
  1. $p \geq 2$: Parviainen (Ann. Mat. Pura Appl., 2009).
  2. $p > \frac{2n}{n+2}$ Parviainen and Bögelein (NoDEA, 2010), Byun, Kim, and Lim (Forum Math., 2020).
- $p(x,t)$-Laplacian: Bögelein and Duzaar (Publ. Mat., 2011).
- Obstacle problems
  1. Bögelein and Scheven (Forum Math., 2012).
  2. $p(x,t)$-Laplacian: Erhardt (JMAA, 2016).

A brief history

Porous medium type equations/systems

- Gianazza and Schwarzacher:
  nonnegative solutions of porous medium type equations

$$\iint_{\Omega_T} u\,\partial_t\phi - m u^{m-1} Du \cdot D\phi \,\mathrm{d}x\mathrm{d}t = 0, \quad \text{for any } \phi \in C_0^\infty(\Omega_T)$$

1. $m > 1$ (Amer. J. Math., 2019):
   $\Longrightarrow \big|Du^{\frac{m+1}{2}}\big| \in L^{2+\varepsilon}_{\mathrm{loc}}(\Omega_T)$ for some $\varepsilon > 0$.
2. $\frac{(n-2)_+}{n+2} < m < 1$ (JFA, 2019):
   $\Longrightarrow \big|Du^m\big| \in L^{2+\varepsilon}_{\mathrm{loc}}(\Omega_T)$ for some $\varepsilon > 0$.
3. (Intrinsic scaling) They consider cylinders $Q_{\varrho,\theta\varrho^2} = B_\varrho \times (-\theta\varrho^2, \theta\varrho^2)$ with

$$⨌_{Q_{\varrho,\theta\varrho^2}} u^{m+1}\,\mathrm{d}x\mathrm{d}t \approx \theta^{\frac{m+1}{1-m}}.$$

Porous medium type equations/systems

- Signed porous medium type systems

$$\iint_{\Omega_T} u\,\partial_t\phi - D\big(|u|^{m-1}u\big)\cdot D\phi\,\mathrm{d}x\mathrm{d}t = 0, \quad \text{for any } \phi \in C_0^\infty(\Omega_T,\mathbb{R}^N).$$

$$\Longrightarrow \ \big|D\boldsymbol{u}^m\big| = \big|D\big(|u|^{m-1}u\big)\big| \in L^{2+\varepsilon}_{\mathrm{loc}}(\Omega_T) \text{ for some } \varepsilon > 0.$$

1. $m \geq 1$: Bögelein, Duzaar, Korte, and Scheven (Adv. Nonlinear Anal., 2019)
2. $\frac{(n-2)_+}{n+2} < m < 1$: Bögelein, Duzaar, and Scheven (J. Reine Angew. Math., 2020).
3. (Intrinsic scaling) They consider cylinders $Q_{r,s} = B_r \times (-s,s)$ such that

$$\frac{s}{r^{\frac{1+m}{m}}} = \theta^{1-m} \quad \text{with } \theta^m \text{ related to } \frac{|\boldsymbol{u}^m|}{r}.$$

- Global higher integrability $(m > 1)$: Moring, Scheven, Schwarzacher, and Singer (CPAA, 2020).

$$\partial_t u - \Delta(|u|^{m-1}u) = 0$$

From the modulus of ellipticity of the equation, we consider cylinders

$$B_\varrho \times (-\lambda\varrho^2, \lambda\varrho^2)$$

with $\lambda \sim |u|^{1-m}$.
However, we are going to prove an estimate for $D\boldsymbol{u}^m$. Now setting $\theta^m \sim \frac{|u|^m}{\varrho}$,

$$\theta^m \sim \frac{|u|^m}{\varrho} \sim \frac{\lambda^{\frac{m}{1-m}}}{\varrho}.$$

This leads to the cylinders

$$Q_\varrho^{(\theta)} := B_\varrho \times (-\theta^{1-m}\varrho^{\frac{m+1}{m}}, \theta^{1-m}\varrho^{\frac{m+1}{m}}).$$

$(m \geq 1)$ We call a cylinder $Q_\varrho^{(\theta)}$ *intrinsic* iff

$$⨌_{Q_\varrho^{(\theta)}} \frac{|u|^{2m}}{\varrho^2} \mathrm{d}x\mathrm{d}t \approx \theta^{2m}.$$

We work with cylinders of the following form:

$$Q_{\varrho}^{(\theta)}(z_o) := B_{\varrho}(x_o) \times \Lambda_{\varrho}^{(\theta)}(t_o),$$

for $z_o = (x_o, t_o) \in \mathbb{R}^n \times (0, T)$, where $B_{\varrho}(x_o)$ denotes the open ball with radius $\varrho > 0$ and center $x_o \in \mathbb{R}^n$ and

$$\Lambda_{\varrho}^{(\theta)}(t_o) := \big(t_o - \theta^{1-m}\varrho^{\frac{m+1}{m}}, t_o + \theta^{1-m}\varrho^{\frac{m+1}{m}}\big)$$

for some scaling factor $\theta > 0$. If $\theta = 1$, we use the following abbreviation:

$$Q_{\varrho}(z_o) := Q_{\varrho}^{(1)}(z_o) \quad \text{and} \quad \Lambda_{\varrho}(t_o) := \Lambda_{\varrho}^{(1)}(t_o).$$

We next define a boundary term

$$\mathfrak{b}[\boldsymbol{u}^m, \boldsymbol{a}^m] := \tfrac{m}{m+1}\big(|a|^{m+1} - |u|^{m+1}\big) - u\big(\boldsymbol{a}^m - \boldsymbol{u}^m\big),$$

for $u, a \in \mathbb{R}$.

Main theorem [C. and Scheven (NoDEA, 2019)]

There exists a constant $\sigma_o = \sigma_o(n, m, \nu, L) \in (0, 1]$ such that if

$$\Psi := |\partial_t \boldsymbol{\psi}^m|^{\frac{m}{2m-1}} + |D\boldsymbol{\psi}^m| + |F| + |g| \in L^{2+\gamma}_{\mathrm{loc}}(\Omega_T)$$

for some $\gamma > 0$, then we have

$$D\boldsymbol{u}^m \in L^{2+\sigma_1}_{\mathrm{loc}}\big(\Omega_T, \mathbb{R}^n\big),$$

where $\sigma_1 := \min\{\sigma_o, \gamma\}$. Furthermore, for any $\sigma \in (0, \sigma_1]$ and any cylinder $Q_{2R}(z_o) \Subset \Omega_T$ with $R \in (0, 1]$, the following quantitative local higher integrability estimate is satisfied:

$$\begin{aligned} &\fint\!\!\!\fint_{Q_R(z_o)} \big|D\boldsymbol{u}^m\big|^{2+\sigma} \,\mathrm{d}x\mathrm{d}t \\ &\qquad \le c\left[1 + \fint\!\!\!\fint_{Q_{2R}(z_o)} \left[\frac{|u|^{2m}}{R^2} + \Psi^2\right]\mathrm{d}x\mathrm{d}t\right]^{\frac{\sigma m}{m+1}} \fint\!\!\!\fint_{Q_{2R}(z_o)} \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t \\ &\qquad\quad + c \fint\!\!\!\fint_{Q_{2R}(z_o)} \Psi^{2+\sigma} \mathrm{d}x\mathrm{d}t, \end{aligned}$$

with a constant $c = c(n, m, \nu, L) \ge 1$, where we considered the parabolic cylinders $Q_R(z_o) := B_R(x_o) \times (t_o - R^{\frac{m+1}{m}}, t_o + R^{\frac{m+1}{m}})$.

