Monda y’s Nonstandard Seminar 2020/21

Samuele Riccò

Dipartimento di Scienze Fisiche, Informatiche e Matematiche
Università di Modena e Reggio Emilia

REGULARITY FOR OBSTACLE PROBLEMS
WITHOUT STRUCTURE CONDITIONS

Monday March 15th, 2021
The aim of this seminar is to deal with the possible occurrence of the \textit{Lavrentiev phenomenon} on a variational obstacle problem with $p, q$–growth.

The main tool used here is a Lemma which let us move from the variational obstacle problem to the one with the relaxed functional, in order to find the solutions’ regularity we want. We assume the Sobolev regularity both for the gradient of the obstacle and for the coefficients.


Joint project with Dr. G. Bertazzoni

\footnote{Regularity for obstacle problems without structure conditions", preprint arXiv:2102.12906}
Outline of the seminar

The aim of this seminar is to deal with the possible occurrence of the Lavrentiev phenomenon on a variational obstacle problem with $p, q$–growth.

The main tool used here is a Lemma which let us move from the variational obstacle problem to the one with the relaxed functional, in order to find the solutions’ regularity we want. We assume the Sobolev regularity both for the gradient of the obstacle and for the coefficients.


Joint project with Dr. G. Bertazzoni

1"Regularity for obstacle problems without structure conditions", preprint arXiv:2102.12906
Outline of the seminar

The aim of this seminar is to deal with the possible occurrence of the Lavrentiev phenomenon on a variational obstacle problem with $p,q$–growth.

The main tool used here is a Lemma which let us move from the variational obstacle problem to the one with the relaxed functional, in order to find the solutions’ regularity we want. We assume the Sobolev regularity both for the gradient of the obstacle and for the coefficients.


Joint project with Dr. G. Bertazzoni

---

1"Regularity for obstacle problems without structure conditions", preprint arXiv:2102.12906
The aim of this seminar is to deal with the possible occurrence of the Lavrentiev phenomenon on a variational obstacle problem with $p, q$–growth.

The main tool used here is a Lemma which let us move from the variational obstacle problem to the one with the relaxed functional, in order to find the solutions’ regularity we want. We assume the Sobolev regularity both for the gradient of the obstacle and for the coefficients.

---


Joint project with Dr. G. Bertazzoni

\[1\] Regularity for obstacle problems without structure conditions", preprint arXiv:2102.12906
Outline of the seminar

- Motivation
- Statement of the problem and main results
- A priori estimate
- Approximation in case of occurrence of Lavrentiev Phenomenon
Outline of the seminar

• Motivation

• Statement of the problem and main results

• A priori estimate

• Approximation in case of occurrence of Lavrentiev Phenomenon
Outline of the seminar

- Motivation
- Statement of the problem and main results
- A priori estimate
- Approximation in case of occurrence of Lavrentiev Phenomenon
Outline of the seminar

- Motivation
- Statement of the problem and main results
- A priori estimate
- Approximation in case of occurrence of Lavrentiev Phenomenon
MOTIVATION
Motivation

This talk is focused on studying the Lipschitz continuity of the solutions to variational obstacle problems of the form

$$\min \left\{ \int_{\Omega} f(x, Dw) : w \in \mathcal{K}_\psi(\Omega) \right\}$$

in the case of $p, q$–growth condition, where Lavrentiev phenomenon may occur.

The relationship between the ellipticity and the growth exponent we impose is the one considered in a series of papers started with

*M. Eleuteri, P. Marcellini, E. Mascolo
Motivation

This talk is focused on studying the Lipschitz continuity of the solutions to variational obstacle problems of the form

$$\min \left\{ \int_{\Omega} f(x, Dw) : w \in K_{\psi}(\Omega) \right\}$$

in the case of $p, q$–growth condition, where Lavrentiev phenomenon may occur.

The relationship between the ellipticity and the growth exponent we impose is the one considered in a series of papers started with

M. Eleuteri, P. Marcellini, E. Mascolo

The **local boundedness of the gradient** $Dw$ is a fundamental property, in fact, thanks to that, the behavior of $|Dw|$ at infinity becomes irrelevant for further regularity.

- C. De Filippis, G. Mingione - preprint (2020)
- M. Eleuteri, P. Marcellini, E. Mascolo, S. Perrotta - submitted (2021)
Lipschitz continuity of solutions

The local boundedness of the gradient $Dw$ is a fundamental property, in fact, thanks to that, the behavior of $|Dw|$ at infinity becomes irrelevant for further regularity.


