The self-improving property of higher integrability in the obstacle problem for the porous medium equation

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(This is a joint work with Christoph Scheven)

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Our problem

Let \( \Omega_T := \Omega \times (0, T) \) for a bounded open domain \( \Omega \subset \mathbb{R}^n, n \geq 2, \) and \( T > 0. \) The signed porous medium equation (PME):

\[
\partial_t u - \Delta (|u|^{m-1}u) = 0 \quad \text{in } \Omega_T.
\]

- \( m > 1: \) degenerate case (slow diffusion)
  In this case, disturbances propagate with finite speed and solutions might vanish outside of a compact subset of the spatial domain.

- \( m < 1: \) singular case (fast diffusion)
  In this case, solutions exhibit infinite propagation speed and we may observe extinction in finite time.

Introduce a vector valued function \( A(x, t, u, \xi) : \Omega_T \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) which is measurable in \((x, t)\) and continuous in \((u, \xi)\). In addition, \( A \) satisfies the following ellipticity and growth conditions with some constants \(0 < \nu \leq L < \infty:\)

\[
\begin{aligned}
A(x, t, u, \xi) \cdot \xi &\geq \nu |\xi|^2, \\
|A(x, t, u, \xi)| &\leq L|\xi|,
\end{aligned}
\]

for a.e. \((x, t) \in \Omega_T\) and all \((u, \xi) \in \mathbb{R} \times \mathbb{R}^n\).

We can consider a generalization of the signed PME,

\[
\partial_t u - \text{div } A(x, t, u, Du^m) = 0 \quad \text{in } \Omega_T,
\]

where \( u^m := |u|^{m-1}u. \)
Our problem

Let $\Omega_T := \Omega \times (0, T)$ for a bounded open domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and $T > 0$. The signed porous medium equation (PME):

$$\partial_t u - \Delta (|u|^{m-1}u) = 0 \quad \text{in } \Omega_T.$$

- $m > 1$: degenerate case (slow diffusion)
  In this case, disturbances propagate with finite speed and solutions might vanish outside of a compact subset of the spatial domain.
- $m < 1$: singular case (fast diffusion)
  In this case, solutions exhibit infinite propagation speed and we may observe extinction in finite time.

Introduce a vector valued function $A(x, t, u, \xi) : \Omega_T \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ which is measurable in $(x, t)$ and continuous in $(u, \xi)$. In addition, $A$ satisfies the following ellipticity and growth conditions with some constants $0 < \nu \leq L < \infty$:

$$\begin{cases}
A(x, t, u, \xi) \cdot \xi \geq \nu |\xi|^2, & \text{for a.e. } (x, t) \in \Omega_T \text{ and all } (u, \xi) \in \mathbb{R} \times \mathbb{R}^n, \\
|A(x, t, u, \xi)| \leq L|\xi|,
\end{cases}$$

We can consider a generalization of the signed PME,

$$\partial_t u - \text{div} \ A(x, t, u, Du^m) = 0 \quad \text{in } \Omega_T,$$

where $u^m := |u|^{m-1}u$. 
Our problem

We now consider the obstacle problem related to the equation

\[
\partial_t u - \text{div} A(x, t, u, Du^m) = g - \text{div} F \quad \text{in } \Omega_T,
\]

(1)

with an obstacle constraint given by the condition \( u \geq \psi \) a.e. in \( \Omega_T \).

- We restrict our attention to the case \( m > 1 \).
- We consider inhomogeneities

\[
F \in L^2(\Omega_T, \mathbb{R}^n) \quad \text{and} \quad g \in L^2(\Omega_T, \mathbb{R}),
\]

and an obstacle function \( \psi : \Omega_T \rightarrow \mathbb{R} \) with

\[
\psi^m \in L^2(0, T; W^{1,2}(\Omega)) \quad \text{and} \quad \partial_t \psi^m \in L^{2m-1}(\Omega_T).
\]

- We define the function classes

\[
K_\psi := \{ w \in C^0([0, T]; L^{m+1}(\Omega)) : \quad w^m \in L^2(0, T; W^{1,2}(\Omega)), \ w \geq \psi \ \text{a.e. in } \Omega_T \}
\]

and

\[
K'_\psi := \{ w \in K_\psi : \partial_t w^m \in L^{2m-1}(\Omega_T) \}.
\]
Our problem

**Definition**

We say that a function \( u \in K_\psi \) is a local weak solution of the obstacle problem related to the equation (1) if the variational inequality

\[
\langle \partial_t u, \alpha \eta(w^m - u^m) \rangle + \int_{\Omega_T} \alpha A(x, t, u, Du^m) \cdot D(\eta(w^m - u^m)) \, dx \, dt \\
\geq \int_{\Omega_T} \alpha (F \cdot D(\eta(w^m - u^m)) + \eta g(w^m - u^m)) \, dx \, dt
\]

holds true for all comparison maps \( w \in K'_\psi \), any cut-off function \( \alpha \in W^{1,\infty}_0([0, T], \mathbb{R}_{\geq 0}) \) in time, and any cut-off function \( \eta \in W^{1,\infty}_0(\Omega, \mathbb{R}_{\geq 0}) \) in space. Here, the term containing the time derivative is defined by

\[
\langle \partial_t u, \alpha \eta(w^m - u^m) \rangle := \int_{\Omega_T} \eta \left\{ \alpha' \left( \frac{1}{m+1} |u|^{m+1} - uw^m \right) - \alpha u \partial_t w^m \right\} \, dx \, dt.
\]

(Note: A corresponding existence result was showed by Bögelein, Lukkari, and Scheven in Math. Ann, 2015.)
Our goal

\[ |\partial_t \psi^m|^{\frac{m}{2m-1}}, |D\psi^m|, |F|, |g| \in L^{2+\gamma}_{\text{loc}}(\Omega_T) \] for some \( \gamma > 0 \)

\[ \implies \] The gradient \( Du^m \) of a local weak solution is more integrable than assumed in the definition. More precisely, \( |Du^m| \in L^{2+\sigma_1}_{\text{loc}}(\Omega_T) \) for some \( \sigma_1 > 0 \).
Elliptic $p$-Laplace problems ($1 < p < \infty$): Meyers and Elcrat (1975)

Let $u \in W^{1,p}_{\text{loc}}(\Omega)$ be a weak solution of the equation

\[
\text{div}(|Du|^{p-2}Du) = 0 \quad \text{in} \quad \Omega.
\]

There exists $\varepsilon > 0$ such that

\[
\int_{B_r} |Du|^{p+\varepsilon} \, dx \leq c \left( \int_{B_{2r}} |Du|^p \, dx \right)^{\frac{p+\varepsilon}{p}}
\]

for any ball $B_{2r} \subset \Omega$. 
(Idea of the proof)

1. An energy estimate and Sobolev-Poincaré inequality:

\[
\int_{B_r} |Du|^p \, dx \leq \frac{c}{r^p} \int_{B_{2r}} |u - (u)_{B_{2r}}|^p \, dx \\
\leq c \left( \int_{B_{2r}} |Du|^{p^*} \, dx \right)^{\frac{p}{p^*}}.
\]

2. A reverse Hölder inequality:

\[
\int_{B_r} |Du|^p \, dx \leq c \left( \int_{B_{2r}} |Du|^{p^*} \, dx \right)^{\frac{p}{p^*}}
\]

for any ball \( B_{2r} \subset \Omega \).