Sketch of the proof

- Our strategy

1. an energy estimate
2. a Sobolev-Poincaré type inequality $\Leftarrow$ a gluing lemma
3. (1)+(2) $\Rightarrow$ a reverse Hölder type inequality
4. covering argument and the gradient estimate

1. An energy estimate
   There exists a constant $c = c(m, \nu, L) > 0$ such that on any cylinder $Q_\varrho^{(\theta)}(z_o) \Subset \Omega_T$ with $0 < \varrho \le 1$ and $\theta > 0$, the energy estimate

$$\sup_{t \in \Lambda_r^{(\theta)}(t_o)} \fint_{B_r(x_o)} \theta^{m-1} \frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t), \boldsymbol{a}^m\big]}{r^{\frac{m+1}{m}}} \mathrm{d}x + \iint_{Q_r^{(\theta)}(z_o)} \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t$$
$$\le c \iint_{Q_\varrho^{(\theta)}(z_o)} \Bigg[\frac{\big|\boldsymbol{u}^m - \boldsymbol{a}^m\big|^2}{(\varrho - r)^2} + \theta^{m-1} \frac{\mathfrak{b}\big[\boldsymbol{u}^m, \boldsymbol{a}^m\big]}{\varrho^{\frac{m+1}{m}} - r^{\frac{m+1}{m}}} \Bigg] \mathrm{d}x\mathrm{d}t$$
$$+ c \iint_{Q_\varrho^{(\theta)}(z_o)} |F|^2 + |g|^2 + |D\boldsymbol{\psi}^m|^2 + |\partial_t \boldsymbol{\psi}^m|^{\frac{2m}{2m-1}} \,\mathrm{d}x\mathrm{d}t$$

holds true for all $r \in [\varrho/2, \varrho)$ and all $a \in \mathbb{R}$.

(Test function: $\boldsymbol{w}^m := \max\{\boldsymbol{a}^m, \boldsymbol{\psi}^m\} = \boldsymbol{a}^m + (\boldsymbol{\psi}^m - \boldsymbol{a}^m)_+ \ge \boldsymbol{\psi}^m$ a.e.)

2. A gluing lemma: This compares the means at different time slices, and will be used for a Poincaré type inequality.

We consider a cylinder $Q_\varrho^{(\theta)}(z_o) \Subset \Omega_T$ with $0 < \varrho \leq 1$ and $\theta > 0$ and times $t_1, t_2 \in \Lambda_\varrho^{(\theta)}(t_o)$ with $t_1 < t_2$. Then, for a.e. $r \in [\frac{\varrho}{2}, \varrho]$, we have the estimates

$$
\begin{aligned}
&|(u)_{x_o;r}(t_1) - (u)_{x_o;r}(t_2)| \\
&\quad \leq \frac{c\varrho^{\frac{1}{m}}}{\theta^{m-1}} \fint_{\Lambda_\varrho^{(\theta)}(t_o)} \fint_{\partial B_r(x_o)} (|D\boldsymbol{u}^m| + |F|)\, \mathrm{d}\mathcal{H}^{n-1}\mathrm{d}t \\
&\quad + \frac{1}{\mu^m |B_r|} \int_{(B_r(x_o)\times\{t_1\})\cap\{\boldsymbol{u}^m \leq \boldsymbol{\psi}^m + \mu^m\}} \mathfrak{b}[\boldsymbol{u}^m, \boldsymbol{\psi}^m + \mu^m]\, \mathrm{d}x \\
&\quad + \frac{c\varrho^{\frac{m+1}{m}}}{\theta^{m-1}\mu^m |Q_\varrho^{(\theta)}|} \iint_{Q_\varrho^{(\theta)}(z_o)\cap\{\boldsymbol{u}^m \leq \boldsymbol{\psi}^m + \mu^m\}} \left(\mu|\partial_t \boldsymbol{\psi}^m| + |D\boldsymbol{\psi}^m|^2 + |F|^2\right) \mathrm{d}x\mathrm{d}t \\
&\quad + \frac{c\varrho^{\frac{m+1}{m}}}{\theta^{m-1}} \bbint_{Q_\varrho^{(\theta)}(z_o)} |g|\, \mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

where $c = c(n, m, \nu, L)$ and $\mu > 0$ is an arbitrary parameter.

3. A Sobolev-Poincaré type inequality on sub-intrinsic cylinders

Consider cylinders $Q_\varrho^{(\theta)}(z_o) \Subset \Omega_T$ with $0 < \varrho \le 1$ and $\theta > 0$ that are sub-intrinsic in the sense

$$\iint_{Q_\varrho^{(\theta)}(z_o)} \frac{|u|^{2m}}{\varrho^2} \mathrm{d}x\mathrm{d}t \le 2^{d+2}\theta^{2m} \quad \text{for } d = n + 1 + \tfrac{1}{m}. \tag{2}$$

Then, for any given $\varepsilon \in (0,1]$ we have the Sobolev-Poincaré type inequality

$$\begin{aligned} &\iint_{Q_\varrho^{(\theta)}(z_o)} \frac{\left|\boldsymbol{u}^m - (\boldsymbol{u}^m)^{(\theta)}_{z_o;\varrho}\right|^2}{\varrho^2} \mathrm{d}x\mathrm{d}t \\ &\quad \le \varepsilon \sup_{t \in \Lambda_\varrho^{(\theta)}(t_o)} \int_{B_\varrho(x_o)} \theta^{m-1} \frac{\mathfrak{b}\left[\boldsymbol{u}^m(\cdot,t), (\boldsymbol{u}^m)^{(\theta)}_{z_o;\varrho}\right]}{\varrho^{\frac{m+1}{m}}} \mathrm{d}x \\ &\qquad + \frac{c}{\varepsilon^{\frac{2}{n}}} \left[\iint_{Q_\varrho^{(\theta)}(z_o)} \left|D\boldsymbol{u}^m\right|^{2q_o} \mathrm{d}x\mathrm{d}t\right]^{\frac{1}{q_o}} + c \iint_{Q_\varrho^{(\theta)}(z_o)} \Psi^2 \mathrm{d}x\mathrm{d}t, \end{aligned}$$

where c is a universal constant depending only on n, m, ν, and L, and q_o is defined by $q_o := \max\{\frac{m-1}{m}, \frac{1}{2}, \frac{n}{d}\} < 1$.

