Higher regularity in congested traffic dynamics

Verena Bögelein

Paris Lodron Universität Salzburg

Monday’s Nonstandard Seminar

May 17, 2021
Very degenerate PDEs

Model case of a very degenerate PDE

\[
\text{div} \left( (|\nabla u| - 1)^{p-1} \frac{\nabla u}{|\nabla u|} \right) = f
\]

\[\Leftrightarrow \text{Minimizing the variational integral}\]

\[
F(u) = \int_{\Omega} \left[ \frac{1}{p} (|\nabla u| - 1)^p + fu \right] \, dx
\]

\[h(t) = (t - 1)^p_+\]

\[p = \frac{3}{2}, \quad p = 2, \quad p = 4\]
Very degenerate PDEs

Model case of a very degenerate PDE

\[
\text{div} \left( (|\nabla u| - 1)_+^{p-1} \frac{\nabla u}{|\nabla u|} \right) = f
\]

\[
\iff\text{Minimizing the variational integral}
\]

\[
F(u) = \int_{\Omega} \left[ \frac{1}{p}(|\nabla u| - 1)_+^p + fu \right] \, dx
\]

\[
h(t) = (t - 1)^p_+
\]

\[
p = \frac{3}{2},\, p = 2,\, p = 4
\]
Very degenerate PDEs

Model case of a very degenerate PDE

\[
\text{div} \left( (|\nabla u| - 1)_+^{p-1} \frac{\nabla u}{|\nabla u|} \right) = f
\]

⇔ Minimizing the variational integral

\[
F(u) = \int_{\Omega} \left[ \frac{1}{p}(|\nabla u| - 1)_+^p + fu \right] \, dx
\]

\[
h(t) = (t - 1)_+^p
\]

\[
p = \frac{3}{2}, \quad p = 2, \quad p = 4
\]
Wardrop equilibrium (Wardrop 1952)

Relies on two principles:

- **User equilibrium**: each user chooses the route that is the best $\Rightarrow$ journey times in all routes actually used are equal and less than those that would be experienced by a single vehicle on any unused route

- **System optimality**: average journey time is at a minimum (in particular, users behave cooperatively in choosing their routes to ensure the most efficient use of the whole system)
Model by Monge-Kantorovich problem

- $\Omega \subset \mathbb{R}^n$ (\(\Omega\) models the city for \(n = 2\))
- \(\mu_0, \mu_1\) probability measures on \(\Omega\) (distribution of residents and services in the city \(\Omega\))
- \(\Pi(\mu_0, \mu_1)\): set of transportation plans (probability measures on \(\Omega \times \Omega\) having \(\mu_0\) and \(\mu_1\) as marginals)
- \(c \in C(\overline{\Omega} \times \overline{\Omega}, \mathbb{R})\) cost function

\[
\inf_{\gamma \in \Pi(\mu_0, \mu_1)} \int_{\overline{\Omega} \times \overline{\Omega}} c(x, y) \, d\gamma(x, y).
\]

What is not realistic in this model:
- model is path independent (individual’s travelling strategies are irrelevant)
- congestion effects are not considered (the cost \(c(x, y)\) is independent of “how crowded” the used path is)
Carlier, Jimenez, Santambrogio (2008) introduced the notion of a transportation strategy taking into account

- different possible paths
- congestion effects
This model results in the following minimization problem:

$$\min \left\{ \int_{\Omega} \mathcal{H}(\sigma) \, dx : \sigma \in L^q(\Omega, \mathbb{R}^n), \, \text{div} \, \sigma = \mu_0 - \mu_1, \, \sigma \cdot \nu_{\partial \Omega} = 0 \right\},$$

where $\sigma$ represents the traffic flow and

$$\mathcal{H}(\sigma) = H(|\sigma|), \quad \text{with} \quad H(t) = t + \frac{1}{q} t^q \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1.$$  

The function $g(t) = H'(t) = 1 + t^{q-1}$ models the congestion effect.
By duality one can show that $\sigma = \nabla \mathcal{H}^*(\nabla u)$

