A variational approach to fluid-structure interactions

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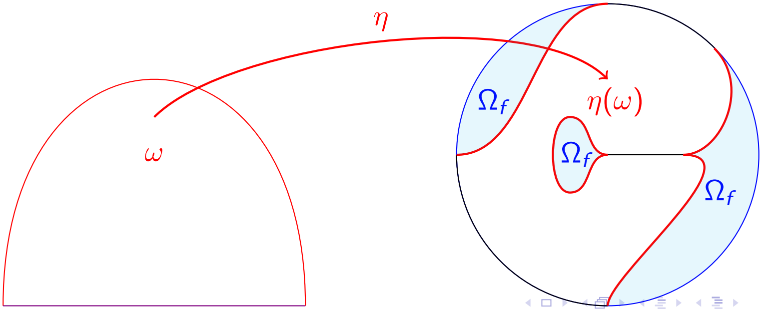
Fluid-Structure interactions

1. $\Omega = \Omega_f \cup \Omega_s \subset \mathbb{R}^3$ is the (Eulerian) domain under investigation.
2. The solid will be situated on Ω_s which is characterized by its **deformation** $\eta : \omega \to \eta(\omega) = \Omega_s \subset \Omega$ in **Lagrangian** coordinates.
3. The fluid will be contained in Ω_f, prescribed in **Eulerian** coordinates by its **velocity** $v : \Omega_f \to \mathbb{R}^3$ and its pressure $p : \Omega_f \to \mathbb{R}$.
4. The velocities and stresses are in equilibrium at the interface.



The solid–and why a variational approach is needed

Unsteady solutions are (formally) given by:

$$\rho_s \partial_t^2 \eta + \operatorname{div}\sigma = \rho_s f \circ \eta \text{ in } [0, T] \times \omega.$$

where $\operatorname{div}\sigma = E'(\eta) + D_2 R(\eta, \partial_t \eta)$.
Here E is the elastic potential of the deformation and R is the dissipation potential.
The (regularized) Saint Venant-Kirchhoff energy as prototype:

$$E(\eta) := \int_Q \frac{1}{8}\left(\mathcal{C}(\nabla\eta^T\nabla\eta - I)\right) \cdot \left(\nabla\eta^T\nabla\eta - I\right) + \frac{1}{(\det \nabla\eta)^a} + \frac{1}{q}\left|\nabla^2\eta\right|^q dx.$$

Steady solutions are considered to be **minimizers** over **non-convex sets** (e.g. $\{\eta \in W^{2,q}(\omega) : \det(\nabla\eta) > 0\}$).
Problem: Energy is **not convex**–minimizers are not unique, no linearisation is possible, no fixed point methods...

Kelvin Voigt dissipation potential: $R(\eta, \partial_t\eta) = \frac{1}{2}\int_Q |\partial_t(\nabla\eta^T\nabla\eta)|^2 dx.$

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The weak formulation for a quasi-steady model

Strong formulation: For all $t \in [0, T]$ (with simplified dissipation):

$$
\begin{aligned}
\mathrm{div}\sigma(\eta, \partial_t \eta) &= \rho_s f \circ \eta && \text{in } \omega, \\
\mathrm{div}\sigma(\eta, \partial_t \eta) &= DE(\eta) - \Delta \partial_t \eta && \text{in } \omega, \\
-\Delta v + \nabla p &= \rho_f f && \text{on } \Omega \setminus \eta(t, \omega), \\
\mathrm{div} v &= 0 && \text{on } \Omega \setminus \eta(t, \omega).
\end{aligned}
$$

At the interface: $\partial_t \eta(t, x) = v \circ \eta(t, x)$
and $\sigma(t, x) n(x) = \left(\nabla_{sym} v(t, \eta(t, x)) \cdot \hat{n} + p(t, \eta(t, x)) I \right) \hat{n}$.
Coupled weak formulation:

$$
\int_0^T \langle E'(\eta), \varphi \rangle_\omega + \langle \nabla \partial_t \eta, \nabla \varphi \rangle_\omega + \langle \nabla_{sym} v, \nabla \xi \rangle_{\Omega_f(t)} - \langle p, \mathrm{div} \xi \rangle_{\Omega_f(t)} \, dt
$$

$$
= \int_0^T \rho_f \langle f, \xi \rangle_{\Omega_f(t)} + \rho_s \langle f \circ \eta, \varphi \rangle_\omega dt
$$

for all smooth (φ, ξ) satisfying $\varphi = \xi \circ \eta$ on ω.