Proof of a Sobolev-Poincaré type inequality

(1) Applying the gluing lemma, the sub-intrinsic property (2),Poincaré's inequality, and Mean value's theorem, there exists $\hat{\varrho} \in [\frac{\varrho}{2}, \varrho]$ such that

$$\frac{1}{\varrho^2} \fint_{\Lambda_\varrho^{(\theta)}(t_o)} \fint_{\Lambda_\varrho^{(\theta)}(t_o)} \left| (\boldsymbol{u})^m_{\boldsymbol{x_o};\hat{\boldsymbol{\varrho}}}(t) - (\boldsymbol{u})^m_{\boldsymbol{x_o};\hat{\boldsymbol{\varrho}}}(\tau) \right|^2 \mathrm{d}t\mathrm{d}\tau$$

$$\leq c \left(\fint\!\!\!\!\fint_{Q_\varrho^{(\theta)}(z_o)} |D\boldsymbol{u}^m|^{2q} \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{q}} + c \fint\!\!\!\!\fint_{Q_\varrho^{(\theta)}(z_o)} \Psi^2 \mathrm{d}x\mathrm{d}t,$$

for $q := \max\{\frac{m-1}{m}, \frac{1}{2}\} < 1$ and a constant $c = c(n, m, \nu, L)$.

(2) We then add and subtract the slice-wise means $(\boldsymbol{u})^m_{\boldsymbol{x_o};\hat{\boldsymbol{\varrho}}}(t)$, to obtain

$$\fint\!\!\!\!\fint_{Q_\varrho^{(\theta)}(z_o)} \frac{\left|\boldsymbol{u}^m - (\boldsymbol{u}^m)^{(\theta)}_{z_o;\varrho}\right|^2}{\varrho^2} \mathrm{d}x\mathrm{d}t$$

$$\leq 3 \Bigg[\fint\!\!\!\!\fint_{Q_\varrho^{(\theta)}(z_o)} \frac{\left|\boldsymbol{u}^m - (\boldsymbol{u})^m_{\boldsymbol{x_o};\hat{\boldsymbol{\varrho}}}(t)\right|^2}{\varrho^2} \mathrm{d}x\mathrm{d}t$$

$$+ \frac{1}{\varrho^2} \fint_{\Lambda_\varrho^{(\theta)}(t_o)} \left| \fint_{\Lambda_\varrho^{(\theta)}(t_o)} \left[(\boldsymbol{u})^m_{\boldsymbol{x_o};\hat{\boldsymbol{\varrho}}}(t) - (\boldsymbol{u})^m_{\boldsymbol{x_o};\hat{\boldsymbol{\varrho}}}(\tau) \right] \mathrm{d}\tau \right|^2 \mathrm{d}t$$

$$+ \frac{1}{\varrho^2} \left| \fint_{\Lambda_\varrho^{(\theta)}(t_o)} (\boldsymbol{u})^m_{\boldsymbol{x_o};\hat{\boldsymbol{\varrho}}}(\tau)\mathrm{d}\tau - (\boldsymbol{u}^m)^{(\theta)}_{z_o;\varrho} \right|^2 \Bigg] =: 3\left[\mathrm{I} + \mathrm{II} + \mathrm{III}\right].$$

Proof of a Sobolev-Poincaré type inequality

(1) Applying the gluing lemma, the sub-intrinsic property (2),Poincaré's inequality, and Mean value's theorem, there exists $\hat\varrho \in [\frac{\varrho}{2}, \varrho]$ such that

$$\frac{1}{\varrho^2} \fint_{\Lambda_\varrho^{(\theta)}(t_o)} \fint_{\Lambda_\varrho^{(\theta)}(t_o)} \left| (\boldsymbol{u})^m_{\boldsymbol{x_o};\hat{\boldsymbol{\varrho}}}(t) - (\boldsymbol{u})^m_{\boldsymbol{x_o};\hat{\boldsymbol{\varrho}}}(\tau) \right|^2 \mathrm{d}t\mathrm{d}\tau$$

$$\leq c \left(\fint\!\!\fint_{Q_\varrho^{(\theta)}(z_o)} |D\boldsymbol{u}^m|^{2q} \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{q}} + c \fint\!\!\fint_{Q_\varrho^{(\theta)}(z_o)} \Psi^2 \mathrm{d}x\mathrm{d}t,$$

for $q := \max\{\frac{m-1}{m}, \frac{1}{2}\} < 1$ and a constant $c = c(n, m, \nu, L)$.

(2) We then add and subtract the slice-wise means $(\boldsymbol{u})^m_{\boldsymbol{x_o};\hat{\boldsymbol{\varrho}}}(t)$, to obtain

$$\fint\!\!\fint_{Q_\varrho^{(\theta)}(z_o)} \frac{\left|\boldsymbol{u}^m - (\boldsymbol{u}^m)^{(\theta)}_{z_o;\varrho}\right|^2}{\varrho^2} \mathrm{d}x\mathrm{d}t$$

$$\leq 3 \Bigg[\fint\!\!\fint_{Q_\varrho^{(\theta)}(z_o)} \frac{\left|\boldsymbol{u}^m - (\boldsymbol{u})^m_{\boldsymbol{x_o};\hat{\boldsymbol{\varrho}}}(t)\right|^2}{\varrho^2} \mathrm{d}x\mathrm{d}t$$

$$+ \frac{1}{\varrho^2} \fint_{\Lambda_\varrho^{(\theta)}(t_o)} \left| \fint_{\Lambda_\varrho^{(\theta)}(t_o)} \left[(\boldsymbol{u})^m_{\boldsymbol{x_o};\hat{\boldsymbol{\varrho}}}(t) - (\boldsymbol{u})^m_{\boldsymbol{x_o};\hat{\boldsymbol{\varrho}}}(\tau) \right] \mathrm{d}\tau \right|^2 \mathrm{d}t$$

$$+ \frac{1}{\varrho^2} \left| \fint_{\Lambda_\varrho^{(\theta)}(t_o)} (\boldsymbol{u})^m_{\boldsymbol{x_o};\hat{\boldsymbol{\varrho}}}(\tau) \mathrm{d}\tau - (\boldsymbol{u}^m)^{(\theta)}_{z_o;\varrho} \right|^2 \Bigg] =: 3[\mathrm{I} + \mathrm{II} + \mathrm{III}].$$