3. Gehring’s lemma:

There exists \( \varepsilon > 0 \) such that

\[
\int_{B_r} |Du|^{p+\varepsilon} \, dx \leq c \left( \int_{B_{2r}} |Du|^p \, dx \right)^{\frac{p+\varepsilon}{p}}.
\]
Parabolic problems with $p$-Laplacian type

- **Parabolic systems**
  1. $p = 2$: Giaquinta and Struwe (Math. Z., 1982).
    - an intrinsic scaling method

- **Global higher integrability**
  2. $p > \frac{2n}{n+2}$: Parviainen and Bögelein (NoDEA, 2010), Byun, Kim, and Lim (Forum Math., 2020).

- $p(x, t)$-Laplacian: Bögelein and Duzaar (Publ. Mat., 2011).

- **Obstacle problems**
  2. $p(x, t)$-Laplacian: Erhardt (JMAA, 2016).
A brief history

Porous medium type equations/systems

- **Gianazza and Schwarzacher:**
  nonnegative solutions of porous medium type equations

\[
\int_{\Omega_T} u \partial_t \phi - mu^{m-1} Du \cdot D\phi \, dx dt = 0, \quad \text{for any } \phi \in C_0^\infty(\Omega_T)
\]

1. \(m > 1\) (Amer. J. Math., 2019):
   \[\Rightarrow |Du^{m+1}/2| \in L^{2+\varepsilon}_{1oc}(\Omega_T) \text{ for some } \varepsilon > 0.\]

2. \(\frac{(n-2)+}{n+2} < m < 1\) (JFA, 2019):
   \[\Rightarrow |Du^m| \in L^{2+\varepsilon}_{1oc}(\Omega_T) \text{ for some } \varepsilon > 0.\]

3. (Intrinsic scaling) They consider cylinders \(Q_{\varrho,\theta\varrho^2} = B_\varrho \times (-\theta \varrho^2, \theta \varrho^2)\) with
   \[\int_{Q_{\varrho,\theta\varrho^2}} u^{m+1} \, dx dt \approx \theta^{m+1/1-m}.\]
Porous medium type equations/systems

- Signed porous medium type systems

\[
\iint_{\Omega_T} u \partial_t \phi - D(|u|^{m-1} u) \cdot D\phi \, dx \, dt = 0, \quad \text{for any } \phi \in C_0^\infty(\Omega_T, \mathbb{R}^N).
\]

\[\Rightarrow \quad |D u^m| = |D(|u|^{m-1} u)| \in L^{2+\varepsilon}_{\text{loc}}(\Omega_T) \text{ for some } \varepsilon > 0.\]

1. \(m \geq 1\): Bögelein, Duzaar, Korte, and Scheven (Adv. Nonlinear Anal., 2019)
2. \(\frac{(n-2)+}{n+2} < m < 1\): Bögelein, Duzaar, and Scheven (J. Reine Angew. Math., 2020).
3. (Intrinsic scaling) They consider cylinders \(Q_{r,s} = B_r \times (-s,s)\) such that

\[
\frac{s}{r^{1+m}} = \theta^{1-m} \quad \text{with } \theta^m \text{ related to } \frac{|u^m|}{r^m}.
\]

- Global higher integrability \((m > 1)\): Moring, Scheven, Schwarzacher, and Singer (CPAA, 2020).
\[ \partial_t u - \Delta(|u|^{m-1}u) = 0 \]

From the modulus of ellipticity of the equation, we consider cylinders

\[ B_\varrho \times (-\lambda \varrho^2, \lambda \varrho^2) \]

with \( \lambda \sim |u|^{1-m} \).

However, we are going to prove an estimate for \( Du^m \). Now setting

\[ \theta^m \sim \frac{|u|^m}{\varrho}, \]

\[ \theta^m \sim \frac{|u|^m}{\varrho} \sim \frac{\lambda^{\frac{m}{1-m}}}{\varrho}. \]

This leads to the cylinders

\[ Q_\varrho^{(\theta)} := B_\varrho \times (-\theta^{1-m} \varrho^{\frac{m+1}{m}}, \theta^{1-m} \varrho^{\frac{m+1}{m}}). \]

\((m \geq 1)\) We call a cylinder \( Q_\varrho^{(\theta)} \) intrinsic iff

\[ \iint_{Q_\varrho^{(\theta)}} \frac{|u|^{2m}}{\varrho^2} \, dx \, dt \approx \theta^{2m}. \]
We work with cylinders of the following form:

$$Q_\varrho^{(\theta)}(z_o) := B_\varrho(x_o) \times \Lambda_\varrho^{(\theta)}(t_o),$$

for $z_o = (x_o, t_o) \in \mathbb{R}^n \times (0, T)$, where $B_\varrho(x_o)$ denotes the open ball with radius $\varrho > 0$ and center $x_o \in \mathbb{R}^n$ and

$$\Lambda_\varrho^{(\theta)}(t_o) := (t_o - \theta^{1-m} \varrho \frac{m+1}{m}, t_o + \theta^{1-m} \varrho \frac{m+1}{m})$$

for some scaling factor $\theta > 0$. If $\theta = 1$, we use the following abbreviation:

$$Q_\varrho(z_o) := Q_\varrho^{(1)}(z_o) \quad \text{and} \quad \Lambda_\varrho(t_o) := \Lambda_\varrho^{(1)}(t_o).$$

We next define a boundary term

$$b[u^m, a^m] := \frac{m}{m+1} (|a|^{m+1} - |u|^{m+1}) - u(a^m - u^m),$$

for $u, a \in \mathbb{R}$. 