C. De Filippis, G. Mingione - preprint (2020)
M. Eleuteri, P. Marcellini, E. Mascolo, S. Perrotta - submitted (2021)
Lipschitz continuity of solutions

The local boundedness of the gradient $Dw$ is a fundamental property, in fact, thanks to that, the behavior of $|Dw|$ at infinity becomes irrelevant for further regularity.


C. De Filippis, G. Mingione - preprint (2020)
M. Eleuteri, P. Marcellini, E. Mascolo, S. Perrotta - submitted (2021)
Lipschitz continuity of solutions

The local boundedness of the gradient $Dw$ is a fundamental property, in fact, thanks to that, the behavior of $|Dw|$ at infinity becomes irrelevant for further regularity.


C. De Filippis, G. Mingione - preprint (2020)
M. Eleuteri, P. Marcellini, E. Mascolo, S. Perrotta - submitted (2021)
$p, q$–growth


MONDAY’S NONSTANDARD SEMINAR
$p, q$–growth


$p, q$–growth


MONDAY’S NONSTANDARD SEMINAR
The Lavrentiev phenomenon is a surprising result first demonstrated in 1926 by M. Lavrentiev that, in our case, it may occurs due to the nonstandard growth conditions required on the lagrangian.

Under our assumptions, this phenomenon can be reformulated in these terms:

$$\inf_{w \in (W^{1,p} \cap \{w \geq \psi\})} \int_{\Omega} f(x, Dw) \, dx < \inf_{w \in (W^{1,q} \cap \{w \geq \psi\})} \int_{\Omega} f(x, Dw) \, dx$$

This is an obstruction to regularity, since it prevents minimizers to belong to $W^{1,q}$. The basic strategy to get regularity results is to exclude the occurrence of Lavrentiev phenomenon by imposing that the Lavrentiev gap vanishes on solutions.
The Lavrentiev phenomenon is a surprising result first demonstrated in 1926 by M. Lavrentiev that, in our case, it may occur due to the nonstandard growth conditions required on the Lagrangian.

Under our assumptions, this phenomenon can be reformulated in these terms:

\[
\inf_{w \in (W^{1,p} \cap \{w \geq \psi\})} \int_{\Omega} f(x, Dw) \, dx < \inf_{w \in (W^{1,q} \cap \{w \geq \psi\})} \int_{\Omega} f(x, Dw) \, dx
\]

This is an obstruction to regularity, since it prevents minimizers to belong to \(W^{1,q}\). The basic strategy to get regularity results is to exclude the occurrence of Lavrentiev phenomenon by imposing that the Lavrentiev gap vanishes on solutions.
The Lavrentiev phenomenon is a surprising result first demonstrated in 1926 by M. Lavrentiev that, in our case, it may occurs due to the nonstandard growth conditions required on the lagrangian.

Under our assumptions, this phenomenon can be reformulated in these terms:

$$\inf_{w \in (W^{1,p} \cap \{w \geq \psi\})} \int_{\Omega} f(x, Dw) \, dx < \inf_{w \in (W^{1,q} \cap \{w \geq \psi\})} \int_{\Omega} f(x, Dw) \, dx$$

This is an obstruction to regularity, since it prevents minimizers to belong to $W^{1,q}$. The basic strategy to get regularity results is to exclude the occurrence of Lavrentiev phenomenon by imposing that the Lavrentiev gap vanishes on solutions.
Lavrentiev phenomenon

G. Buttazzo, M. Belloni - Mathematical Applications (1995)

Lavrentiev phenomenon

G. Buttazzo, M. Belloni - Mathematical Applications (1995)

However, here we adopt a different viewpoint, following the lines of


We present a general Lipschitz regularity result by covering the case in which the Lavrentiev phenomenon may occur. In this respect, a key role will be played by the relaxed functional and by the crucial Lemma which is the natural counterpart of the necessary and sufficient condition to get the absence of Lavrentiev phenomenon.
However, here we adopt a different viewpoint, following the lines of


We present a general Lipschitz regularity result by covering the case in which the Lavrentiev phenomenon may occur. In this respect, a key role will be played by the relaxed functional and by the crucial Lemma which is the natural counterpart of the necessary and sufficient condition to get the absence of Lavrentiev phenomenon.
Sobolev dependence

We consider Sobolev dependence on the obstacle and the partial map $x \mapsto D_\xi f(x, \xi)$.