Proof of a Sobolev-Poincaré type inequality

(3) We get

$$\mathrm{II} \leq c\left(\iint_{Q_\varrho^{(\theta)}(z_o)} |D\boldsymbol{u}^m|^{2q} \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{q}} + c\iint_{Q_\varrho^{(\theta)}(z_o)} \Psi^2 \mathrm{d}x\mathrm{d}t,$$

and

$$\mathrm{III} \leq \mathrm{I} \leq c\iint_{Q_\varrho^{(\theta)}(z_o)} \frac{\left|\boldsymbol{u}^m - (\boldsymbol{u}^m)_{x_o;\varrho}(t)\right|^2}{\varrho^2} \mathrm{d}x\mathrm{d}t.$$

Finally, we have the inequality

$$\begin{aligned}
&\iint_{Q_\varrho^{(\theta)}(z_o)} \frac{\left|\boldsymbol{u}^m - (\boldsymbol{u}^m)^{(\theta)}_{z_o;\varrho}\right|^2}{\varrho^2} \mathrm{d}x\mathrm{d}t \\
&\quad \leq c\iint_{Q_\varrho^{(\theta)}(z_o)} \frac{\left|\boldsymbol{u}^m - (\boldsymbol{u}^m)_{x_o;\varrho}(t)\right|^2}{\varrho^2} \mathrm{d}x\mathrm{d}t + c\left(\iint_{Q_\varrho^{(\theta)}(z_o)} |D\boldsymbol{u}^m|^{2q} \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{q}} \\
&\qquad + c\iint_{Q_\varrho^{(\theta)}(z_o)} \Psi^2 \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Proof of a Sobolev-Poincaré type inequality

(4) Using the properties for $\mathfrak{b}[\cdot,\cdot]$ and the sub-intrinsic coupling (2), we infer

$$\begin{aligned}
&\iint_{Q_\varrho^{(\theta)}(z_o)} \frac{\left|\boldsymbol{u}^m - (\boldsymbol{u}^m)_{x_o;\varrho}(t)\right|^2}{\varrho^2}\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{\varrho^2}\iint_{Q_\varrho^{(\theta)}(z_o)} \left|\boldsymbol{u}^m - (\boldsymbol{u}^m)_{x_o;\varrho}(t)\right|^{\frac{4}{n+2}}\left|\boldsymbol{u}^m - (\boldsymbol{u}^m)_{x_o;\varrho}(t)\right|^{\frac{2n}{n+2}}\mathrm{d}x\mathrm{d}t \\
&\le c\Bigg\{\int_{\Lambda_\varrho^{(\theta)}(t_o)} \left[\int_{B_\varrho(x_o)} \theta^{m-1}\frac{\mathfrak{b}\left[\boldsymbol{u}^m,(\boldsymbol{u}^m)_{x_o;\varrho}(t)\right]}{\varrho^{\frac{m+1}{m}}}\mathrm{d}x\right]^{\frac{2}{d}} \\
&\qquad\cdot\left[\int_{B_\varrho(x_o)} \frac{\left|\boldsymbol{u}^m - (\boldsymbol{u}^m)_{x_o;\varrho}(t)\right|^{\frac{2n}{d-2}}}{\varrho^{\frac{2n}{d-2}}}\mathrm{d}x\right]^{\frac{d-2}{d}}\mathrm{d}t\Bigg\}^{\frac{d}{n+2}}.
\end{aligned}$$

From Sobolev's inequality slicewise for a.e. $t\in\Lambda_\varrho^{(\theta)}(t_o)$,

$$\begin{aligned}
\iint_{Q_\varrho^{(\theta)}(z_o)} \frac{\left|\boldsymbol{u}^m - (\boldsymbol{u}^m)_{x_o;\varrho}(t)\right|^2}{\varrho^2}\mathrm{d}x\mathrm{d}t &\le c\left[\iint_{Q_\varrho^{(\theta)}(z_o)} \left|D\boldsymbol{u}^m\right|^{\frac{2n}{d}}\mathrm{d}x\mathrm{d}t\right]^{\frac{d}{n+2}} \\
&\qquad\cdot\sup_{t\in\Lambda_\varrho^{(\theta)}(t_o)}\left[\int_{B_\varrho(x_o)} \theta^{m-1}\frac{\mathfrak{b}\left[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)^{(\theta)}_{z_o;\varrho}\right]}{\varrho^{\frac{m+1}{m}}}\mathrm{d}x\right]^{\frac{2}{n+2}}.
\end{aligned}$$

Proof of a Sobolev-Poincaré type inequality

(5) Combining with the estimate

$$\iint_{Q_\varrho^{(\theta)}(z_o)} \frac{\left|\boldsymbol{u}^m - (\boldsymbol{u}^m)^{(\theta)}_{z_o;\varrho}\right|^2}{\varrho^2}\,\mathrm{d}x\mathrm{d}t$$

$$\leq c\iint_{Q_\varrho^{(\theta)}(z_o)} \frac{\left|\boldsymbol{u}^m - (\boldsymbol{u}^m)_{x_o;\varrho}(t)\right|^2}{\varrho^2}\mathrm{d}x\mathrm{d}t + c\Big(\iint_{Q_\varrho^{(\theta)}(z_o)} |D\boldsymbol{u}^m|^{2q}\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{q}}$$

$$+ c\iint_{Q_\varrho^{(\theta)}(z_o)} \Psi^2\mathrm{d}x\mathrm{d}t$$

and applying Young's and Hölder's inequality, this deduces

$$\iint_{Q_\varrho^{(\theta)}(z_o)} \frac{\left|\boldsymbol{u}^m - (\boldsymbol{u}^m)^{(\theta)}_{z_o;\varrho}\right|^2}{\varrho^2}\mathrm{d}x\mathrm{d}t$$

$$\leq \varepsilon \sup_{t\in\Lambda_\varrho^{(\theta)}(t_o)} \int_{B_\varrho(x_o)} \theta^{m-1}\frac{\mathfrak{b}\left[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)^{(\theta)}_{z_o;\varrho}\right]}{\varrho^{\frac{m+1}{m}}}\mathrm{d}x$$

$$+ \frac{c}{\varepsilon^{\frac{2}{n}}}\left[\iint_{Q_\varrho^{(\theta)}(z_o)} \left|D\boldsymbol{u}^m\right|^{2q_o}\mathrm{d}x\mathrm{d}t\right]^{\frac{1}{q_o}} + c\iint_{Q_\varrho^{(\theta)}(z_o)} \Psi^2\mathrm{d}x\mathrm{d}t$$

for any $\varepsilon \in (0,1]$ and for $q_o := \max\{q, \frac{n}{d}\} = \max\{\frac{m-1}{m}, \frac{1}{2}, \frac{n}{d}\} < 1$. □