How to couple fluid and solid variationally

Theorem: *There exists a solution to the quasi-steady FSI (until collision).*
Proof: Via De Giorgi's minimizing movements.
Inductive time-stepping $t_k - t_{k-1} = \tau$, $N\tau = T$.
Principle: Make the scheme *explicit* w.r.t. fluid-domain, but *implicit* w.r.t. the coupled Dirichlet boundary values.
Assume $\exists \eta_{k-1} : \omega \to \Omega$ and $v_{k-1} : \Omega_f^{k-2} \to \mathbb{R}^3$ with
$\frac{\eta_{k-1} - \eta_{k-2}}{\tau} = v_{k-1} \circ \eta_{k-2}$ on $\partial\omega$.
Energy class:

$$\{(\beta, w) \in W^{2,q}_{\mathrm{det}}(\omega) \times W^{1,2}_{\mathrm{div}}(\Omega_f^{k-1}) : \frac{\beta - \eta_{k-1}}{\tau} = w \circ \eta_{k-1} \text{ on } \partial\omega\}.$$

$(\eta_k, v_k) =$ arg min of

$$\int_\omega \frac{|\nabla(\beta - \eta_{k-1})|^2}{2\tau}\, dx + E(\beta) + \tau \int_{\Omega_f^{k-1}} \frac{|\nabla_{sym} w|^2}{2}\, dy - \langle \rho_s f \circ \eta_{k-1}, \beta \rangle_\omega - \tau \langle \rho_f f, w \rangle_{\Omega_f^{k-1}}.$$

Estimates and Euler-Lagrange

We take $(\eta_{k-1}, 0)$ as competitor and find

$$\tau \int_\omega \frac{|\nabla(\eta_k - \eta_{k-1})|^2}{2\tau^2}\, dx + E(\eta_k) + \tau \int_{\Omega_f^{k-1}} \frac{|\nabla_{sym} v_k|^2}{2}\, dy$$

$$\leq E(\eta_{k-1}) + \tau \|f\|_\infty \left(\|v_k\|_{\Omega_f^{k-1}} + \left\| \frac{\eta_k - \eta_{k-1}}{\tau} \right\|_\omega \right)$$

Korn's inequality implies estimates.

Take (ξ, φ) such that $\xi \circ \eta_{k-1} = \varphi$
$\implies v_k \circ \eta_{k-1} + \frac{\xi}{\tau} \circ \eta_{k-1} = \frac{\eta_k - \eta_{k-1} + \varphi}{\tau}$ (on $\partial\omega$).

$$\langle E'(\eta_k), \varphi \rangle_\omega + \left\langle \nabla \frac{\eta_k - \eta_{k-1}}{\tau}, \nabla\varphi \right\rangle_\omega + \langle \nabla_{sym} v_k, \nabla\xi \rangle_{\Omega_f^{k-1}}$$

$$= \rho_f \langle f, \xi \rangle_{\Omega_f^{k-1}} + \rho_s \langle f \circ \eta, \varphi \rangle_\omega$$

$E'(\eta)$ exists due to the regularizing potentials.
No previous literature on FSI involving large deformations.
Small deformation: (Grandemont 2002).

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Hyperbolic minimizing movements

Theorem: *There exists a weak solution to* $\rho_s\partial_t^2\eta + E'(\eta) - \Delta\partial_t\eta = \rho_s f\circ\eta$.

Parabolic De Giorgi's MM. Inductive time-stepping. $t_k - t_{k-1} = \tau$:

$$\eta_k = \arg\min_\beta \int_\omega \frac{|\nabla(\beta-\eta_{k-1})|^2}{2\tau}\,dx + E(\beta) - \rho_s\langle f\circ\eta_{k-1},\beta\rangle$$

$$\text{E-L}\ :\langle\frac{\nabla(\eta_k-\eta_{k-1})}{\tau},\nabla\varphi\rangle + \langle E'(\eta_k),\varphi\rangle = \rho_s\langle f\circ\eta_{k-1},\varphi\rangle.$$

Hyperbolic De Giorgi's MM. Introduce: $h = N\tau$ as *acceleration time-scale.*
Assume $\exists\{\eta_1^{\ell-1},....,\eta_N^{\ell-1}\}$ define:

$$\eta_k^\ell = \arg\min_\beta \int_\omega \rho_s\left|\frac{\frac{\beta-\eta_{k-1}^\ell}{\tau}-\frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}}{2h}\right|^2 + \frac{|\nabla(\beta-\eta_{k-1}^\ell)|^2}{2\tau}\,dx + E(\beta) - \rho_s\langle f\circ\eta_{k-1},\beta\rangle$$

$$\text{E-L: }\rho_s\langle\frac{\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}-\frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}}{h},\varphi\rangle + \langle\frac{\nabla(\eta_k-\eta_{k-1})}{\tau},\nabla\varphi\rangle + \langle E'(\eta_k),\varphi\rangle = \rho_s\langle f\circ\eta_{k-1},\varphi\rangle$$

Important: Hyperbolic a-priori estimates are by E-L Equations!