A. Gentile - preprint (2021)
Sobolev dependence

We consider Sobolev dependence on the obstacle and the partial map $x \mapsto D_{\xi} f(x, \xi)$.


A. Gentile - preprint (2021)
We consider Sobolev dependence on the obstacle and the partial map $x \mapsto D_\xi f(x, \xi)$.


Model functional

Double-phase functional

\[ w \mapsto \int_{\Omega} \left[ |Dw|^p + a(x) (1 + |Dw|^2)^{\frac{q}{2}} \right] \, dx \]

with \( q > p > 1 \) and \( a(\cdot) \) a bounded Sobolev coefficient

Double-phase functional

$$w \mapsto \int_{\Omega} \left[ |Dw|^p + a(x) \left(1 + |Dw|^2\right)^{q/2} \right] \, dx$$

with $q > p > 1$ and $a(\cdot)$ a bounded Sobolev coefficient

STATEMENT OF THE PROBLEM AND MAIN RESULTS
Assumptions

**Variational integral**

\[ F(u) := \int_{\Omega} f(x, Du) \, dx \]

**Obstacle problem**

\[
\min \{ F(u) : u \in \mathcal{K}_\psi(\Omega) \}
\]

- \( \Omega \) is a bounded open set of \( \mathbb{R}^n, \ n \geq 2 \)
- \( f : \Omega \times \mathbb{R}^n \to [0, +\infty) \) is a Carathéodory function, convex and of class \( C^2 \) with respect to the second variable
- \( \psi : \Omega \to [-\infty, +\infty) \), called obstacle, belongs to the Sobolev space \( W^{1,p}(\Omega) \)
- \( \mathcal{K}_\psi(\Omega) := \{ w \in u_0 + W_0^{1,p}(\Omega) : w \geq \psi \text{ a.e. in } \Omega \} \)
- \( u_0 \) is a fixed boundary value. We need to assume \( u_0 \in W^{1,q}(\Omega) \)
Assumptions

\begin{itemize}
  \item \(\Omega\) is a bounded open set of \(\mathbb{R}^n, \ n \geq 2\)
  \item \(f : \Omega \times \mathbb{R}^n \to [0, +\infty)\) is a Carathéodory function, convex and of class \(C^2\) with respect to the second variable
  \item \(\psi : \Omega \to [-\infty, +\infty),\) called obstacle, belongs to the Sobolev space \(W^{1,p}(\Omega)\)
  \item \(K_\psi(\Omega) := \{ w \in u_0 + W^{1,p}_0(\Omega) : w \geq \psi \ \text{a.e. in} \ \Omega \}\)
  \item \(u_0\) is a fixed boundary value. We need to assume \(u_0 \in W^{1,q}(\Omega)\)
\end{itemize}
Assumptions

**Variational integral**

\[ F(u) := \int_{\Omega} f(x, Du) \, dx \]

**Obstacle problem**

\[ \min \{ F(u) : u \in \mathcal{K}_\psi(\Omega) \} \]

- \( \Omega \) is a bounded open set of \( \mathbb{R}^n, n \geq 2 \)
- \( f : \Omega \times \mathbb{R}^n \to [0, +\infty) \) is a Carathéodory function, convex and of class \( C^2 \) with respect to the second variable
- \( \psi : \Omega \to [-\infty, +\infty) \), called obstacle, belongs to the Sobolev space \( W^{1,p}(\Omega) \)
- \( \mathcal{K}_\psi(\Omega) := \{ w \in u_0 + W^{1,p}_0(\Omega) : w \geq \psi \text{ a.e. in } \Omega \} \)
- \( u_0 \) is a fixed boundary value. We need to assume \( u_0 \in W^{1,q}(\Omega) \)
Assumptions

**Variational integral**

\[
F(u) := \int_{\Omega} f(x, Du) \, dx
\]

**Obstacle problem**

\[
\min \{ F(u) : u \in K_\psi(\Omega) \}
\]

- \( \Omega \) is a bounded open set of \( \mathbb{R}^n, \ n \geq 2 \)
- \( f : \Omega \times \mathbb{R}^n \to [0, +\infty) \) is a Carathéodory function, convex and of class \( C^2 \) with respect to the second variable
- \( \psi : \Omega \to [-\infty, +\infty) \), called obstacle, belongs to the Sobolev space \( W^{1,p}(\Omega) \)
- \( K_\psi(\Omega) := \{ w \in u_0 + W_0^{1,p}(\Omega) : w \geq \psi \ a.e. \ in \ \Omega \} \)
- \( u_0 \) is a fixed boundary value. We need to assume \( u_0 \in W^{1,q}(\Omega) \)
Assumptions