4. A reverse Hölder type inequality

Let $Q_{2\varrho}^{(\theta)}(z_o) \Subset \Omega_T$ for $0 < \varrho \le \frac{1}{2}$ and $\theta > 0$. Whenever the cylinder $Q_{2\varrho}^{(\theta)}(z_o)$ fulfills the following couplings:

either

$$\iint_{Q_{2\varrho}^{(\theta)}(z_o)} \frac{|u|^{2m}}{(2\varrho)^2} \mathrm{d}x\mathrm{d}t \le \theta^{2m} \le \iint_{Q_{\varrho}^{(\theta)}(z_o)} \frac{|u|^{2m}}{\varrho^2} \mathrm{d}x\mathrm{d}t \tag{3}$$

or

$$\iint_{Q_{2\varrho}^{(\theta)}(z_o)} \frac{|u|^{2m}}{(2\varrho)^2} \mathrm{d}x\mathrm{d}t \le \theta^{2m} \le K \iint_{Q_{\varrho}^{(\theta)}(z_o)} \Big[\big| D\boldsymbol{u}^m \big|^2 + \Psi^2 \Big] \mathrm{d}x\mathrm{d}t \tag{4}$$

for some constant $K \ge 1$,

we have the reverse Hölder type inequality

$$\iint_{Q_{\varrho}^{(\theta)}(z_o)} \big| D\boldsymbol{u}^m \big|^2 \mathrm{d}x\mathrm{d}t \le c \Big[\iint_{Q_{2\varrho}^{(\theta)}(z_o)} \big| D\boldsymbol{u}^m \big|^{2q_o} \mathrm{d}x\mathrm{d}t \Big]^{\frac{1}{q_o}} + c \iint_{Q_{2\varrho}^{(\theta)}(z_o)} \Psi^2 \mathrm{d}x\mathrm{d}t,$$

for some constant $c = c(n, m, \nu, L, (K)) > 0$ and $q_o := \max\{\frac{m-1}{m}, \frac{1}{2}, \frac{n}{d}\} < 1$.

Proof of a reverse Hölder inequality

For radii r, s with $\varrho \le r < s \le 2\varrho$,

- the energy estimate:

$$\sup_{t\in\Lambda_r^{(\theta)}(t_o)} \fint_{B_r(x_o)} \theta^{m-1} \frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t), (\boldsymbol{u}^m)^{(\theta)}_{z_o;r}\big]}{r^{\frac{m+1}{m}}} \mathrm{d}x + \iint_{Q_r^{(\theta)}(z_o)} \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t$$
$$\le c \iint_{Q_s^{(\theta)}(z_o)} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)^{(\theta)}_{z_o;r}\big|^2}{(s-r)^2} \mathrm{d}x\mathrm{d}t + c \iint_{Q_s^{(\theta)}(z_o)} \theta^{m-1} \frac{\mathfrak{b}\big[\boldsymbol{u}^m, (\boldsymbol{u}^m)^{(\theta)}_{z_o;r}\big]}{s^{\frac{m+1}{m}} - r^{\frac{m+1}{m}}} \mathrm{d}x\mathrm{d}t$$
$$+ c \iint_{Q_s^{(\theta)}(z_o)} \Psi^2 \,\mathrm{d}x\mathrm{d}t$$
$$=: \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

- We use the notation

$$\mathcal{R}_{r,s} := \frac{s^{\frac{m+1}{2m}}}{s^{\frac{m+1}{2m}} - r^{\frac{m+1}{2m}}},$$

- I is estimated by

$$\mathrm{I} \le c\,\mathcal{R}_{r,s}^{\frac{4m}{m+1}} \iint_{Q_s^{(\theta)}(z_o)} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)^{(\theta)}_{z_o;s}\big|^2}{s^2} \mathrm{d}x\mathrm{d}t.$$

Proof of a reverse Hölder inequality

For radii r, s with $\varrho \le r < s \le 2\varrho$,

- the energy estimate:

$$\sup_{t\in\Lambda_r^{(\theta)}(t_o)} \fint_{B_r(x_o)} \theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)^{(\theta)}_{z_o;r}\big]}{r^{\frac{m+1}{m}}}\mathrm{d}x + \iint_{Q_r^{(\theta)}(z_o)} \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t$$
$$\le c \iint_{Q_s^{(\theta)}(z_o)} \frac{\big|\boldsymbol{u}^m-(\boldsymbol{u}^m)^{(\theta)}_{z_o;r}\big|^2}{(s-r)^2}\mathrm{d}x\mathrm{d}t + c\iint_{Q_s^{(\theta)}(z_o)} \theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m,(\boldsymbol{u}^m)^{(\theta)}_{z_o;r}\big]}{s^{\frac{m+1}{m}}-r^{\frac{m+1}{m}}}\mathrm{d}x\mathrm{d}t$$
$$+ c\iint_{Q_s^{(\theta)}(z_o)} \Psi^2\,\mathrm{d}x\mathrm{d}t$$
$$=: \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

- We use the notation

$$\mathcal{R}_{r,s} := \frac{s^{\frac{m+1}{2m}}}{s^{\frac{m+1}{2m}} - r^{\frac{m+1}{2m}}},$$

- I is estimated by

$$\mathrm{I} \le c\,\mathcal{R}_{r,s}^{\frac{4m}{m+1}} \iint_{Q_s^{(\theta)}(z_o)} \frac{\big|\boldsymbol{u}^m-(\boldsymbol{u}^m)^{(\theta)}_{z_o;s}\big|^2}{s^2}\mathrm{d}x\mathrm{d}t.$$

Proof of a reverse Hölder inequality

- For the second term II,

(1) $\theta^{2m} \le \iint_{Q_\varrho^{(\theta)}(z_o)} \frac{|u|^{2m}}{\varrho^2} \mathrm{d}x\mathrm{d}t$ –(3)$_2$:

$$\mathrm{II} = c \iint_{Q_s^{(\theta)}(z_o)} \theta^{m-1} \frac{\mathfrak{b}\big[\boldsymbol{u}^m, (\boldsymbol{u}^m)_{z_o;r}^{(\theta)}\big]}{s^{\frac{m+1}{m}} - r^{\frac{m+1}{m}}} \mathrm{d}x\mathrm{d}t$$

$$\le c\mathcal{R}_{r,s}^2 \iint_{Q_s^{(\theta)}(z_o)} \theta^{m-1} \frac{\mathfrak{b}\big[\boldsymbol{u}^m, (\boldsymbol{u}^m)_{z_o;r}^{(\theta)}\big]}{s^{\frac{m+1}{m}}} \mathrm{d}x\mathrm{d}t$$

$$\le c\mathcal{R}_{r,s}^2 \left[\iint_{Q_s^{(\theta)}(z_o)} \frac{|\boldsymbol{u}^m|^2}{s^2} \mathrm{d}x\mathrm{d}t\right]^{\frac{m-1}{2m}} \iint_{Q_s^{(\theta)}(z_o)} \frac{\mathfrak{b}\big[\boldsymbol{u}^m, (\boldsymbol{u}^m)_{z_o;r}^{(\theta)}\big]}{s^{\frac{m+1}{m}}} \mathrm{d}x\mathrm{d}t$$

$$\le c\mathcal{R}_{r,s}^2 \iint_{Q_s^{(\theta)}(z_o)} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)_{z_o;r}^{(\theta)}\big|^2}{s^2} \mathrm{d}x\mathrm{d}t$$

$$\le c\mathcal{R}_{r,s}^2 \iint_{Q_s^{(\theta)}(z_o)} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)_{z_o;s}^{(\theta)}\big|^2}{s^2} \mathrm{d}x\mathrm{d}t.$$