- $\mathcal{H}^*$ is the Legendre transform of $\mathcal{H}$
- $u$ solves the Neumann problem

\[
\begin{aligned}
\text{div } \nabla \mathcal{H}^*(\nabla u) &= \mu_0 - \mu_1 & \text{in } \Omega, \\
\nabla \mathcal{H}^*(\nabla u) \cdot \nu &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]
Literature on traffic congestion problem models:

- Warprop (1952)
- Carlier, Jimenez, Santambrogio (2008)
  Derivation of the model and existence of minimizers
- Brasco, Carlier, Santambrogio (2010)
  Characterization by the very degenerate elliptic PDE
- ...
Since

$$\mathcal{H}(\zeta) = \frac{1}{q} |\zeta|^q + |\zeta|$$

we compute

$$\mathcal{H}^*(z) = \frac{1}{p} (|z| - 1)^p_+, \quad \text{where } p = \frac{q}{q-1}.$$ 

This results in the very degenerate PDE

$$\text{div} \left( (|\nabla u| - 1)^{p-1}_+ \frac{\nabla u}{|\nabla u|} \right) = f$$
Weak solutions are **Lipschitz continuous**

- Scalar setting: Brasco & Carlier & Santambrogio, Brasco
- Vectorial setting: Clop & Giova & Hathami & Passarelli di Napoli
- As special case of an asymptotically regular problem: Chipot & Evans, Raymond, Foss, Foss & Passarelli di Papoli & Verde, …

Even if \( f \equiv 0 \): **better than Lipschitz is not possible:**

\[
(\nabla u - 1)_+ = 0 \quad \text{if } |\nabla u| \leq 1
\]
**Sobolev regularity:** $\mathcal{G} \in W^{1,2}$ for $\mathcal{G} = (|\nabla u| - 1)^{\frac{p}{2}} + \frac{\nabla u}{|\nabla u|}$

- Brasco & Carlier & Santambrogio (2010)
- Clop & Giova & Hatami & Passarelli di Napoli (2019): Vector valued case

**Continuity:** $g(\nabla u)$ is continuous for any continuous function $g : \mathbb{R}^n \to \mathbb{R}$ with $g = 0$ on $B_1$

- Santambrogio & Vespri (2010): $n = 2$
- Colombo & Figalli (2014): $n \geq 2$
Vectorial setting

Consider weak solutions $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ with $N \geq 1$ of

$$\text{div} \left( (|Du| - 1)^{p-1} \frac{Du}{|Du|} \right) = f,$$

where $p > 1$ and $f : \Omega \to \mathbb{R}^N$.

What is the optimal regularity?

- **Contra:** Solutions are less regular in the vectorial case (even unbounded; counterexample by De Giorgi)
- **Pro:** Solutions of the $p$-Laplace system

$$\Delta_p u = 0$$

are of class $C^{1,\alpha}$ for some $\alpha > 0$ (first proof by Uhlenbeck)

$$\frac{(|Du|-1)^{p-1}}{|Du|}$$

depends only on the modulus of $Du$

$\Rightarrow$ there is some hope for regularity
Consider weak solutions \( u: \mathbb{R}^n \supset \Omega \to \mathbb{R}^N \) with \( N \geq 1 \) of

\[
\text{div} \left( (|Du| - 1)^{p-1}_+ \frac{Du}{|Du|} \right) = f.
\]

**Theorem (B., Duzaar, Giova, Passarelli di Napoli)**

Let \( p > 1 \) and \( f \in L^{n+\sigma}(\Omega, \mathbb{R}^N) \) for some \( \sigma > 0 \).

Then \( g(Du) \) is continuous for any continuous function \( g: \mathbb{R}^{Nn} \to \mathbb{R} \) vanishing on \( \{||\xi|| \leq 1\} \).
Optimality of the result

- We treat any $p > 1$
- On the set where $|Du| \leq 1$, $Du$ could be discontinuous
- Hölder continuity of $g(Du)$ is not true: counterexample
- $f \in L^n$ is not enough: $Du$ possibly unbounded
Lipschitz regularity. $Du$ is bounded on any compact subset of $\Omega$.