Variational integral

\[ F(u) := \int_{\Omega} f(x, Du) \, dx \]

Obstacle problem

\[ \min \{ F(u) : u \in \mathcal{K}_\psi(\Omega) \} \]

- \( \Omega \) is a bounded open set of \( \mathbb{R}^n \), \( n \geq 2 \)
- \( f : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty) \) is a Carathéodory function, convex and of class \( C^2 \) with respect to the second variable
- \( \psi : \Omega \rightarrow [-\infty, +\infty) \), called obstacle, belongs to the Sobolev space \( W^{1,p}(\Omega) \)
- \( \mathcal{K}_\psi(\Omega) := \{ w \in u_0 + W_0^{1,p}(\Omega) : w \geq \psi \text{ a.e. in } \Omega \} \)
- \( u_0 \) is a fixed boundary value. We need to assume \( u_0 \in W^{1,q}(\Omega) \)
Assumptions

Variational integral

\[ \mathcal{F}(u) := \int_{\Omega} f(x, Du) \, dx \]

Obstacle problem

\[ \min \{ \mathcal{F}(u) : u \in \mathcal{K}_\psi(\Omega) \} \]

- \( \Omega \) is a bounded open set of \( \mathbb{R}^n \), \( n \geq 2 \)
- \( f : \Omega \times \mathbb{R}^n \to [0, +\infty) \) is a Carathéodory function, convex and of class \( C^2 \) with respect to the second variable
- \( \psi : \Omega \to [-\infty, +\infty) \), called obstacle, belongs to the Sobolev space \( W^{1,p}(\Omega) \)
- \( \mathcal{K}_\psi(\Omega) := \{ w \in u_0 + W^{1,p}_0(\Omega) : w \geq \psi \text{ a.e. in } \Omega \} \)
- \( u_0 \) is a fixed boundary value. We need to assume \( u_0 \in W^{1,q}(\Omega) \)
### Assumptions

<table>
<thead>
<tr>
<th>Variational integral</th>
<th>Obstacle problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{F}(u) := \int_{\Omega} f(x, Du) , dx )</td>
<td>( \min { \mathcal{F}(u) : u \in \mathcal{K}_\psi(\Omega) } )</td>
</tr>
</tbody>
</table>

- \( \Omega \) is a bounded open set of \( \mathbb{R}^n, n \geq 2 \)
- \( f : \Omega \times \mathbb{R}^n \to [0, +\infty) \) is a Carathéodory function, convex and of class \( C^2 \) with respect to the second variable
- \( \psi : \Omega \to [-\infty, +\infty), \) called obstacle, belongs to the Sobolev space \( W^{1,p}(\Omega) \)
- \( \mathcal{K}_\psi(\Omega) := \{ w \in u_0 + W^{1,p}_0(\Omega) : w \geq \psi \text{ a.e. in } \Omega \} \)
- \( u_0 \) is a fixed boundary value. We need to assume \( u_0 \in W^{1,q}(\Omega) \)
Hypothesis

Exponents condition

\[ 1 \leq \frac{q}{p} < 1 + \frac{r - n}{r \cdot n} = 1 + \frac{1}{n} - \frac{1}{r} \]

where we consider \( q > p \geq 2 \) and where \( r > n \)

Hypothesis

\[ 1 \leq \frac{q}{p} < 1 + \frac{r - n}{r n} = 1 + \frac{1}{n} - \frac{1}{r} \]

where we consider \( q > p \geq 2 \) and where \( r > n \)

We suppose that there exist:

- \( \nu > 0 \) and \( L > 0 \)
- \( h : \Omega \rightarrow [0, +\infty) \) such as \( h(x) \in L^{r}_{\text{loc}}(\Omega) \)

### Hypothesis on functional

\[
\nu \left(1 + |\xi|^2\right)^{\frac{p}{2}} \leq f(x, \xi) \leq L \left(1 + |\xi|^2\right)^{\frac{q}{2}}
\]
\[
\nu \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} |\lambda|^2 \leq \sum_{i,j} f_{\xi_i \xi_j}(x, \xi) \lambda_i \lambda_j \leq L \left(1 + |\xi|^2\right)^{\frac{q-2}{2}} |\lambda|^2
\]
\[
|f_{x \xi}(x, \xi)| \leq h(x) \left(1 + |\xi|^2\right)^{\frac{q-1}{2}}
\]

for all \( \lambda, \xi \in \mathbb{R}^n, \lambda = \lambda_i, \xi = \xi_i, i = 1, 2, \ldots, n \) a.e. in \( \Omega \)
Hypothesis

