Gradient estimates of very weak solutions to nonlinear equations with nonstandard growth

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Consider variational integrals

\[ w \in W^{1,1} \rightarrow \int_{\Omega} f(x, Dw(x)) \, dx, \]

\( f(x, \xi) \) denoting a given Lagrangian and \( \Omega \) a bounded domain in \( \mathbb{R}^n \) with \( n \geq 2 \).

Standard growth condition is

\[ |\xi|^p \lesssim f(x, \xi) \lesssim |\xi|^p + 1 \]

for \( 1 < p < \infty \).
Nonstandard growth condition is

$$|\xi|^p \lesssim f(x, \xi) \lesssim |\xi|^q + 1$$

for $1 < p < q < \infty$.

Regularity is to be expected if $p$ and $q$ are not too far away, as observed by Paolo Marcellini in the late ’80s.

A natural and the best/simplist example is the $G$-Laplacian considered by Gary M. Lieberman in the early ’90s.
Assume $G \in C^2(0, \infty)$, $g = G'$, is an N-function such that $0 < \delta_0 \leq \frac{g'(t)t}{g(t)} \leq g_0$ for some constants $\delta_0$ and $g_0$.

The problem under consideration is

$$\begin{cases}
\operatorname{div} a(x, Dx) = \operatorname{div} \left( \frac{g(|F|)}{|F|} F \right) & \text{in} \quad \Omega, \\
u \geq 0 & \text{on} \quad \partial\Omega.
\end{cases}$$

(1)

The given Carathéodory function $a = a(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to satisfy

1. $|a(x, \xi)| + |\xi||D_\xi a(x, \xi)| \leq Lg(|\xi|)$ and
2. $D_\xi a(x, \xi)z \cdot z \geq \nu \frac{g(|\xi|)}{|\xi|} |z|^2$

for all $x, \xi \neq 0$, $z \in \mathbb{R}^n$ and some constants $0 < \nu \leq 1 \leq L$.

$F \in L^g(\Omega, \mathbb{R}^n)$ is given.
Nonlinear elliptic problems of G-Laplacian type

- $G \in \Delta_2 \cap \nabla_2$.
- There exists a small constant $\delta_1 = \delta_1(\delta_0, g_0) \in (0, \min\{1, g_0\})$ such that $G^{1-\delta_1} \in \Delta_2 \cap \nabla_2$.
- For $g(t) = t^{p-1}$, it becomes the $p$-Laplacian.
Definition

\( u \in W_{0}^{1,g}(\Omega) \) is a very weak solution of (1) if

\[
\int_{\Omega} a(x, Du) D\varphi \, dx = \int_{\Omega} \frac{g(|F|)}{|F|} FD\varphi \, dx
\]

for all \( \varphi \in C_{0}^{\infty}(\Omega) \).
Theorem

There exists a small positive constant $\tilde{\delta} = \tilde{\delta}(n, \delta_0, g_0, \nu, L)$ such that if $F \in W^{1, G^{1-\tilde{\delta}}}(\Omega, \mathbb{R}^n)$ for all $\delta \in \left(0, \frac{1}{2}\tilde{\delta}\right)$, then any very weak solution $u \in W^{1, G^{1-\tilde{\delta}}}_{loc}(\Omega)$ to the problem (1) satisfies

$$u \in W^{1, G^{1-\delta}}_{loc}(\Omega)$$

and for each open $\Omega' \subset \subset \Omega$ we have the estimate

$$\|u\|_{W^{1, G^{1-\delta}}(\Omega')} \leq c \left(\|u\|_{W^{1, G^{1-\tilde{\delta}}}(\Omega)} + \|F\|_{W^{1, G^{1-\delta}}(\Omega)}\right),$$

the constant $c$ independent of $F$ and $u$. 
We look at

\[ \text{div} a(x, Dx) = \mu \text{ in } \Omega \]

whose distributional formulation is

\[ \int_{\Omega} a(x, Du) D\varphi \, dx = \langle \mu, \varphi \rangle \quad (\varphi \in C^\infty_0(\Omega)). \]

Due to monotone operators theory, one can assert existence, uniqueness and regularity of weak solutions in \( W^{1,G}_0(\Omega) \) when \( \mu \in \left( W^{1,G}_0(\Omega) \right)^* \). This is the dual case and it lies in the realm of weak solutions.
We first look at the case when $a(x, \xi) \approx |\xi|^{p-2} \xi$, and $g(t) = t^{p-1}$.