Regularization. Consider solution $u_\varepsilon$ of

$$
\begin{aligned}
\text{div} \left( (|Du_\varepsilon| - 1)^{p-1} \frac{Du_\varepsilon}{|Du_\varepsilon|} \right) + \varepsilon \Delta u_\varepsilon &= f, \\
&\quad \text{in } B_R \subset \Omega, \\
&\quad u_\varepsilon = u, \\
&\quad \text{on } \partial B_R.
\end{aligned}
$$
Lipschitz regularity. $Du$ is bounded on any compact subset of $\Omega$.

Regularization. Consider solution $u_\varepsilon$ of

$$
\begin{aligned}
&\text{div} \left( (|Du_\varepsilon| - 1)^{p-1} \frac{Du_\varepsilon}{|Du_\varepsilon|} \right) + \varepsilon \Delta u_\varepsilon = f, \quad \text{in } B_R \subset \Omega, \\
&u_\varepsilon = u, \quad \text{on } \partial B_R.
\end{aligned}
$$
Hölder-continuity. For any $\delta \in (0, 1]$ 

$G_\delta(Du_\varepsilon)$ is Hölder continuous with exponent $\alpha_\delta$

where $G_\delta(\xi) := \frac{(|\xi|-1-\delta)+}{|\xi|} \xi$.

Constants are independent of $\varepsilon$!

Passage to the limit.
- $\varepsilon \to 0$: $G_\delta(Du)$ is Hölder continuous with exponent $\alpha_\delta$
- $\delta \to 0$: Continuity of 

$$\frac{(|Du|-1)+}{|Du|} Du$$

Continuity of $g(Du)$.
- $\xi \mapsto \frac{(1-|\xi|)+}{|\xi|} \xi$ is invertible on the set $\{|\xi| > 1\}$
Hölder-continuity. For any $\delta \in (0, 1]$

$G_\delta(Du_\varepsilon)$ is Hölder continuous with exponent $\alpha_\delta$

where $G_\delta(\xi) := \frac{|\xi|^{-1-\delta} + |\xi|}{|\xi|} \xi$.

Constants are independent of $\varepsilon$!

Passage to the limit.

- $\varepsilon \to 0$: $G_\delta(Du)$ is Hölder continuous with exponent $\alpha_\delta$
- $\delta \to 0$: Continuity of

$$\frac{(|Du| - 1) + |Du|}{|Du|} Du$$

Continuity of $g(Du)$.

- $\xi \mapsto \frac{(|\xi|^{-1}) + |\xi|}{|\xi|} \xi$ is invertible on the set $\{|\xi| > 1\}$
Hölder-continuity. For any \( \delta \in (0, 1] \)

\[
G_\delta(Du_\varepsilon) \text{ is Hölder continuous with exponent } \alpha_\delta
\]

where \( G_\delta(\xi) := \frac{|\xi|^{-1-\delta}+}{|\xi|} \xi. \)

Constants are independent of \( \varepsilon \)!

Passage to the limit.

\( \varepsilon \to 0: G_\delta(Du) \) is Hölder continuous with exponent \( \alpha_\delta \)

\( \delta \to 0: \) Continuity of

\[
\frac{|Du| - 1}{|Du|} Du
\]

Continuity of \( g(Du) \).

\( \xi \mapsto \frac{|\xi|^{-1}+}{|\xi|} \xi \) is invertible on the set \( \{|\xi| > 1\} \)
Hölder-continuity of $G_\delta(Du_\epsilon)$

Our goal (abbreviate $Du = Du_\epsilon$):

$$\int_{B_r(x_0)} |G_\delta(Du) - \Gamma_{x_0}|^2 \, dx \leq c \left( \frac{r}{\varrho} \right)^{2\alpha} \quad \forall B_r(x_0) \subset B_\varrho(x_0) \subset B_R$$

- Suppose

  $$\sup_{B_\varrho(x_0)} |G_\delta(Du)| \leq \mu$$

- Distinguish between two regimes ($0 < \nu \ll 1$):

  (D) $$|E_\nu(x_0)| \leq (1 - \nu)|B_\varrho(x_0)|,$$

  (ND) $$|E_\nu(x_0)| > (1 - \nu)|B_\varrho(x_0)| \quad \text{and} \quad \mu \geq \delta,$$

where

$$E_\nu(x_0) := B_\varrho(x_0) \cap \{|G_\delta(Du)| > (1 - \nu)\mu\}.$$
Universal energy inequality