We suppose that there exist:

- $\nu > 0$ and $L > 0$
- $h : \Omega \rightarrow [0, +\infty)$ such as $h(x) \in L^r_{\text{loc}}(\Omega)$

Hypothesis on functional

$$
\nu (1 + |\xi|^2)^{\frac{p}{2}} \leq f(x, \xi) \leq L (1 + |\xi|^2)^{\frac{q}{2}}
$$

$$
\nu (1 + |\xi|^2)^{\frac{p-2}{2}} |\lambda|^2 \leq \sum_{i,j} f_{\xi_i \xi_j}(x, \xi) \lambda_i \lambda_j \leq L (1 + |\xi|^2)^{\frac{q-2}{2}} |\lambda|^2
$$

$$
|f_{x\xi}(x, \xi)| \leq h(x) (1 + |\xi|^2)^{\frac{q-1}{2}}
$$

for all $\lambda, \xi \in \mathbb{R}^n$, $\lambda = \lambda_i$, $\xi = \xi_i$, $i = 1, 2, \ldots, n$ a.e. in $\Omega$
Hypothesis

We suppose that there exist:

- $\nu > 0$ and $L > 0$
- $h : \Omega \rightarrow [0, +\infty)$ such as $h(x) \in L^r_{loc}(\Omega)$

Hypothesis on functional

\[
\nu \left( 1 + |\xi|^2 \right)^{\frac{p}{2}} \leq f(x, \xi) \leq L \left( 1 + |\xi|^2 \right)^{\frac{q}{2}} \\
\nu \left( 1 + |\xi|^2 \right)^{\frac{p-2}{2}} |\lambda|^2 \leq \sum_{i,j} f_{\xi i \xi j}(x, \xi) \lambda_i \lambda_j \leq L \left( 1 + |\xi|^2 \right)^{\frac{q-2}{2}} |\lambda|^2 \\
|f_{x \xi}(x, \xi)| \leq h(x) \left( 1 + |\xi|^2 \right)^{\frac{q-1}{2}}
\]

for all $\lambda, \xi \in \mathbb{R}^n$, $\lambda = \lambda_i$, $\xi = \xi_i$, $i = 1, 2, \ldots, n$ a.e. in $\Omega$
Hypothesis

We suppose that there exist:

- \( \nu > 0 \) and \( L > 0 \)
- \( h : \Omega \to [0, +\infty) \) such as \( h(x) \in L^r_{\text{loc}}(\Omega) \)

Hypothesis on functional

\[
\nu \left(1 + |\xi|^2\right)^{\frac{p}{2}} \leq f(x, \xi) \leq L \left(1 + |\xi|^2\right)^{\frac{q}{2}} \\
\nu \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} |\lambda|^2 \leq \sum_{i,j} f_{\xi_i \xi_j}(x, \xi) \lambda_i \lambda_j \leq L \left(1 + |\xi|^2\right)^{\frac{q-2}{2}} |\lambda|^2 \\
|f_{x\xi}(x, \xi)| \leq h(x) \left(1 + |\xi|^2\right)^{\frac{q-1}{2}}
\]

for all \( \lambda, \xi \in \mathbb{R}^n, \lambda = \lambda_i, \xi = \xi_i, i = 1, 2, \ldots, n \) a.e. in \( \Omega \)
Theorem (A priori estimate)

Let $u \in K_\psi(\Omega)$ be a smooth solution to the obstacle problem under the assumptions of growth and ellipticity stated before. If $\psi \in W^{2,r}_{loc}(\Omega)$, then $u \in W^{1,\infty}_{loc}(\Omega)$ and the following estimate

$$\|Du\|_{L^\infty(B_\rho)} \leq C \left\{ \int_{B_R} [1 + f(x, Du)] \, dx \right\}^\beta$$

holds for every $0 < \rho < R$ and with positive constants $C$ and $\beta$ depending on $n, r, p, q, \nu, L, R, \rho$ and on the local bounds for $\|D\psi\|_{W^{1,r}}$ and $\|h\|_{L^r}$. 
Now we want to present a meaningful definition of relaxation for the variational obstacle problem we are focusing about.