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Notations
Main theorem [C. and Scheven (NoDEA, 2019)]

There exists a constant $\sigma_o = \sigma_o(n, m, \nu, L) \in (0, 1]$ such that if

$$\Psi := |\partial_t \psi^m|^{\frac{m}{2m-1}} + |D\psi^m| + |F| + |g| \in L^{2+\gamma}_{loc}(\Omega_T)$$

for some $\gamma > 0$, then we have

$$Du^m \in L^{2+\sigma_1}_{loc}(\Omega_T, \mathbb{R}^n),$$

where $\sigma_1 := \min\{\sigma_o, \gamma\}$. Furthermore, for any $\sigma \in (0, \sigma_1]$ and any cylinder $Q_{2R}(z_o) \subseteq \Omega_T$ with $R \in (0, 1]$, the following quantitative local higher integrability estimate is satisfied:

$$\iint_{Q_R(z_o)} |Du^m|^{2+\sigma} \, dx \, dt \leq c \left[ 1 + \iint_{Q_{2R}(z_o)} \left[ \frac{|u|^{2m}}{R^2} + \Psi^2 \right] \, dx \, dt \right]^\frac{\sigma m}{m+1} \iint_{Q_{2R}(z_o)} |Du^m|^2 \, dx \, dt$$

$$+ c \iint_{Q_{2R}(z_o)} \Psi^{2+\sigma} \, dx \, dt,$$

with a constant $c = c(n, m, \nu, L) \geq 1$, where we considered the parabolic cylinders $Q_R(z_o) := B_R(x_o) \times (t_o - R \frac{m+1}{m}, t_o + R \frac{m+1}{m})$. 
- Our strategy
  1. an energy estimate
  2. a Sobolev-Poincaré type inequality ⇐ a gluing lemma
  3. (1)+(2) ⇒ a reverse Hölder type inequality
  4. covering argument and the gradient estimate
1. An energy estimate

There exists a constant $c = c(m, \nu, L) > 0$ such that on any cylinder $Q_{\varrho}^{(\theta)}(z_o) \subseteq \Omega_T$ with $0 < \varrho \leq 1$ and $\theta > 0$, the energy estimate

$$\sup_{t \in \Lambda_r^{(\theta)}(t_o)} \int_{B_r(x_o)} \theta^{m-1} \frac{b[\mathbf{u}^m(\cdot, t), \mathbf{a}^m]}{r^{\frac{m+1}{m}}} \, dx + \iint_{Q_r^{(\theta)}(z_o)} |D\mathbf{u}^m|^2 \, dx \, dt$$

$$\leq c \iint_{Q_{\varrho}^{(\theta)}(z_o)} \left[ \frac{|\mathbf{u}^m - \mathbf{a}^m|^2}{(\varrho - r)^2} + \theta^{m-1} \frac{b[\mathbf{u}^m, \mathbf{a}^m]}{\varrho^{\frac{m+1}{m}} - r^{\frac{m+1}{m}}} \right] \, dx \, dt$$

$$+ c \iint_{Q_{\varrho}^{(\theta)}(z_o)} |F|^2 + |g|^2 + |D\psi^m|^2 + |\partial_t \psi^m|^{\frac{2m}{2m-1}} \, dx \, dt$$

holds true for all $r \in [\varrho/2, \varrho)$ and all $a \in \mathbb{R}$.

(Test function: $\mathbf{w}^m := \max\{\mathbf{a}^m, \psi^m\} = \mathbf{a}^m + (\psi^m - \mathbf{a}^m)_+ \geq \psi^m$ a.e.)
2. A gluing lemma: This compares the means at different time slices, and will be used for a Poincaré type inequality.

We consider a cylinder $Q_{\bar{\rho}}^{(\theta)}(z_o) \subseteq \Omega_T$ with $0 < \bar{\rho} \leq 1$ and $\theta > 0$ and times $t_1, t_2 \in \Lambda^{(\theta)}(t_o)$ with $t_1 < t_2$. Then, for a.e. $r \in [\frac{\rho}{2}, \rho]$, we have the estimates

$$
|(u)_{\ell \ell} : r(t_1) - (u)_{\ell \ell} : r(t_2)| \\
\leq c\rho \frac{1}{\theta^{m-1}} \int_{\Lambda^{(\theta)}(t_o)} \int_{\partial B_r(x_o)} (|Du^m| + |F|) \, d\mathcal{H}^{n-1} \, dt \\
+ \frac{1}{\mu^m |B_r|} \int_{(B_r(x_o) \times \{t_1\}) \cap \{u^m \leq \psi^m + \mu^m\}} b[u^m, \psi^m + \mu^m] \, dx \\
+ \frac{c\rho}{\theta^{m-1} \mu^m |Q_{\bar{\rho}}^{(\theta)}|} \iint_{Q_{\bar{\rho}}^{(\theta)}(z_o) \cap \{u^m \leq \psi^m + \mu^m\}} (\mu |\partial_t \psi^m| + |D\psi^m|^2 + |F|^2) \, dx \, dt \\
+ \frac{c\rho}{\theta^{m-1}} \iint_{Q_{\bar{\rho}}^{(\theta)}(z_o)} |g| \, dx \, dt,
$$

where $c = c(n, m, \nu, L)$ and $\mu > 0$ is an arbitrary parameter.
Sketch of the proof

3. A Sobolev-Poincaré type inequality on sub-intrinsic cylinders

Consider cylinders $Q_{\varrho}^{(\theta)}(z_o) \subseteq \Omega_T$ with $0 < \varrho \leq 1$ and $\theta > 0$ that are sub-intrinsic in the sense

$$\iint_{Q_{\varrho}^{(\theta)}(z_o)} \left| \frac{u}{\varrho^2} \right|^2 dxdt \leq 2^{d+2} \theta^{2m} \quad \text{for } d = n + 1 + \frac{1}{m}. \quad (2)$$