In the weak form use the test-function

\[ \varphi = \zeta \phi(|Du|) D_\beta u, \]

where \( \beta \in \{1, \ldots, n\} \), \( \zeta \) cut-off function, \( \phi \) non-negative and non-decreasing. We obtain

\[
\int_{B_R} \left[ \mathcal{A}(D^2 u, D^2 u) \phi(|Du|) + \mathcal{B}(\nabla|Du|, \nabla|Du|) \phi'(|Du|)|Du| \right] \zeta \, dx \\
+ \int_{B_R} \mathcal{B}(\nabla|Du|, \nabla \zeta) \phi(|Du|)|Du| \, dx \leq 0,
\]

where \( \mathcal{A} = \mathcal{A}(Du) \) and \( \mathcal{B} = \mathcal{B}(Du) \) are bilinear forms.
Universal energy inequality

In the weak form use the test-function

\[ \varphi = \zeta \phi(|Du|) D_\beta u, \]

where \( \beta \in \{1, \ldots, n\} \), \( \zeta \) cut-off function, \( \phi \) non-negative and non-decreasing. We obtain

\[
\int_{B_R} \left[ \mathcal{A}(D^2 u, D^2 u) \phi(|Du|) + \mathcal{B}(\nabla|Du|, \nabla|Du|) \phi'(|Du|) |Du| \right] \zeta \, dx \geq \nu |D^2 u|^2
\]

\[
+ \int_{B_R} \mathcal{B}(\nabla|Du|, \nabla \zeta) \phi(|Du|) |Du| \, dx \leq 0,
\]

where \( \mathcal{A} = \mathcal{A}(Du) \) and \( \mathcal{B} = \mathcal{B}(Du) \) are bilinear forms.
Universal energy inequality

In the weak form use the test-function

$$\varphi = \zeta \phi(|Du|) D_\beta u,$$

where $\beta \in \{1, \ldots, n\}$, $\zeta$ cut-off function, $\phi$ non-negative and non-decreasing. We obtain

$$\int_{B_R} \left[ \mathcal{A}(D^2 u, D^2 u) \phi(|Du|) + \mathcal{B}(\nabla|Du|, \nabla|Du|) \phi'(|Du|)|Du| \right] \zeta \, dx \geq 0$$

$$+ \int_{B_R} \mathcal{B}(\nabla|Du|, \nabla \zeta) \phi(|Du|)|Du| \, dx \leq 0,$$

where $\mathcal{A} = \mathcal{A}(Du)$ and $\mathcal{B} = \mathcal{B}(Du)$ are bilinear forms.

$\nabla|Du|$ is subsolution
\( \nabla |Du| \) is a subsolution to an elliptic equation.

Reduction of the supremum by a De Giorgi type argument:

\[
\sup_{B_{\rho/2}(x_0)} |G_\delta(Du)| \leq \kappa \mu, \quad \kappa < 1
\]
Non-degenerate regime

Define the excess

$$\Phi(x_o, \varrho) := \int_{B_{\varrho}(x_o)} \left| Du - (Du)_{x_o, \varrho} \right|^2 \, dx$$

The measure theoretic information yields

$$\Phi(x_o, \varrho) \ll 1 \quad \text{and} \quad |(Du)_{x_o, \varrho}| \geq 1 + \delta + \frac{1}{2} \mu$$

Compare $u$ with the solution $v$ of a linear elliptic system

$$\int_{B_{\varrho/2}(x_o)} |Du - Dv|^2 \, dx \leq c \Phi(x_o, \varrho)^{1+\vartheta} \quad \text{for some } \vartheta > 0$$

Excess decay

$$\Phi(x_o, \tau \varrho) \leq c \tau^2 \Phi(x_o, \varrho) \quad \text{for } \tau \in (0, 1)$$

Iteration

- The limit $\Gamma_{x_o} := \lim_{i \to \infty} \left( G_\delta(Du) \right)_{\tau_i \varrho}$ exists
- Campanato-type estimate: $\int_{B_r} |G_\delta(Du) - \Gamma_{x_o}|^2 \, dx \leq c \left( \frac{r}{\varrho} \right)^{2\beta}$
Non-degenerate regime