Proof of a reverse Hölder inequality

- For the second term II,

  (2) $\theta^{2m} \le K \iint_{Q_\varrho^{(\theta)}(z_o)} \Big[\big|D\boldsymbol{u}^m\big|^2 + \Psi^2\Big] \mathrm{d}x\mathrm{d}t$ –(4)$_2$:

  We first use Young's inequality, to infer for any $\tau \in (0,1]$

$$
\begin{aligned}
\mathrm{II} &\le \mathcal{R}_{r,s}^2 \iint_{Q_s^{(\theta)}(z_o)} \theta^{m-1} \frac{\mathfrak{b}\big[\boldsymbol{u}^m, (\boldsymbol{u}^m)_{z_o;r}^{(\theta)}\big]}{s^{\frac{m+1}{m}}} \mathrm{d}x\mathrm{d}t \\
&\le \tau\theta^{2m} + \frac{\mathcal{R}_{r,s}^{\frac{4m}{m+1}}}{\tau^{\frac{m-1}{m+1}}} \iint_{Q_s^{(\theta)}(z_o)} \frac{\mathfrak{b}\big[\boldsymbol{u}^m, (\boldsymbol{u}^m)_{z_o;r}^{(\theta)}\big]^{\frac{2m}{m+1}}}{s^2} \mathrm{d}x\mathrm{d}t \\
&\le \tau\theta^{2m} + \frac{c\,\mathcal{R}_{r,s}^{\frac{4m}{m+1}}}{\tau^{\frac{m-1}{m+1}}} \iint_{Q_s^{(\theta)}(z_o)} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)_{z_o;r}^{(\theta)}\big|^2}{s^2} \mathrm{d}x\mathrm{d}t \\
&\le \tau\theta^{2m} + \frac{c\,\mathcal{R}_{r,s}^{\frac{4m}{m+1}}}{\tau^{\frac{m-1}{m+1}}} \iint_{Q_s^{(\theta)}(z_o)} \frac{\big|\boldsymbol{u}^m - (\boldsymbol{u}^m)_{z_o;s}^{(\theta)}\big|^2}{s^2} \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

  Meanwhile, the coupling (4)$_2$ and $\varrho < r < 2\varrho$ lead to

$$
\theta^{2m} \le 2^d K \iint_{Q_r^{(\theta)}(z_o)} \big[|D\boldsymbol{u}^m|^2 + \Psi^2\big] \mathrm{d}x\mathrm{d}t.
$$

Proof of a reverse Hölder inequality

- We obtain for both cases

$$\sup_{t\in\Lambda_r^{(\theta)}(t_o)} \fint_{B_r(x_o)} \theta^{m-1} \frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)^{(\theta)}_{z_o;r}\big]}{r^{\frac{m+1}{m}}}\mathrm{d}x + \iint_{Q_r^{(\theta)}(z_o)} \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t$$
$$\le c\,\mathcal{R}_{r,s}^{\frac{4m}{m+1}} \iint_{Q_s^{(\theta)}(z_o)} \frac{\big|\boldsymbol{u}^m-(\boldsymbol{u}^m)^{(\theta)}_{z_o;s}\big|^2}{s^2}\mathrm{d}x\mathrm{d}t + c\iint_{Q_s^{(\theta)}(z_o)} \Psi^2\,\mathrm{d}x\mathrm{d}t.$$

- Observe that $Q_s^{(\theta)}(z_o)$ is sub-intrinsic, and use the Sobolev-Poincaré type inequality:

$$\iint_{Q_s^{(\theta)}(z_o)} \frac{\big|\boldsymbol{u}^m-(\boldsymbol{u}^m)^{(\theta)}_{z_o;s}\big|^2}{s^2}\mathrm{d}x\mathrm{d}t$$
$$\le \varepsilon \sup_{t\in\Lambda_s^{(\theta)}(t_o)} \fint_{B_s(x_o)} \theta^{m-1} \frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)^{(\theta)}_{z_o;s}\big]}{s^{\frac{m+1}{m}}}\mathrm{d}x$$
$$+\frac{c}{\varepsilon^{\frac{2}{n}}}\left[\iint_{Q_s^{(\theta)}(z_o)} \big|D\boldsymbol{u}^m\big|^{2q_o}\mathrm{d}x\mathrm{d}t\right]^{\frac{1}{q_o}} + c\iint_{Q_s^{(\theta)}(z_o)} \Psi^2\mathrm{d}x\mathrm{d}t.$$

Proof of a reverse Hölder inequality

- Combining the previous estimates and choosing a suitable ε,

$$\sup_{t\in\Lambda_r^{(\theta)}(t_o)} \fint_{B_r(x_o)} \theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)_r^{(\theta)}\big]}{r^{\frac{m+1}{m}}}\mathrm{d}x + \iint_{Q_r^{(\theta)}(z_o)} \big|D\boldsymbol{u}^m\big|^2 \mathrm{d}x\mathrm{d}t$$

$$\le \tfrac{1}{2}\sup_{t\in\Lambda_s^{(\theta)}(t_o)} \fint_{B_s(x_o)} \theta^{m-1}\frac{\mathfrak{b}\big[\boldsymbol{u}^m(\cdot,t),(\boldsymbol{u}^m)_s^{(\theta)}\big]}{s^{\frac{m+1}{m}}}\mathrm{d}x$$

$$+ c\,\mathcal{R}_{r,s}^{\frac{4m(n+2)}{n(m+1)}}\left[\iint_{Q_{2\varrho}^{(\theta)}(z_o)} \big|D\boldsymbol{u}^m\big|^{2q_o}\mathrm{d}x\mathrm{d}t\right]^{\frac{1}{q_o}}$$

$$+ c\,\mathcal{R}_{r,s}^{\frac{4m}{m+1}}\iint_{Q_{2\varrho}^{(\theta)}(z_o)} \Psi^2\mathrm{d}x\mathrm{d}t.$$

- We finally apply an iteration lemma to get the result. □

5. Covering argument and the gradient estimate

- Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.