Then, for any given $\varepsilon \in (0, 1]$ we have the Sobolev-Poincaré type inequality

$$\iint_{Q_{\varrho}^{(\theta)}(z_o)} \left| \frac{u^m - (u^m)^{(\theta)}_{z_o;\varrho}}{\varrho^2} \right|^2 dxdt \leq \varepsilon \sup_{t \in \Lambda_{\varrho}^{(\theta)}(t_o)} \int_{B_{\varrho}(x_o)} \theta^{m-1} \frac{b\left[u^m(\cdot, t), (u^m)^{(\theta)}_{z_o;\varrho}\right]}{\varrho^{m+1}} dx$$

$$+ \frac{c}{\varepsilon^{\frac{n}{2}}} \left[ \iint_{Q_{\varrho}^{(\theta)}(z_o)} \left| D u^m \right|^2 qo dxdt \right]^{\frac{1}{2qo}} + c \iint_{Q_{\varrho}^{(\theta)}(z_o)} \Psi^2 dxdt,$$

where $c$ is a universal constant depending only on $n, m, \nu, \text{ and } L$, and $q_o$ is defined by $q_o := \max\{ \frac{m-1}{m}, \frac{1}{2}, \frac{n}{d} \} < 1$. 
Proof of a Sobolev-Poincaré type inequality

(1) Applying the gluing lemma, the sub-intrinsic property (2), Poincaré’s inequality, and Mean value’s theorem, there exists $\hat{\varrho} \in \left[\frac{\varrho}{2}, \varrho\right]$ such that

$$\frac{1}{\varrho^2} \int_{\Lambda_{\hat{\varrho}}^{(\theta)}(t_0)} \int_{\Lambda_{\hat{\varrho}}^{(\theta)}(t_0)} \left| (u)^m_{x_0;\hat{\varrho}}(t) - (u)^m_{x_0;\hat{\varrho}}(\tau) \right|^2 dt d\tau$$

$$\leq c \left( \int_{Q_{\hat{\varrho}}^{(\theta)}(z_0)} |Du|^q dx dt \right)^{\frac{1}{q}} + c \int_{Q_{\hat{\varrho}}^{(\theta)}(z_0)} \Psi dx dt,$$

for $q := \max\{\frac{m-1}{m}, \frac{1}{2}\} < 1$ and a constant $c = c(n, m, \nu, L)$.

(2) We then add and subtract the slice-wise means $(u)^m_{x_0;\hat{\varrho}}(t)$, to obtain

$$\int_{Q_{\hat{\varrho}}^{(\theta)}(z_0)} \left| u^m - (u^m)^{(\theta)}_{z_0;\hat{\varrho}} \right|^2 dx dt$$

$$\leq 3 \left[ \int_{Q_{\hat{\varrho}}^{(\theta)}(z_0)} \left| u^m - (u)^m_{x_0;\hat{\varrho}}(t) \right|^2 dx dt$$

$$+ \frac{1}{\varrho^2} \int_{\Lambda_{\hat{\varrho}}^{(\theta)}(t_0)} \int_{\Lambda_{\hat{\varrho}}^{(\theta)}(t_0)} \left| (u)^m_{x_0;\hat{\varrho}}(t) - (u)^m_{x_0;\hat{\varrho}}(\tau) \right|^2 d\tau dt$$

$$+ \frac{1}{\varrho^2} \left| \int_{\Lambda_{\hat{\varrho}}^{(\theta)}(t_0)} (u)^m_{x_0;\hat{\varrho}}(\tau) d\tau - (u)^m_{z_0;\hat{\varrho}}(\tau) \right|^2 \right] = 3[I + \ II + \ III].$$
Sketch of the proof

Proof of a Sobolev-Poincaré type inequality

1. Applying the gluing lemma, the sub-intrinsic property (2), Poincaré’s inequality, and Mean value’s theorem, there exists \( \hat{\varrho} \in [\frac{\varrho}{2}, \varrho] \) such that

\[
\frac{1}{\varrho^2} \int_{\Lambda_{\hat{\varrho}}^{(\theta)}(t_0)} \int_{\Lambda_{\hat{\varrho}}^{(\theta)}(t_0)} \left| (u)^m_{x_0; \hat{\varrho}}(t) - (u)^m_{x_0; \hat{\varrho}}(\tau) \right|^2 \, dt \, d\tau
\]

\[
\leq c \left( \iint_{Q_{\hat{\varrho}}^{(\theta)}(z_0)} |Du|^2q \, dx \, dt \right) \frac{1}{q} + c \iint_{Q_{\hat{\varrho}}^{(\theta)}(z_0)} \Psi^2 \, dx \, dt,
\]

for \( q := \max\{\frac{m-1}{m}, \frac{1}{2}\} < 1 \) and a constant \( c = c(n, m, \nu, L) \).

2. We then add and subtract the slice-wise means \( (u)^m_{x_0; \hat{\varrho}}(t) \), to obtain

\[
\iint_{Q_{\hat{\varrho}}^{(\theta)}(z_0)} \frac{|u^m - (u^m)^{(\theta)}_{z_0; \hat{\varrho}}|^2}{\varrho^2} \, dx \, dt
\]

\[
\leq 3 \left[ \iint_{Q_{\hat{\varrho}}^{(\theta)}(z_0)} \frac{|u^m - (u)^m_{x_0; \hat{\varrho}}(t)|^2}{\varrho^2} \, dx \, dt
\]

\[
+ \frac{1}{\varrho^2} \int_{\Lambda_{\hat{\varrho}}^{(\theta)}(t_0)} \int_{\Lambda_{\hat{\varrho}}^{(\theta)}(t_0)} \left| (u)^m_{x_0; \hat{\varrho}}(t) - (u)^m_{x_0; \hat{\varrho}}(\tau) \right| \, d\tau \, dt
\]

\[
+ \frac{1}{\varrho^2} \int_{\Lambda_{\hat{\varrho}}^{(\theta)}(t_0)} (u)^m_{x_0; \hat{\varrho}}(\tau) \, d\tau - (u^m)^{(\theta)}_{z_0; \hat{\varrho}} \right|^2 \right] =: 3[I + II + III].
\]
Proof of a Sobolev-Poincaré type inequality

(3) We get

\[ \text{II} \leq c \left( \iint_{Q_{\varrho}(\theta) (z_0)} |Du^m|^2 q \, dx \, dt \right)^{\frac{1}{q}} + c \iint_{Q_{\varrho}(\theta) (z_0)} \Psi^2 \, dx \, dt, \]

and

\[ \text{III} \leq I \leq c \iint_{Q_{\varrho}(\theta) (z_0)} \left| \frac{u^m - (u^m)_{x_0; \theta} (t)}{\varrho^2} \right|^2 \, dx \, dt. \]