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Solids coupled to Navier Stokes equations

Aim: Find solutions to:

$$\begin{aligned} \rho_s \partial_t^2 \eta + E'(\eta) - \Delta \partial_t \eta &= \rho_s f \circ \eta && \text{in } \omega, \\ \rho_f(\partial_t v + [\nabla v]v) &= \Delta v - \nabla p + \rho_f f && \text{on } \Omega \setminus \eta(t,\omega), \\ \operatorname{div} v &= 0 && \text{on } \Omega \setminus \eta(t,\omega). \end{aligned}$$

And coupling of velocities and stresses at the interface.

Introduce the global velocity:

$$u : [0,T] \times \Omega \to \mathbb{R}^d, \quad u|_{\Omega_f} = v|_{\Omega_f}, \quad u|_{\Omega_s} = \partial_t \eta \circ \eta^{-1}.$$

Define $\Phi : [0,T] \times \Omega \to \Omega$, such that $\partial_t \Phi(t,x) = u(t, \Phi(t,x))$.

For $y = \Phi(t,x) \in \Omega_f$: $\partial_t(v(t,\Phi(t,x))) = \partial_t v(t,y) + [\nabla_y v(t,y)]v(t,y)$.

For $x \in \omega$ we find $\Phi(t, \eta(0,x)) = \eta(t,x)$.

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Navier Stokes as *hyperbolic* Minimizing Movements

Let $v_k^{\ell-1} : \Omega_k^{\ell-1} \to \mathbb{R}^d$ be given **and** $\Phi_k^{\ell-1} : \Omega_f(0) \to \Omega_k^{\ell-1} = \Omega \setminus \eta_k^{\ell-1}(\omega)$.
Introduce inductively:

$$v_k^\ell = \arg \min_{w:\Omega_{k-1}^\ell \to \mathbb{R}^d} \int_{\Omega_f(0)} \rho_f \left| \frac{w \circ \Phi_{k-1}^\ell - v^{(\ell-1)} \circ \Phi_{k-1}^{\ell-1}}{2h} \right|^2 dx + \int_{\Omega_{k-1}^\ell} \frac{|\nabla w|^2}{2} \, dy - \rho_f \langle f, w \rangle.$$

Next introduce Φ_k^ℓ such that $\partial_t \Phi_k^\ell = v_k^\ell \circ \Phi_{k-1}^\ell$.
For fixed domains see: (Gigli, Mosconi, 2012).

Important for FSI:

- Make the scheme *explicit* w.r.t. fluid-domain, but implicit w.r.t. the coupled Dirichlet boundary values.
- Do not change (mollify) the domain since this masses with the flow map Φ.
- Beware that the domain where the global velocity is divergence free changes in time.

Open Problem: Existence of steady solutions to fluid-structure interactions.

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Main Result (Benešová, Kampschulte, Sch 2020)

There exist $\eta : [0, T] \times \omega \to \Omega$, $v : [0, T] \times \Omega(t) \to \mathbb{R}^n$ and $p : [0, T] \times \Omega(t) \to \mathbb{R}$, satisfying an energy inequality. And

$$\int_0^T -\rho_s \langle \partial_t \eta, \partial_t \varphi \rangle_\omega - \rho_s \langle v, \partial_t \xi - v \cdot \nabla \xi \rangle_{\Omega_f} \, dt$$
$$+ \int_0^T \langle E'(\eta), \varphi \rangle_\omega + \langle D_2 R(\eta, \partial_t \eta), \varphi \rangle_\omega + \langle \nabla_{sym} v, \nabla_{sym} \xi \rangle_{\Omega_f(t)}$$
$$- \langle p, \mathrm{div} \xi \rangle_{\Omega_f} dt = \int_0^T \rho_s \langle f \circ \eta, \varphi \rangle_\omega + \rho_f \langle f, \xi \rangle_{\Omega_f} dt + \text{initial conditions.}$$

for all (φ, ξ) smooth, with $\varphi(t) = \xi(t) \circ \eta(t)$ on $\partial\omega$.