– We have the reverse Hölder type inequality on the cylinder $Q_\varrho^{(\theta)}(z_o)$, whenever the cylinder $Q_\varrho^{(\theta)}(z_o)$ fulfills the following couplings:
either

$$\iint_{Q_{2\varrho}^{(\theta)}(z_o)} \frac{|u|^{2m}}{(2\varrho)^2} \mathrm{d}x\mathrm{d}t \le \theta^{2m} \le \iint_{Q_\varrho^{(\theta)}(z_o)} \frac{|u|^{2m}}{\varrho^2} \mathrm{d}x\mathrm{d}t$$

or

$$\iint_{Q_{2\varrho}^{(\theta)}(z_o)} \frac{|u|^{2m}}{(2\varrho)^2} \mathrm{d}x\mathrm{d}t \le \theta^{2m} \le K \iint_{Q_\varrho^{(\theta)}(z_o)} \Big[\big|D\boldsymbol{u}^m\big|^2 + \Psi^2\Big] \mathrm{d}x\mathrm{d}t$$

for some constant $K \ge 1$.

5. Covering argument and the gradient estimate

- Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.

– We have the reverse Hölder type inequality on the cylinder $Q_\varrho^{(\theta)}(z_o)$, whenever the cylinder $Q_\varrho^{(\theta)}(z_o)$ fulfills the following couplings:
either

$$\iint_{Q_{2\varrho}^{(\theta)}(z_o)} \frac{|u|^{2m}}{(2\varrho)^2} \mathrm{d}x\mathrm{d}t \le \theta^{2m} \le \iint_{Q_{\varrho}^{(\theta)}(z_o)} \frac{|u|^{2m}}{\varrho^2} \mathrm{d}x\mathrm{d}t$$

or

$$\iint_{Q_{2\varrho}^{(\theta)}(z_o)} \frac{|u|^{2m}}{(2\varrho)^2} \mathrm{d}x\mathrm{d}t \le \theta^{2m} \le K \iint_{Q_{\varrho}^{(\theta)}(z_o)} \left[\left|D\boldsymbol{u}^m\right|^2 + \Psi^2\right] \mathrm{d}x\mathrm{d}t$$

for some constant $K \ge 1$.

5. Covering argument and the gradient estimate

- Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.
- For a given center z and radius ϱ, we select $\tilde\theta_{z;\varrho}$ with

$$\iint_{Q_\varrho^{(\tilde\theta_{z;\varrho})}(z)} \frac{|u|^{2m}}{\varrho^2}\,\mathrm{d}x\mathrm{d}t = \tilde\theta_{z;\varrho}^{2m}.$$

However, the mapping $\varrho \mapsto \tilde\theta_{z;\varrho}$ might not be monotone, and so we introduce the modification

$$\theta_{z;\varrho} := \max_{r\geq\varrho} \tilde\theta_{z;r}.$$

Then the mapping $\varrho \mapsto \theta_{z;\varrho}$ has the following properties:

1. monotonically decreasing (i.e., $Q_\varrho^{(\theta_{z;\varrho})}(z) \subset Q_s^{(\theta_{z;s})}(z)$ if $\varrho < s$).
2. $Q_r^{(\theta_{z;\varrho})}(z)$ are sub-intrinsic for all $r \geq \varrho$.
3. $\exists \tilde\varrho \geq \varrho$ s.t. $Q_{\tilde\varrho}^{(\theta_{z;\varrho})}(z)$ is intrinsic.

Now, we choose ϱ_z so that either

$$\theta_{z;\varrho_z}^{2m} \leq c \iint_{Q_{\varrho_z}^{(\theta_{z;\varrho_z})}(z)} \Big[\big|Du^m\big|^2 + \Psi^2\Big]\mathrm{d}x\mathrm{d}t$$

or

$$Q_{\varrho_z}^{(\theta_{z;\varrho_z})}(z) \subset Q_{\tilde\varrho_z}^{(\theta_{z;\varrho_z})}(z) \subset Q_{2\tilde\varrho_z}^{(\theta_{z;\varrho_z})}(z) \subset Q_{4\varrho_z}^{(\theta_{z;\varrho_z})}(z).$$

5. Covering argument and the gradient estimate

- Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.
  - For a given center z and radius ϱ, we select $\tilde{\theta}_{z;\varrho}$ with

$$\iint_{Q_\varrho^{(\tilde{\theta}_{z;\varrho})}(z)} \frac{|u|^{2m}}{\varrho^2} \mathrm{d}x\mathrm{d}t = \tilde{\theta}_{z;\varrho}^{2m}.$$

However, the mapping $\varrho \mapsto \tilde{\theta}_{z;\varrho}$ might not be monotone, and so we introduce the modification

$$\theta_{z;\varrho} := \max_{r \geq \varrho} \tilde{\theta}_{z;r}.$$

Then the mapping $\varrho \mapsto \theta_{z;\varrho}$ has the following properties:

1. monotonically decreasing (i.e., $Q_\varrho^{(\theta_{z;\varrho})}(z) \subset Q_s^{(\theta_{z;s})}(z)$ if $\varrho < s$).
2. $Q_r^{(\theta_{z;\varrho})}(z)$ are sub-intrinsic for all $r \geq \varrho$.
3. $\exists \tilde{\varrho} \geq \varrho$ s.t. $Q_{\tilde{\varrho}}^{(\theta_{z;\varrho})}(z)$ is intrinsic.

Now, we choose ϱ_z so that either

$$\theta_{z;\varrho_z}^{2m} \leq c \iint_{Q_{\varrho_z}^{(\theta_{z;\varrho_z})}(z)} \left[\left| Du^m \right|^2 + \Psi^2 \right] \mathrm{d}x\mathrm{d}t$$

or

$$Q_{\varrho_z}^{(\theta_{z;\varrho_z})}(z) \subset Q_{\tilde{\varrho}_z}^{(\theta_{z;\varrho_z})}(z) \subset Q_{2\tilde{\varrho}_z}^{(\theta_{z;\varrho_z})}(z) \subset Q_{4\varrho_z}^{(\theta_{z;\varrho_z})}(z).$$

5. Covering argument and the gradient estimate

- Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.
- – For a given center z and radius ϱ, we select $\tilde{\theta}_{z;\varrho}$ with

$$\iint_{Q_\varrho^{(\tilde{\theta}_{z;\varrho})}(z)} \frac{|u|^{2m}}{\varrho^2} \mathrm{d}x\mathrm{d}t = \tilde{\theta}_{z;\varrho}^{2m}.$$

However, the mapping $\varrho \mapsto \tilde{\theta}_{z;\varrho}$ might not be monotone, and so we introduce the modification

$$\theta_{z;\varrho} := \max_{r \geq \varrho} \tilde{\theta}_{z;r}.$$

Then the mapping $\varrho \mapsto \theta_{z;\varrho}$ has the following properties:

1. monotonically decreasing (i.e., $Q_\varrho^{(\theta_{z;\varrho})}(z) \subset Q_s^{(\theta_{z;s})}(z)$ if $\varrho < s$).
2. $Q_r^{(\theta_{z;\varrho})}(z)$ are sub-intrinsic for all $r \geq \varrho$.
3. $\exists \tilde{\varrho} \geq \varrho$ s.t. $Q_{\tilde{\varrho}}^{(\theta_{z;\varrho})}(z)$ is intrinsic.