Finally, we have the inequality

\[ \iint_{Q_{\varrho}(\theta) (z_0)} \left| \frac{u^m - (u^m)_{x_0; \theta} (t)}{\varrho^2} \right|^2 \, dx \, dt \]

\[ \leq c \iint_{Q_{\varrho}(\theta) (z_0)} \left| \frac{u^m - (u^m)_{x_0; \theta} (t)}{\varrho^2} \right|^2 \, dx \, dt + c \left( \iint_{Q_{\varrho}(\theta) (z_0)} |Du^m|^2 q \, dx \, dt \right)^{\frac{1}{q}} \]

\[ + c \iint_{Q_{\varrho}(\theta) (z_0)} \Psi^2 \, dx \, dt. \]
Proof of a Sobolev-Poincaré type inequality

(4) Using the properties for \( b[\cdot, \cdot] \) and the sub-intrinsic coupling (2), we infer

\[
\iint_{Q_{\theta}(z_0)} \frac{|u^m - (u^m)_{x_o;\varrho}(t)|^2}{\varrho^2} \, dx \, dt
= \frac{1}{\varrho^2} \iint_{Q_{\theta}(z_0)} |u^m - (u^m)_{x_o;\varrho}(t)| \frac{4}{n+2} |u^m - (u^m)_{x_o;\varrho}(t)|^{\frac{2n}{n+2}} \, dx \, dt
\leq c \left\{ \int_{\Lambda_{\theta}(t_o)} \left[ \int_{B_{\varrho}(x_o)} \theta^{m-1} b[u^m, (u^m)_{x_o;\varrho}(t)] \frac{\varrho^m + 1}{m} \, dx \right]^{\frac{2}{d}} dx \right. \\
\left. \cdot \left[ \int_{B_{\varrho}(x_o)} \frac{|u^m - (u^m)_{x_o;\varrho}(t)|^{\frac{2n}{d-2}}}{\varrho^{\frac{2n}{d-2}}} \, dx \right]^{\frac{d-2}{d}} dt \right\}^{\frac{d}{n+2}}.
\]

From Sobolev’s inequality slicewise for a.e. \( t \in \Lambda_{\theta}(t_o) \),

\[
\iint_{Q_{\theta}(z_0)} \frac{|u^m - (u^m)_{x_o;\varrho}(t)|^2}{\varrho^2} \, dx \, dt \leq c \left[ \iint_{Q_{\theta}(z_0)} |D u^m|^{\frac{2n}{d}} \, dx \, dt \right]^{\frac{d}{n+2}} \\
\cdot \sup_{t \in \Lambda_{\theta}(t_o)} \left[ \int_{B_{\varrho}(x_o)} \theta^{m-1} b[u^m(\cdot, t), (u^m)_{x_o;\varrho}(t)] \frac{\varrho^m + 1}{m} \, dx \right]^{\frac{2}{n+2}}.
\]
Sketch of the proof

Proof of a Sobolev-Poincaré type inequality

(5) Combining with the estimate

$$\iint_{Q_e^{(\theta)}(z_0)} \frac{|u^m - (u^m)_{z_0;\theta}|^2}{\rho^2} \; dxdt \leq c \iint_{Q_e^{(\theta)}(z_0)} \frac{|u^m - (u^m)_{x_0;\theta}(t)|^2}{\rho^2} \; dxdt + c \left( \iint_{Q_e^{(\theta)}(z_0)} |Du^m|^{2q} \; dxdt \right)^{\frac{1}{q}} + c \iint_{Q_e^{(\theta)}(z_0)} \Psi^2 \; dxdt$$

and applying Young’s and Hölder’s inequality, this deduces

$$\iint_{Q_e^{(\theta)}(z_0)} \frac{|u^m - (u^m)_{z_0;\theta}|^2}{\rho^2} \; dxdt \leq \varepsilon \sup_{t \in \Lambda_{e}^{(\theta)}(t_0)} \int_{B_{e}(x_0)} \theta^{m-1} \frac{b[u^m(\cdot, t), (u^m)^{(\theta)}_{z_0;\theta}]}{\rho^{m+1}} \; dx$$

$$+ \frac{c}{\varepsilon^{\frac{2}{n}}} \left[ \iint_{Q_e^{(\theta)}(z_0)} |Du^m|^{2q_0} \; dxdt \right]^{\frac{1}{q_0}} + c \iint_{Q_e^{(\theta)}(z_0)} \Psi^2 \; dxdt$$

for any $\varepsilon \in (0, 1]$ and for $q_0 := \max\{q, \frac{n}{d}\} = \max\{\frac{m-1}{m}, \frac{1}{2}, \frac{n}{d}\} < 1$. □
4. A reverse Hölder type inequality

Let $Q^{(\theta)}_{2\varrho}(z_o) \subseteq \Omega_T$ for $0 < \varrho \leq \frac{1}{2}$ and $\theta > 0$. Whenever the cylinder $Q^{(\theta)}_{2\varrho}(z_o)$ fulfills the following couplings:

either

$$\iiint_{Q^{(\theta)}_{2\varrho}(z_o)} \frac{|u|^{2m}}{(2\varrho)^2} \, dx \, dt \leq \theta^{2m} \leq \iiint_{Q^{(\theta)}_{\varrho}(z_o)} \frac{|u|^{2m}}{\varrho^2} \, dx \, dt$$

or

$$\iiint_{Q^{(\theta)}_{2\varrho}(z_o)} \frac{|u|^{2m}}{(2\varrho)^2} \, dx \, dt \leq \theta^{2m} \leq K \iiint_{Q^{(\theta)}_{\varrho}(z_o)} \left[ |Du|^2 + \Psi^2 \right] \, dx \, dt$$

for some constant $K \geq 1$,

we have the reverse Hölder type inequality

$$\iiint_{Q^{(\theta)}_{\varrho}(z_o)} |Du|^2 \, dx \, dt \leq c \left[ \iiint_{Q^{(\theta)}_{2\varrho}(z_o)} |Du|^{2q_o} \, dx \, dt \right]^{\frac{1}{q_o}}$$

$$+ c \iiint_{Q^{(\theta)}_{2\varrho}(z_o)} \Psi^2 \, dx \, dt,$$

for some constant $c = c(n, m, \nu, L, (K)) > 0$ and $q_o := \max\{\frac{m-1}{m}, \frac{1}{2}, \frac{n}{d}\} < 1$. 
Proof of a reverse Hölder inequality