**Class of solutions**

\[ K^*_\psi(\Omega) := \{ w \in u_0 + W^{1,q}_0(\Omega) : w \geq \psi \text{ a.e. in } \Omega \} \]
Now we want to present a meaningful definition of relaxation for the variational obstacle problem we are focusing about.

**Class of solutions**

\[ \mathcal{K}^*_\psi(\Omega) := \{ w \in u_0 + W^{1,q}_0(\Omega) : w \geq \psi \text{ a.e. in } \Omega \} \]
Relaxation

Now we want to present a meaningful definition of relaxation for the variational obstacle problem we are focusing about.

### Class of solutions

\[ \mathcal{K}_\psi^*(\Omega) := \{ w \in u_0 + W_0^{1,q}(\Omega) : w \geq \psi \text{ a.e. in } \Omega \} \]

Relaxation

**Relaxed functional**

\[
\overline{F}(u) := \inf_{C(u)} \liminf_{j \to +\infty} F(u_j)
\]

\[
C(u) := \{ \{ u_j \}_{j \in \mathbb{N}} \subset \mathcal{K}_\psi^*(\Omega) : u_j \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega) \}
\]


Relaxed functional

\[ \overline{F}(u) := \inf_{C(u)} \{ \liminf_{j \to +\infty} F(u_j) \} \]

\[ C(u) := \{ \{ u_j \}_{j \in \mathbb{N}} \subset \mathcal{K}_{\psi}^*(\Omega) : u_j \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega) \} \]


Relaxed functional

\[ \bar{F}(u) := \inf_{C(u)} \{ \liminf_{j \to +\infty} F(u_j) \} \]

\[ C(u) := \{ \{ u_j \}_{j \in \mathbb{N}} \subset K^*_\psi(\Omega) : u_j \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega) \} \]


Second main result

Theorem

Assume that $f$ satisfies the hypothesis we stated before. The Dirichlet problem

$$\min \left\{ \overline{F}(u) : u \in K_\psi(\Omega) \right\}$$

with $\overline{F}$ defined above and $u_0 \in W^{1,q}(\Omega)$, has at least one locally Lipschitz continuous solution.
A PRIORI ESTIMATE
The linearization procedure

M. Fuchs - Analysis (1985)

M. Fuchs - Advanced Lectures in Mathematics (1994)

C. Benassi, M. Caselli - Rendiconti Lincei (2020)
The linearization procedure

M. Fuchs - Analysis (1985)

M. Fuchs - Advanced Lectures in Mathematics (1994)

C. Benassi, M. Caselli - Rendiconti Lincei (2020)
The linearization procedure

**Variational inequality**

\[
\int_{\Omega} D_\xi f(x, Du) \cdot D(\varphi - u) \, dx \geq 0
\]

*that holds true for all $\varphi \in W^{1,q}_{loc}(\Omega)$, $\varphi \geq \psi$*

\[
g := -\text{div}(D_\xi f(x, Du)) \chi_{[u=\psi]}
\]

**Higher differentiability**

\[
D\psi \in W^{1,r}_{loc}(\Omega) \quad \implies \quad (1 + |Du|^2)^{\frac{p-2}{4}} Du \in W^{1,2}_{loc}(\Omega)
\]

The linearization procedure

**Variational inequality**

\[ \int_{\Omega} D_\xi f(x, Du) \cdot D(\varphi - u) \, dx \geq 0 \]

that holds true for all \( \varphi \in W^{1,q}_{\text{loc}}(\Omega), \varphi \geq \psi \)

\[ g := -\text{div}(D_\xi f(x, Du)) \chi[u=\psi] \]

**Higher differentiability**

\[ D\psi \in W^{1,r}_{\text{loc}}(\Omega) \quad \Rightarrow \quad (1 + |Du|^2)^{\frac{p-2}{4}} \, Du \in W^{1,2}_{\text{loc}}(\Omega) \]

The linearization procedure

Variational inequality

\[ \int_{\Omega} D_{\xi} f(x, Du) \cdot D(\varphi - u) \, dx \geq 0 \]

that holds true for all \( \varphi \in W_{\text{loc}}^{1,q}(\Omega), \varphi \geq \psi \)

\[ g := -\text{div}(D_{\xi} f(x, Du)) \chi_{[u=\psi]} \]

Higher differentiability

\[ D\psi \in W_{\text{loc}}^{1,r}(\Omega) \implies (1 + |Du|^2)^{\frac{p-2}{4}} Du \in W_{\text{loc}}^{1,2}(\Omega) \]