Now, we choose ϱ_z so that either

$$\theta_{z;\varrho_z}^{2m} \leq c \iint_{Q_{\varrho_z}^{(\theta_{z;\varrho_z})}(z)} \left[\left|Du^m\right|^2 + \Psi^2 \right] \mathrm{d}x\mathrm{d}t$$

or

$$Q_{\varrho_z}^{(\theta_{z;\varrho_z})}(z) \subset Q_{\tilde{\varrho}_z}^{(\theta_{z;\varrho_z})}(z) \subset Q_{2\tilde{\varrho}_z}^{(\theta_{z;\varrho_z})}(z) \subset Q_{4\varrho_z}^{(\theta_{z;\varrho_z})}(z).$$

5. Covering argument and the gradient estimate

- Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.

– For a given center z and radius ϱ, we select $\tilde{\theta}_{z;\varrho}$ with

$$\iint_{Q_\varrho^{(\tilde{\theta}_{z;\varrho})}(z)} \frac{|u|^{2m}}{\varrho^2} \mathrm{d}x\mathrm{d}t = \tilde{\theta}_{z;\varrho}^{2m}.$$

However, the mapping $\varrho \mapsto \tilde{\theta}_{z;\varrho}$ might not be monotone, and so we introduce the modification

$$\theta_{z;\varrho} := \max_{r \geq \varrho} \tilde{\theta}_{z;r}.$$

Then the mapping $\varrho \mapsto \theta_{z;\varrho}$ has the following properties:

1. monotonically decreasing (i.e., $Q_\varrho^{(\theta_{z;\varrho})}(z) \subset Q_s^{(\theta_{z;s})}(z)$ if $\varrho < s$).
2. $Q_r^{(\theta_{z;\varrho})}(z)$ are sub-intrinsic for all $r \geq \varrho$.
3. $\exists \tilde{\varrho} \geq \varrho$ s.t. $Q_{\tilde{\varrho}}^{(\theta_{z;\varrho})}(z)$ is intrinsic.

Now, we choose ϱ_z so that either

$$\theta_{z;\varrho_z}^{2m} \leq c \iint_{Q_{\varrho_z}^{(\theta_{z;\varrho_z})}(z)} \left[|D\boldsymbol{u}^m|^2 + \Psi^2 \right] \mathrm{d}x\mathrm{d}t$$

or

$$Q_{\varrho_z}^{(\theta_{z;\varrho_z})}(z) \subset Q_{\tilde{\varrho}_z}^{(\theta_{z;\varrho_z})}(z) \subset Q_{2\tilde{\varrho}_z}^{(\theta_{z;\varrho_z})}(z) \subset Q_{4\varrho_z}^{(\theta_{z;\varrho_z})}(z).$$

5. Covering argument and the gradient estimate
   - Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.
   - Cover the super-level set of $|D\boldsymbol{u}^m|$ with these cylinders in the sense of a Vitali-type covering.
   - Applying the reverse Hölder inequality on each of the cylinders, we get a quantitative estimate for $|D\boldsymbol{u}^m|^2$ on the super-level sets in terms of $|D\boldsymbol{u}^m|^{2q_o}$ and $\Psi^2 = \left(|\partial_t \psi^m|^{\frac{m}{2m-1}} + |D\psi^m| + |F| + |g|\right)^2$.
   - In a standard way, the estimate on the super-level sets leads to the higher integrability estimate for $D\boldsymbol{u}^m$.

- Remark

  We have a similar result for the singular case in [C. and Scheven, JMAA, 2020].

5. Covering argument and the gradient estimate
   - Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.
   - Cover the super-level set of $|D\boldsymbol{u}^m|$ with these cylinders in the sense of a Vitali-type covering.
   - Applying the reverse Hölder inequality on each of the cylinders, we get a quantitative estimate for $|D\boldsymbol{u}^m|^2$ on the super-level sets in terms of $|D\boldsymbol{u}^m|^{2q_o}$ and $\Psi^2 = \left(|\partial_t \boldsymbol{\psi}^m|^{\frac{m}{2m-1}} + |D\boldsymbol{\psi}^m| + |F| + |g|\right)^2$.
   - In a standard way, the estimate on the super-level sets leads to the higher integrability estimate for $D\boldsymbol{u}^m$.

- Remark

  We have a similar result for the singular case in [C. and Scheven, JMAA, 2020].

5. Covering argument and the gradient estimate
   - Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.
   - Cover the super-level set of $|D\boldsymbol{u}^m|$ with these cylinders in the sense of a Vitali-type covering.
   - Applying the reverse Hölder inequality on each of the cylinders, we get a quantitative estimate for $|D\boldsymbol{u}^m|^2$ on the super-level sets in terms of $|D\boldsymbol{u}^m|^{2q_o}$ and $\Psi^2 = \left(|\partial_t \boldsymbol{\psi}^m|^{\frac{m}{2m-1}} + |D\boldsymbol{\psi}^m| + |F| + |g|\right)^2$.
   - In a standard way, the estimate on the super-level sets leads to the higher integrability estimate for $D\boldsymbol{u}^m$.

- Remark

  We have a similar result for the singular case in [C. and Scheven, JMAA, 2020].

5. Covering argument and the gradient estimate
    - Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.
    - Cover the super-level set of $|D\boldsymbol{u}^m|$ with these cylinders in the sense of a Vitali-type covering.
    - Applying the reverse Hölder inequality on each of the cylinders, we get a quantitative estimate for $|D\boldsymbol{u}^m|^2$ on the super-level sets in terms of $|D\boldsymbol{u}^m|^{2q_o}$ and $\Psi^2 = \left(|\partial_t \boldsymbol{\psi}^m|^{\frac{m}{2m-1}} + |D\boldsymbol{\psi}^m| + |F| + |g|\right)^2$.
    - In a standard way, the estimate on the super-level sets leads to the higher integrability estimate for $D\boldsymbol{u}^m$.

- Remark
  We have a similar result for the singular case in [C. and Scheven, JMAA, 2020].

Thank you for your attention!

Dziękuję Ci!

감사합니다[gamsahamnida]!