- Define the excess
  \[
  \Phi(x_o, \varrho) := \int_{B_{\varrho}(x_o)} |Du - (Du)_{x_o, \varrho}|^2 \, dx
  \]

- The measure theoretic information yields
  \[
  \Phi(x_o, \varrho) \ll 1 \quad \text{and} \quad |(Du)_{x_o, \varrho}| \geq 1 + \delta + \frac{1}{2} \mu
  \]

- Compare \( u \) with the solution \( v \) of a linear elliptic system
  \[
  \int_{B_{\varrho/2}(x_o)} |Du - Dv|^2 \, dx \leq c \Phi(x_o, \varrho)^{1+\vartheta} \quad \text{for some } \vartheta > 0
  \]

- Excess decay
  \[
  \Phi(x_o, \tau \varrho) \leq c \tau^2 \Phi(x_o, \varrho) \quad \text{for } \tau \in (0, 1)
  \]

- Iteration
  - The limit \( \Gamma_{x_o} := \lim_{i \to \infty} \left( G_\delta(Du) \right)_{\tau_i \varrho} \) exists
  - Campanato-type estimate: \( \int_{B_r} |G_\delta(Du) - \Gamma_{x_o}|^2 \, dx \leq c \left( \frac{r}{\varrho} \right)^{2\beta} \)
Define the excess

\[ \Phi(x_0, \varrho) := \int_{B_\varrho(x_0)} \left| Du - (Du)_{x_0, \varrho} \right|^2 \, dx \]

The measure theoretic information yields

\[ \Phi(x_0, \varrho) \ll 1 \quad \text{and} \quad |(Du)_{x_0, \varrho}| \geq 1 + \delta + \frac{1}{2} \mu \]

Compare \( u \) with the solution \( v \) of a linear elliptic system

\[ \int_{B_{\varrho/2}(x_0)} |Du - Dv|^2 \, dx \leq c \Phi(x_0, \varrho)^{1+\vartheta} \quad \text{for some } \vartheta > 0 \]

Excess decay

\[ \Phi(x_0, \tau \varrho) \leq c \tau^2 \Phi(x_0, \varrho) \quad \text{for } \tau \in (0, 1) \]

Iteration

- The limit \( \Gamma_{x_0} := \lim_{i \to \infty} \left(G_\delta(Du)\right)_{\tau_i \varrho} \) exists
- Campanato-type estimate: \( \int_{B_r} \left| G_\delta(Du) - \Gamma_{x_0} \right|^2 \, dx \leq c \left(\frac{r}{\varrho}\right)^{2\beta} \)
Define the excess

\[ \Phi(x_o, \varrho) := \int_{B_{\varrho}(x_o)} \left| Du - (Du)_{x_o, \varrho} \right|^2 \, dx \]

The measure theoretic information yields

\[ \Phi(x_o, \varrho) \ll 1 \quad \text{and} \quad |(Du)_{x_o, \varrho}| \geq 1 + \delta + \frac{1}{2} \mu \]

Compare \( u \) with the solution \( v \) of a linear elliptic system

\[ \int_{B_{\varrho/2}(x_o)} |Du - Dv|^2 \, dx \leq c \Phi(x_o, \varrho)^{1+\vartheta} \quad \text{for some } \vartheta > 0 \]

Excess decay

\[ \Phi(x_o, \tau \varrho) \leq c \tau^2 \Phi(x_o, \varrho) \quad \text{for } \tau \in (0, 1) \]

Iteration

- The limit \( \Gamma_{x_o} := \lim_{i \to \infty} \left( G_\delta(Du) \right)_{\tau_i, \varrho} \) exists
- Campanato-type estimate: \( \int_{B_r} \left| G_\delta(Du) - \Gamma_{x_o} \right|^2 \, dx \leq c \left( \frac{r}{\varrho} \right)^{2\beta} \)
Non-degenerate regime

- Define the excess
  \[ \Phi(x_o, \varrho) := \int_{B_\varrho(x_o)} |Du - (Du)_{x_o, \varrho}|^2 \, dx \]