For radii \( r, s \) with \( \varrho \leq r < s \leq 2\varrho \),

- the energy estimate:

\[
\begin{align*}
\sup_{t \in \Lambda_r^\theta(x_0)} & \int_{B_r(x_0)} \theta^{m-1} \frac{b[\mathbf{u}^m(\cdot, t), (\mathbf{u}^m)^{\theta}_{z_0;r}]}{r^{\frac{m+1}{m}}} \, dx + \iint_{Q_r^\theta(z_0)} |D\mathbf{u}^m|^2 \, dx \, dt \\
& \leq c \iint_{Q_s^\theta(z_0)} \frac{|\mathbf{u}^m - (\mathbf{u}^m)^{\theta}_{z_0;r}|^2}{(s-r)^2} \, dx \, dt + c \iint_{Q_s^\theta(z_0)} \theta^{m-1} \frac{b[\mathbf{u}^m, (\mathbf{u}^m)^{\theta}_{z_0;r}]}{s^{\frac{m+1}{m}} - r^{\frac{m+1}{m}}} \, dx \, dt \\
& \quad + c \iint_{Q_s^\theta(z_0)} \Psi^2 \, dx \, dt \\
& =: I + II + III.
\end{align*}
\]

- We use the notation

\[
\mathcal{R}_{r,s} := \frac{s^{\frac{m+1}{2m}}}{s^{\frac{m+1}{2m}} - r^{\frac{m+1}{2m}}},
\]

- \( I \) is estimated by

\[
I \leq c \mathcal{R}_{r,s}^{\frac{4m}{m+1}} \iint_{Q_s^\theta(z_0)} \frac{|\mathbf{u}^m - (\mathbf{u}^m)^{\theta}_{z_0;s}|^2}{s^2} \, dx \, dt.
\]
Proof of a reverse Hölder inequality

For radii $r, s$ with $\varrho \leq r < s \leq 2\varrho$,

- the energy estimate:

\[
\sup_{t \in \Lambda_r^{(\theta)}(t_0)} \int_{B_r(x_0)} \theta^{m-1} \frac{b \left[ u^m(\cdot, t), (u^m)^{(\theta)}_{z_0;r} \right]}{r^{m+1}} \frac{m+1}{m} \, dx + \iint_{Q_r^{(\theta)}(z_0)} \left| Du^m \right|^2 \, dx \, dt \\
\leq c \iint_{Q_s^{(\theta)}(z_0)} \frac{|u^m - (u^m)^{(\theta)}_{z_0;r}|^2}{(s - r)^2} \, dx \, dt + c \iint_{Q_s^{(\theta)}(z_0)} \theta^{m-1} \frac{b \left[ u^m, (u^m)^{(\theta)}_{z_0;r} \right]}{s^{m+1} - r^{m+1}} \, dx \, dt \\
+ c \iint_{Q_s^{(\theta)}(z_0)} \Psi^2 \, dx \, dt \\
=: \ I + \ II + \ III.
\]

- We use the notation

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\]

- $I$ is estimated by

\[
I \leq c \mathcal{R}_{r,s}^{\frac{4m}{m+1}} \iint_{Q_s^{(\theta)}(z_0)} \frac{|u^m - (u^m)^{(\theta)}_{z_0;s}|^2}{s^2} \, dx \, dt.
\]
Sketch of the proof

Proof of a reverse H"older inequality

For the second term II,

\[ \theta^{2m} \leq \mathcal{II} \leq c \frac{\theta^{m-1}}{s^{\frac{m+1}{m}} - r^{\frac{m+1}{m}}} \mathcal{B}[u^m, (u^m)_{z_0; \theta}] \]

\[ \mathcal{II} = c \mathcal{R}_r^2 \frac{\theta^{m-1}}{s^{\frac{m+1}{m}} - r^{\frac{m+1}{m}}} \mathcal{B}[u^m, (u^m)_{z_0; \theta}] \]

\[ \leq c \mathcal{R}_r^2 \mathcal{B}[u^m, (u^m)_{z_0; \theta}] \frac{\theta^{m-1}}{s^{\frac{m+1}{m}}} \mathcal{B}[u^m, (u^m)_{z_0; r}] \]

\[ \leq c \mathcal{R}_r^2 \left[ \frac{\theta^{m-1}}{s^{\frac{m+1}{m}}} \mathcal{B}[u^m, (u^m)_{z_0; r}] \right] \frac{\theta^{m-1}}{s^{\frac{m+1}{m}}} \mathcal{B}[u^m, (u^m)_{z_0; r}] \]

\[ \leq c \mathcal{R}_r^2 \frac{\theta^{m-1}}{s^{\frac{m+1}{m}}} \mathcal{B}[u^m, (u^m)_{z_0; r}] \]

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\[ \leq c \mathcal{R}_r^2 \frac{\theta^{m-1}}{s^{\frac{m+1}{m}}} \mathcal{B}[u^m, (u^m)_{z_0; r}] \]

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\[ \leq c \mathcal{R}_r^2 \frac{\theta^{m-1}}{s^{\frac{m+1}{m}}} \mathcal{B}[u^m, (u^m)_{z_0; r}] \]
Proof of a reverse Hölder inequality

For the second term II,

\[ \theta^{2m} \leq K \iint_{Q_{\varrho}(\theta)(z_0)} |Du|^2 + \Psi^2 \, dx \, dt \]  \quad (2)

We first use Young’s inequality, to infer for any \( \tau \in (0, 1] \)

\[ II \leq \mathcal{R}_{r,s}^2 \iint_{Q_s(\theta)(z_0)} \theta^{m-1} \frac{b[u^m, (u^m)^{(\theta)} s_{m+1}^{m+1}]}{s} \, dx \, dt \]

\[ \leq \tau \theta^{2m} + \mathcal{R}_{r,s} \iint_{Q_s(\theta)(z_0)} \frac{b[u^m, (u^m)^{(\theta)} s_{m+1}^{m+1}]}{s^2} \, dx \, dt \]

\[ \leq \tau \theta^{2m} + \frac{4m}{\tau^m \tau_{m+1}^{m+1}} \iint_{Q_s(\theta)(z_0)} \frac{|u^m - (u^m)^{(\theta)} s_{m+1}^{m+1}|^2}{s^2} \, dx \, dt \]

Meanwhile, the coupling (4) and \( \varrho < r < 2\varrho \) lead to

\[ \theta^{2m} \leq 2^d K \iint_{Q_{\tau}(\theta)(z_0)} [|Du|^2 + \Psi^2] \, dx \, dt. \]
Proof of a reverse Hölder inequality