The linearization procedure

Starting point

\[ \int_{\Omega} D_\xi f(x, Du) \cdot D \eta \, dx = \int_{\Omega} g \, \eta \, dx \quad \forall \eta \in C^1_0(\Omega) \]

Estimate on \( g \)

\[ |g| \leq h(x) \left(1 + |D\psi|^2\right)^{\frac{q-1}{2}} + L \left(1 + |D\psi|^2\right)^{\frac{q-2}{2}} |D^2\psi| \]
The linearization procedure

Starting point

\[ \int_{\Omega} D_{\xi} f(x, Du) \cdot D\eta \, dx = \int_{\Omega} g \, \eta \, dx \quad \forall \eta \in C^1_0(\Omega) \]

Estimate on \( g \)

\[ |g| \leq h(x) \left( 1 + |D\psi|^2 \right)^{\frac{q-1}{2}} + L \left( 1 + |D\psi|^2 \right)^{\frac{q-2}{2}} |D^2\psi| \]
A priori estimate

"Second variation" system

\[
\int_{\Omega} \left( \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(x, Du) \, x_j \, x_s \, D_{x_i} \varphi + \sum_{i=1}^{n} f_{\xi_i \, x_s}(x, Du) \, D_{x_i} \varphi \right) \, dx = \int_{\Omega} g \, D_{x_s} \varphi \, dx
\]

for all \( s = 1, \ldots, n \) and for all \( \varphi \in W^{1,2}_0(\Omega) \).

- \( 0 < \rho < R \) with \( B_R \) compactly contained in \( \Omega \)
- \( \eta \in C^1_0(\Omega) \) such that \( 0 \leq \eta \leq 1 \), \( \eta \equiv 1 \) on \( B_\rho \), \( \eta \equiv 0 \) outside \( B_R \), \( |D\eta| \leq \frac{c}{R-\rho} \)
- \( \gamma \geq 0 \)

\[
\varphi = \eta^2 \left( 1 + |Du|^2 \right)^\gamma \, u_{x_s}
\]
"Second variation" system

\[
\int_\Omega \left( \sum_{i,j=1}^{n} f_{\xi_i\xi_j}(x, Du) u_{x_jx_s} \, D_{x_i} \varphi + \sum_{i=1}^{n} f_{\xi_i} (x, Du) \, D_{x_i} \varphi \right) \, dx = \int_\Omega g \, D_{x_s} \varphi \, dx
\]

for all \( s = 1, \ldots, n \) and for all \( \varphi \in W^{1,2}_0(\Omega) \).

- \( 0 < \rho < R \) with \( B_R \) compactly contained in \( \Omega \)
- \( \eta \in C^1_0(\Omega) \) such that \( 0 \leq \eta \leq 1 \), \( \eta \equiv 1 \) on \( B_\rho \), \( \eta \equiv 0 \) outside \( B_R \), \( |D\eta| \leq \frac{C}{R-\rho} \)
- \( \gamma \geq 0 \)

\[
\varphi = \eta^2 (1 + |Du|^2)^\gamma \, u_{x_s}
\]
A priori estimate

"Second variation" system

\[
\int_{\Omega} \left( \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(x, Du) u_{x_j x_s} D_{x_i} \varphi + \sum_{i=1}^{n} f_{\xi_i x_s}(x, Du) D_{x_i} \varphi \right) dx = \int_{\Omega} g D_{x_s} \varphi \, dx
\]

for all \( s = 1, \ldots, n \) and for all \( \varphi \in W^{1,2}_0(\Omega) \).

- \( 0 < \rho < R \) with \( B_R \) compactly contained in \( \Omega \)
- \( \eta \in C^1_0(\Omega) \) such that \( 0 \leq \eta \leq 1 \), \( \eta \equiv 1 \) on \( B_{\rho} \), \( \eta \equiv 0 \) outside \( B_R \), \( |D\eta| \leq \frac{C}{R-\rho} \)
- \( \gamma \geq 0 \)

\[
\varphi = \eta^2 \left( 1 + |Du|^2 \right)^\gamma u_{x_s}
\]
A priori estimate

"Second variation" system

\[
\int_{\Omega} \left( \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(x, Du) u_{x_j x_s} D_{x_i} \varphi + \sum_{i=1}^{n} f_{\xi_i x_s}(x, Du) D_{x_i} \varphi \right) \, dx = \int_{\Omega} g \, D_{x_s} \varphi \, dx
\]

for all \( s = 1, \ldots, n \) and for all \( \varphi \in W^{1,2}_0(\Omega) \).