We consider when the right hand side is in divergence form

$$\mu = \text{div}(|F|^{p-2}F) \quad (F \in L^q(\Omega) \text{ with } p \leq q < \infty).$$

An issue is to show that

$$F \in L^q \implies Du \in L^q.$$

Iwaniec, Studia Math. ’83 for the elliptic case.
The dual case

- We next look at the case when $a(x, \xi) \approx \frac{g(|F|)}{|F|} F$.
- The right hand side is in divergence form

$$\mu = \text{div} \left( \frac{g(|\xi|)}{|\xi|} \xi \right) \quad \left( F \in L^H(\Omega) \text{ with } G \prec H \right).$$

- An issue is to show that

$$F \in L^H \implies Du \in L^H.$$

- Anna Verde, J. Convex Anal. ’05 for the elliptic case.
We return again to the case when \( a(x, \xi) \approx |\xi|^{p-2} \xi \), and \( g(t) = t^{p-1} \).

The distributional formulation,

\[
\int_{\Omega} a(x, Du) D\varphi \, dx = \langle \mu, \varphi \rangle \quad (\varphi \in C_0^\infty(\Omega)),
\]

is well-defined even when \( Du \in L^{p-1} \) and we may not have finite \( L^p \)-energy.

This is the case below the duality exponent.
We consider again when the right hand side is in divergence form

$$\mu = \text{div}(|F|^{p-2}F) \quad (F \in L^q(\Omega) \text{ with } p - 1 < q < p).$$

An issue is to show that

$$F \in L^q \implies Du \in L^q.$$
Gradient Estimates for very weak solutions

Very weak solution

\[ p - \delta \leq q \] with \( \delta \) very small

Gradient Estimates for very weak solutions

Very weak solution

$$\text{div} a(x, Du) = \mu$$

- When $$\mu \in L^\gamma$$ for $$1 < \gamma < (p^*)'$$, Mingione, Math. Ann. ’10 for the $$p$$-Laplacian type.
- Chlebicka, Nonlinear Analysis 20 for the $$G$$-Laplacian type.
- Of course, there have many noteworthy works when $$\mu$$ is a bounded Radon measure or $$L^1$$-data.
\[ \text{div} a(x, du) = \text{div} \left( \frac{g(|F|)}{|F|} F \right) \]

- There are few results when \( F \in L^{G^1-\delta} \).
- References:
  - Diening, Malek and Steinhauer, On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications. ESAIM Control Optim. Calc. Var. 14 (2008).
Let $0 < \delta \leq \frac{1}{2} \delta_1$, being determined later.

Let $u \in W_0^{1,G^{1-\delta}}(\Omega)$ be a very weak solution.

Let $B_{2r} = B_{2r}(x_0) \subset \subset \Omega$ be a ball.

Choose a standard cut-off function $\eta \in C_0^1(B_{2r})$ such that $\eta = 1$ on $B_r$, $0 \leq \eta \leq 1$ and $|D\eta| \leq \frac{2}{r}$.

Set $\tilde{u} = (u - \bar{u}_{B_{2r}}) \eta \in W_0^{1,G^{1-2\delta}}(B_{2r})$.

For any $q \in (1 - \delta_1, 1 - 2\delta]$, define $h(x) = M_{\frac{1}{q}} \left( G^q(|D\tilde{u}|) \right)(x)$ for $x \in \Omega$, where $M$ is the Hardy-Littlewood maximal operator.