We obtain for both cases

\[
\sup_{t \in \Lambda_r(t_o)} \int_{B_r(x_o)} \theta^{m-1} \frac{b[u^m(\cdot, t), (u^m)_{z_o}; r]}{r^{m+1} m} dx + \iiint_{Q^{(\theta)}(z_o)} |Du^m|^2 dxdt \\
\leq c R^{\frac{4m}{m+1}} \iiint_{Q^{(\theta)}(z_o)} \frac{|u^m - (u^m)_{z_o}; s|^2}{s^2} dxdt + c \iiint_{Q^{(\theta)}(z_o)} \Psi^2 dxdt.
\]

Observe that \(Q^{(\theta)}(z_o)\) is sub-intrinsic, and use the Sobolev-Poincaré type inequality:

\[
\iiint_{Q^{(\theta)}(z_o)} \frac{|u^m - (u^m)_{z_o}; s|^2}{s^2} dxdt \\
\leq \varepsilon \sup_{t \in \Lambda_s(t_o)} \int_{B_s(x_o)} \theta^{m-1} \frac{b[u^m(\cdot, t), (u^m)_{z_o}; s]}{s^{m+1} m} dx \\
+ \frac{c}{\varepsilon^{\frac{2}{n}}} \left[ \iiint_{Q^{(\theta)}(z_o)} |Du^m|^{2q_o} dxdt \right]^{\frac{1}{q_o}} + c \iiint_{Q^{(\theta)}(z_o)} \Psi^2 dxdt.
\]
Proof of a reverse Hölder inequality

- Combining the previous estimates and choosing a suitable $\varepsilon$,

$$
\sup_{t \in \Lambda_r^{(\theta)}(t_0)} \int_{B_r(x_0)} \theta^{m-1} \frac{b[u^m(\cdot, t), (u^m)_r^{(\theta)}]}{r^{\frac{m+1}{m}}} \, dx + \iint_{Q_r^{(\theta)}(z_0)} |Du^m|^2 \, dx \, dt
$$

$$
\leq \frac{1}{2} \sup_{t \in \Lambda_s^{(\theta)}(t_0)} \int_{B_s(x_0)} \theta^{m-1} \frac{b[u^m(\cdot, t), (u^m)_s^{(\theta)}]}{s^{\frac{m+1}{m}}} \, dx
$$

$$
+ c R_{r,s}^{\frac{4m(n+2)}{n(m+1)}} \left[ \iint_{Q_2^{(\theta)}(z_0)} |Du^m|^{2q_o} \, dx \, dt \right]^{\frac{1}{q_o}}
$$

$$
+ c R_{r,s}^{\frac{4m}{m+1}} \iint_{Q_2^{(\theta)}(z_0)} \Psi^2 \, dx \, dt.
$$

- We finally apply an iteration lemma to get the result. $\square$
5. Covering argument and the gradient estimate

- Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.
  - We have the reverse Hölder type inequality on the cylinder $Q^{(\theta)}_{\varrho}(z_0)$, whenever the cylinder $Q^{(\theta)}_{\varrho}(z_0)$ fulfills the following couplings:
    
    \[
    \iint_{Q^{(\theta)}_{2\varrho}(z_0)} \frac{|u|^{2m}}{(2\varrho)^2} \, dx \, dt \leq \theta^{2m} \leq \iint_{Q^{(\theta)}_{\varrho}(z_0)} \frac{|u|^{2m}}{\varrho^2} \, dx \, dt
    \]
    
    or
    
    \[
    \iint_{Q^{(\theta)}_{2\varrho}(z_0)} \frac{|u|^{2m}}{(2\varrho)^2} \, dx \, dt \leq \theta^{2m} \leq K \iint_{Q^{(\theta)}_{\varrho}(z_0)} \left[ |Du|^2 + \Psi^2 \right] \, dx \, dt
    \]
    
    for some constant $K \geq 1$. 

Sketch of the proof

5. Covering argument and the gradient estimate

- Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.
- We have the reverse Hölder type inequality on the cylinder $Q_{\theta}(\zeta(z_o))$, whenever the cylinder $Q_{\theta}(\zeta(z_o))$ fulfills the following couplings:

  
  
  
  \[
  \iint_{Q_{\theta}(\zeta(z_o))} \frac{|u|^{2m}}{(2\zeta)^2} \, dx \, dt \leq \theta^{2m} \leq \iint_{Q_{\theta}(\zeta(z_o))} \frac{|u|^{2m}}{\zeta^2} \, dx \, dt
  \]

  or

  
  
  
  \[
  \iint_{Q_{\theta}(\zeta(z_o))} \frac{|u|^{2m}}{(2\zeta)^2} \, dx \, dt \leq \theta^{2m} \leq K \iint_{Q_{\theta}(\zeta(z_o))} \left[ |Du|^2 + \Psi^2 \right] \, dx \, dt
  \]

  for some constant $K \geq 1$. 

5. Covering argument and the gradient estimate

- Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.
  - For a given center $z$ and radius $\varrho$, we select $\tilde{\theta}_{z;\varrho}$ with
    \[
    \iint_{Q_{\tilde{\varrho}}^{\tilde{\theta}_{z;\tilde{\varrho}}}(z)} \frac{|u|^{2m}}{\tilde{\varrho}^2} \, dx \, dt = \tilde{\theta}_{z;\tilde{\varrho}}^{2m}.
    \]
    However, the mapping $\varrho \mapsto \tilde{\theta}_{z;\varrho}$ might not be monotone, and so we introduce the modification
    \[
    \theta_{z;\varrho} := \max_{r \geq \varrho} \tilde{\theta}_{z;r}.
    \]
    Then the mapping $\varrho \mapsto \theta_{z;\varrho}$ has the following properties:
    1. monotonically decreasing (i.e., $Q_{\tilde{\varrho}}^{\tilde{\theta}_{z;\tilde{\varrho}}}(z) \subset Q_{\varrho}^{\theta_{z;\varrho}}(z)$ if $\varrho < s$).
    2. $Q_{r}^{\theta_{z;\varrho}}(z)$ are sub-intrinsic for all $r \geq \varrho$.
    3. $\exists \tilde{\varrho} \geq \varrho$ s.t. $Q_{\tilde{\varrho}}^{\tilde{\theta}_{z;\tilde{\varrho}}}(z)$ is intrinsic.