- The measure theoretic information yields
  \[ \Phi(x_o, \varrho) \ll 1 \quad \text{and} \quad |(Du)_{x_o, \varrho}| \geq 1 + \delta + \frac{1}{2} \mu \]

- Compare \( u \) with the solution \( v \) of a linear elliptic system
  \[ \int_{B_{\varrho/2}(x_o)} |Du - Dv|^2 \, dx \leq c \Phi(x_o, \varrho)^{1+\vartheta} \quad \text{for some } \vartheta > 0 \]

- Excess decay
  \[ \Phi(x_o, \tau \varrho) \leq c \tau^2 \Phi(x_o, \varrho) \quad \text{for } \tau \in (0, 1) \]

- Iteration
  - The limit \( \Gamma_{x_o} := \lim_{i \to \infty} \left( G_\delta(Du) \right)_{\tau_i \varrho} \) exists
  - Campanato-type estimate: \( \int_{B_r} |G_\delta(Du) - \Gamma_{x_o}|^2 \, dx \leq c \left( \frac{r}{\varrho} \right)^{2\beta} \)
Define $\varrho^i = 2^{-i} \varrho$

Suppose that (D) is satisfied on $B_{\varrho_i}(x_o)$ for $i = 0, \ldots, i_o - 1$:

$$\sup_{B_{\varrho_i}(x_o)} |G_\delta(Du)| \leq \kappa^i \mu =: \mu_i$$

Suppose that (D) is not satisfied on $B_{\varrho_{i_o}}(x_o)$

$$\int_{B_r} |G_\delta(Du) - \Gamma_{x_o}|^2 \, dx \leq c \left( \frac{r}{\varrho_{i_o}} \right)^{2\beta} \text{ for } r \leq \varrho_{i_o}$$

$\Rightarrow$ Campanato type estimate for $G_\delta(Du)$
Combining both regimes

- Define $\varrho^i = 2^{-i} \varrho$

- Suppose that (D) is satisfied on $B_{\varrho_i}(x_o)$ for $i = 0, \ldots, i_o - 1$:

  $$\sup_{B_{\varrho_i}(x_o)} |G_\delta(Du)| \leq \kappa^i \mu =: \mu_i$$

- Suppose that (D) is not satisfied on $B_{\varrho_{i_o}}(x_o)$

  $$\int_{B_r} |G_\delta(Du) - \Gamma_{x_o}|^2 \, dx \leq c \left( \frac{r}{\varrho_{i_o}} \right)^{2\beta} \text{ for } r \leq \varrho_{i_o}$$

- $\Rightarrow$ Campanato type estimate for $G_\delta(Du)$
Combining both regimes

- Define $\varrho^i = 2^{-i} \varrho$

- Suppose that (D) is satisfied on $B_{\varrho^i}(x_o)$ for $i = 0, \ldots, i_o - 1$:
  \[
  \sup_{B_{\varrho^i}(x_o)} |G_\delta(Du)| \leq \kappa^i \mu =: \mu_i
  \]

- Suppose that (D) is not satisfied on $B_{\varrho_{i_o}}(x_o)$
  \[
  \int_{B_r} |G_\delta(Du) - \Gamma_{x_o}|^2 \, dx \leq c \left( \frac{r}{\varrho_{i_o}} \right)^{2\beta} \text{ for } r \leq \varrho_{i_o}
  \]

\[\Rightarrow\] Campanato type estimate for $G_\delta(Du)$
Combining both regimes

- Define \( \varrho^i = 2^{-i} \varrho \)

- Suppose that (D) is satisfied on \( B_{\varrho^i}(x_0) \) for \( i = 0, \ldots, i_o - 1 \):
  \[
  \sup_{B_{\varrho^i}(x_0)} |G_\delta(Du)| \leq \kappa^i \mu =: \mu_i
  \]

- Suppose that (D) is not satisfied on \( B_{\varrho_{i_o}}(x_0) \)

  \[
  \int_{B_r} |G_\delta(Du) - \Gamma_{x_0}|^2 \, dx \leq c \left( \frac{r}{\varrho_{i_o}} \right)^{2\beta} \quad \text{for } r \leq \varrho_{i_o}
  \]

- \( \Rightarrow \) Campanato type estimate for \( G_\delta(Du) \)