Note that $\frac{1}{q} > 1$ and $h^{-\delta}$ is in the Muckenhoupt class $A_{\frac{1}{q}}$. 
Lemma

Let $\lambda > 0$ and set

$$E_\lambda = E_\lambda(h, B_{2r}) = \{x \in B_{2r} : h(x) \leq \lambda\}.$$ 

Then there exists a Lipschitz truncation $\tilde{u}_\lambda$ of $\tilde{u}$ such that

- $\tilde{u}_\lambda = \tilde{u}$ and $D\tilde{u}_\lambda = D\tilde{u}$ a.e. in $E_\lambda$,
- $\tilde{u}_\lambda$ has support within $E_\lambda$, and
- $G(|D\tilde{u}_\lambda|) \leq c\lambda$ a.e. in $\mathbb{R}^n$ and for some constant $c(\delta_0, g_0, n) > 1$. 
Take $\tilde{u}_\lambda$ as a test function, multiply the resulting distributional formulation by $\lambda^{-(1+\delta)}$, and integrate with respect to $\lambda$, to discover

$$\int_0^\infty \lambda^{-(1+\delta)} \int_{E_\lambda} a(Du, x) D\tilde{u}_\lambda \, dx \, d\lambda$$

$$I_1 = - \int_0^\infty \lambda^{-(1+\delta)} \int_{B_{2r} \setminus E_\lambda} a(Du, x) D\tilde{u}_\lambda \, dx \, d\lambda$$

$$I_2 = \int_0^\infty \lambda^{-(1+\delta)} \int_{E_\lambda} g(|F|) \frac{|F|}{|F|} FD\tilde{u}_\lambda \, dx \, d\lambda$$

$$I_3 = \int_0^\infty \lambda^{-(1+\delta)} \int_{B_{2r} \setminus E_\lambda} g(|F|) \frac{|F|}{|F|} FD\tilde{u}_\lambda \, dx \, d\lambda.$$

$$I_4$$
Estimate of $I_3$

\[
I_3 \leq \int_0^\infty \lambda^{-(1+\delta)} \int_{E_\lambda} g(|F|) |D\tilde{u}| \, dx \, d\lambda \\
= \int_{B_{2r}} g(|F|) |D\tilde{u}| \int_{h(x)}^\infty \lambda^{-(1+\delta)} \, d\lambda \, dx \\
= \frac{1}{\delta} \int_{B_{2r}} g(|F|) |D\tilde{u}| h^{-1}(1+\delta) \, dx \\
\ll \frac{1}{\delta} \int_{B_{2r}} g(|F|) |D\tilde{u}| G^{-(1+\delta)}(|D\tilde{u}|) \, dx \\
\ll \epsilon \frac{1}{\delta} \int_{B_{2r}} G^{1-\delta}(|Du|) \, dx \\
+ c(\epsilon) \frac{1}{\delta} \int_{B_{2r}} G^{1-\delta}(|F|) \, dx.
\]
Very weak solution

Estimate of $I_4$

\[ I_4 \leq \int_0^\infty \lambda^{-(1+\delta)} \int_{B_{2r} \setminus E_\lambda} g(|F|)|D\tilde{u}_\lambda| dxd\lambda \]

\[ = \int_{B_{2r}} g(|F|) \int_0^{h(x)} \lambda^{-(1+\delta)} |D\tilde{u}_\lambda| d\lambda dx \]

\[ \lesssim \int_{B_{2r}} g(|F|) \int_0^{h(x)} \lambda^{-(1+\delta)} G^{-1}(\lambda) d\lambda dx \]

\[ \lesssim \int_{B_{2r}} g(|F|) h^{-\delta} G^{-1}(h) dx \]

\[ \lesssim \int_{B_{2r}} G^{1-\delta}(|Du|) dx + \int_{B_{2r}} G^{1-\delta}(|F|) dx. \]
Estimate of $I_2$

\[
I_2 \leq \int_{0}^{\infty} \lambda^{-(1+\delta)} \int_{B_{2r} \setminus E_{\lambda}} g(|Du|)|D\tilde{u}_\lambda| \, dx \, d\lambda
\]

\[
= \int_{B_{2r}} g(|Du|) \int_{0}^{h(x)} \lambda^{-(1+\delta)}|D\tilde{u}_\lambda| \, d\lambda \, dx
\]

\[
\lesssim \int_{B_{2r}} g(|Du|) \int_{0}^{h(x)} \lambda^{-(1+\delta)} G^{-1}(\lambda) \, d\lambda \, dx
\]