Now, we choose $\varrho_z$ so that either
\[
\theta_{z;\varrho_z}^{2m} \leq c \iint_{Q_{\varrho_z}^{\theta_{z;\varrho_z}}(z)} \left[ |Du|^m \right]^2 + \Psi^2 \, dx \, dt
\]
or
\[
Q_{\varrho_z}^{\theta_{z;\varrho_z}}(z) \subset Q_{\tilde{\varrho}_z}^{\tilde{\theta}_{z;\tilde{\varrho}_z}}(z) \subset Q_{2\tilde{\varrho}_z}^{\tilde{\theta}_{z;2\tilde{\varrho}_z}}(z) \subset Q_{4\varrho_z}^{\theta_{z;4\varrho_z}}(z).
\]
5. Covering argument and the gradient estimate

- Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.
  - For a given center $z$ and radius $\varrho$, we select $\tilde{\theta}_{z;\varrho}$ with
    \[
    \iint_{Q_{\varrho}^{(\tilde{\theta}_{z;\varrho})}(z)} |u|^{2m} \frac{1}{\varrho^2} \,dxdt = \tilde{\theta}^{2m}_{z;\varrho}.
    \]

However, the mapping $\varrho \mapsto \tilde{\theta}_{z;\varrho}$ might not be monotone, and so we introduce the modification

\[
\theta_{z;\varrho} := \max_{r \geq \varrho} \tilde{\theta}_{z;r}.
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1. monotonically decreasing (i.e., $Q_{\varrho}^{(\theta_{z;\varrho})}(z) \subset Q_{s}^{(\theta_{z;s})}(z)$ if $\varrho < s$).
2. $Q_{r}^{(\theta_{z;\varrho})}(z)$ are sub-intrinsic for all $r \geq \varrho$.
3. $\exists \tilde{\varrho} \geq \varrho$ s.t. $Q_{\tilde{\varrho}}^{(\theta_{z;\varrho})}(z)$ is intrinsic.

Now, we choose $\varrho_{z}$ so that either

\[
\theta_{z;\varrho_{z}}^{2m} \leq c \iint_{Q_{\varrho_{z}}^{(\theta_{z;\varrho_{z}})}(z)} \left[ ||Du||^{m}|^{2} + \Psi^{2} \right] \,dxdt
\]

or

\[
Q_{\varrho_{z}}^{(\theta_{z;\varrho_{z}})}(z) \subset Q_{\varrho_{z}}^{(\theta_{z;\varrho_{z}})}(z) \subset Q_{2\varrho_{z}}^{(\theta_{z;\varrho_{z}})}(z) \subset Q_{4\varrho_{z}}^{(\theta_{z;\varrho_{z}})}(z).
\]
5. Covering argument and the gradient estimate

- Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.
  - For a given center $z$ and radius $\varrho$, we select $\tilde{\theta}_{z;\varrho}$ with
    \[
    \iint_{Q_{\tilde{\varrho}}(\tilde{\theta}_{z;\tilde{\varrho}})} \frac{|u|^{2m}}{\varrho^2} \, dx \, dt = \tilde{\theta}_{z;\tilde{\varrho}}^{2m}.
    \]

However, the mapping $\varrho \mapsto \tilde{\theta}_{z;\varrho}$ might not be monotone, and so we introduce the modification

\[
\theta_{z;\varrho} := \max_{r \geq \varrho} \tilde{\theta}_{z;r}.
\]

Then the mapping $\varrho \mapsto \theta_{z;\varrho}$ has the following properties:

1. monotonically decreasing (i.e., $Q_{\varrho}^{(\theta_{z;\varrho})}(z) \subset Q_{s}^{(\theta_{z;s})}(z)$ if $\varrho < s$).
2. $Q_{r}^{(\theta_{z;\varrho})}(z)$ are sub-intrinsic for all $r \geq \varrho$.
3. $\exists \varrho \geq \varrho$ s.t. $Q_{\varrho}^{(\theta_{z;\varrho})}(z)$ is intrinsic.

Now, we choose $\varrho_z$ so that either

\[
\theta_{z;\varrho_z}^{2m} \leq c \iint_{Q_{\varrho_z}^{(\theta_{z;\varrho_z})}} \left[ |Du|^m + \varPsi^2 \right] \, dx \, dt
\]

or

\[
Q_{\varrho_z}^{(\theta_{z;\varrho_z})}(z) \subset Q_{\varrho_{z+1}}^{(\theta_{z;\varrho_{z+1}})}(z) \subset Q_{\varrho_{z+2}}^{(\theta_{z;\varrho_{z+2}})}(z) \subset Q_{\varrho_{z+4}}^{(\theta_{z;\varrho_{z+4}})}(z).
\]
5. Covering argument and the gradient estimate

- Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.

  For a given center $z$ and radius $\varrho$, we select $\tilde{\theta}_{z;\varrho}$ with

  \[
  \iint_{Q_{\varrho}(\tilde{\theta}_{z;\varrho})}(z) \frac{|u|^{2m}}{\varrho^2} \, dx \, dt = \tilde{\theta}_{z;\varrho}^{2m}.
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\[
\theta_{z;\varrho_{z}}^{2m} \leq c \iint_{Q_{\varrho_{z}}(\theta_{z;\varrho_{z}})} (z) \left[ |Du|^2 + \Psi^2 \right] \, dx \, dt
\]

or

\[
Q_{\varrho_{z}}(\theta_{z;\varrho_{z}}) (z) \subset Q_{\tilde{\varrho}_{z}}(\theta_{z;\tilde{\varrho}_{z}}) (z) \subset Q_{2\tilde{\varrho}_{z}}(\theta_{z;2\tilde{\varrho}_{z}}) (z) \subset Q_{4\tilde{\varrho}_{z}}(\theta_{z;4\tilde{\varrho}_{z}}) (z).
\]
5. Covering argument and the gradient estimate

- Construct a suitable system of cylinders on which the reverse Hölder inequality can be applied.
- Cover the super-level set of $|Du^m|$ with these cylinders in the sense of a Vitali-type covering.
- Applying the reverse Hölder inequality on each of the cylinders, we get a quantitative estimate for $|Du^m|^2$ on the super-level sets in terms of $|Du^m|^{2q_o}$ and $\Psi^2 = (|\partial_t \psi^m|^{\frac{2m}{m-1}} + |D\psi^m| + |F| + |g|)^2$.
- In a standard way, the estimate on the super-level sets leads to the higher integrability estimate for $Du^m$.

Remark
We have a similar result for the singular case in [C. and Scheven, JMAA, 2020].
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and

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Thank you for your attention!
Dziękuję Ci!
감사합니다[gamsahamnida]!