\[
\lesssim \int_{B_{2r}} g(|Du|) h^{-\delta} G^{-1}(h) \, dx
\]

\[
\lesssim \int_{B_{2r}} G^{1-\delta}(|Du|) \, dx.
\]
Partition $B_{2r}$ into $B_r$,
$$D_1 = \{ x \in B_{2r} \setminus B_r : M_{q}^{\frac{1}{q}}(G^{q}(\|D\tilde{u}\|))(x) \leq \delta M_{q}^{\frac{1}{q}}(G^{q}(\|Du\|))(x) \}$$
and
$$D_2 = \{ x \in B_{2r} \setminus B_r : M_{q}^{\frac{1}{q}}(G^{q}(\|D\tilde{u}\|))(x) > \delta M_{q}^{\frac{1}{q}}(G^{q}(\|Du\|))(x) \}$$
to see

$$\delta l_1 = \int_{B_{2r}} a(x, Du) D\tilde{u} h^{-\delta} \, dx$$

$$= \int_{B_r} a(x, Du) D\tilde{u} h^{-\delta} \, dx$$

$$= \underbrace{\int_{B_r} a(x, Du) D\tilde{u} h^{-\delta} \, dx}_{l_{11}}$$

$$+ \int_{D_1} a(x, Du) D\tilde{u} h^{-\delta} \, dx + \int_{D_2} a(x, Du) D\tilde{u} h^{-\delta} \, dx .$$

$$= \underbrace{\int_{D_1} a(x, Du) D\tilde{u} h^{-\delta} \, dx}_{l_{12}} + \underbrace{\int_{D_2} a(x, Du) D\tilde{u} h^{-\delta} \, dx}_{l_{13}}.$$
Estimate of $I_{11}$

Recall $h(x) = M^{\frac{1}{q}}(G^q(|D\tilde{u}|))(x)$ and that $h^{-\delta}$ is in the Muckenhoupt class $A^{\frac{1}{q}}$. Consequently,

$$I_{11} = \int_{B_r} a(x, Du) Du h^{-\delta} dx$$

$$\geq \nu \int_{B_r} G(|Du|) h^{-\delta} dx$$

$$\geq \nu \int_{B_r} M^{\frac{1}{q}}(G^q(|D\tilde{u}|)) h^{-\delta} dx$$

$$\geq \nu \int_{B_{\frac{r}{2}}} G^{1-\delta}(|Du|) dx$$

$$- r^n \left[ \frac{1}{r^n} \int_{B_{2r}} G^q(|Du|) dx \right]^{\frac{1-\delta}{q}}.$$
Estimates of $I_{12}$ & $I_{12}$

$$|I_{12}| \lesssim \delta^q \int_{B_{2r}} G^{1-\delta}(|Du|) dx.$$  

$$|I_{13}| \lesssim \epsilon \int_{B_{2r}} G^{1-\delta}(|Du|) dx$$

$$+ c(\epsilon) r^n \left[ \frac{1}{r^n} \int_{B_{2r}} G^q(|Du|) dx \right]^{\frac{1-\delta}{q}}.$$
Gradient Estimates for very weak solutions

**Very weak solution**

### Interior $G^{1-\delta}(|Du|)$ estimates

We combine all estimates to conclude that there exists a positive constant $c_* = c_* (n, \nu, L, \delta_0, g_0)$ such that

$$
\int_{B_{\frac{r}{2}}} G^{1-\delta}(|Du|) \, dx \leq c_* (\epsilon + \delta^q) \int_{B_{2r}} G^{1-\delta}(|Du|) \, dx
$$

$$
+ [1 + c(\epsilon)] r^n \left[ \frac{1}{r^n} \int_{B_{2r}} G^q(|Du|) \, dx \right]^{\frac{1-\delta}{q}}
$$

$$
+ c(\epsilon) \int_{B_{2r}} G^{1-\delta}(|F|) \, dx
$$

First choose $\epsilon = \delta^q$ and then select a small constant $\delta$ so that

$$
0 < 2c_* \delta^p < 1
$$

to derive the desired estimate.
Related works

- Boundary $G^{1-\delta}(|Du|)$ estimates.
- Parabolic problems.