- \( 0 < \rho < R \) with \( B_R \) compactly contained in \( \Omega \)
- \( \eta \in C^1_0(\Omega) \) such that \( 0 \leq \eta \leq 1 \), \( \eta \equiv 1 \) on \( B_\rho \), \( \eta \equiv 0 \) outside \( B_R \), \( |D\eta| \leq \frac{C}{R-\rho} \)
- \( \gamma \geq 0 \)

\[
\varphi = \eta^2 (1 + |Du|^2)^\gamma u_{x_s}
\]
A priori estimate

"Second variation" system

\[ \int_{\Omega} \left( \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(x, Du) u_{x_j x_s} D_{x_i} \varphi + \sum_{i=1}^{n} f_{\xi_i x_s}(x, Du) D_{x_i} \varphi \right) dx = \int_{\Omega} g D_{x_s} \varphi \, dx \]

for all \( s = 1, \ldots, n \) and for all \( \varphi \in W_{0}^{1,2}(\Omega) \).

\[ \varphi = \eta^2 \left( 1 + |Du|^2 \right)^\gamma u_{x_s} \]

- 0 < \( \rho < R \) with \( B_R \) compactly contained in \( \Omega \)
- \( \eta \in C^1_0(\Omega) \) such that \( 0 \leq \eta \leq 1 \), \( \eta \equiv 1 \) on \( B_\rho \), \( \eta \equiv 0 \) outside \( B_R \), \( |D\eta| \leq \frac{c}{R-\rho} \)
- \( \gamma \geq 0 \)
A priori estimate

Summing up the 9 integrals and using the hypothesis we obtain

\[
\int_\Omega \eta^2 (1 + |Du|^2)^{\frac{p-2}{2} + \gamma} |D^2 u|^2 \, dx \\
\leq C \Theta (1 + \gamma^2) \left[ \int_\Omega (\eta^2 m + |D\eta|^2 m) (1 + |Du|^2)^{(q-\frac{p}{2} + \gamma)m} \, dx \right]^{\frac{1}{m}}
\]

where the constant $C$ depends on $\nu, L, n, p, q$ but it is independent of $\gamma$

\[
\Theta = 1 + \|g\|^2_{L^r(\Omega)} + \|h\|^2_{L^r(\Omega)}
\]

\[
m = \frac{r}{r - 2}
\]
A priori estimate

Summing up the 9 integrals and using the hypothesis we obtain

\[
\int_{\Omega} \eta^2 \left(1 + |Du|^2\right)^{\frac{p-2}{2} + \gamma} |D^2 u|^2 \, dx \\
\leq C \Theta (1 + \gamma^2) \left[ \int_{\Omega} (\eta^{2m} + |D\eta|^{2m}) (1 + |Du|^2)^{\left(q - \frac{p}{2} + \gamma\right)m} \, dx \right] \frac{1}{m}
\]

where the constant $C$ depends on $\nu, L, n, p, q$ but it is independent of $\gamma$

\[\Theta = 1 + \|g\|_{L^r(\Omega)}^2 + \|h\|_{L^r(\Omega)}^2\]

\[m = \frac{r}{r-2}\]
A priori estimate

Summing up the 9 integrals and using the hypothesis we obtain

$$\int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{p-2}{2} + \gamma} |D^2 u|^2 \, dx \leq C \Theta (1 + \gamma^2) \left[ \int_{\Omega} (\eta^{2m} + |D\eta|^{2m}) (1 + |Du|^2)^{(q-\frac{p}{2} + \gamma)m} \, dx \right]^\frac{1}{m}$$

where the constant $C$ depends on $\nu, L, n, p, q$ but it is independent of $\gamma$

$$\Theta = 1 + \|g\|^2_{L^r(\Omega)} + \|h\|^2_{L^r(\Omega)}$$

$$m = \frac{r}{r - 2}$$
The iteration process

\[ \|Du\|_{L^\infty(B_\rho)} \leq C \left\{ \int_{B_R} [1 + f(x, Du)] \, dx \right\}^\beta \]

Holds for every \( 0 < \rho < R \) and with positive constants \( C \) and \( \beta \) depending on \( n, r, p, q, \nu, L, R, \rho \) and on the local bounds for \( \|D\psi\|_{W^{1,r}} \) and \( \|h\|_{L^r} \).
The iteration process

Final result

\[ \|Du\|_{L^\infty(B_\rho)} \leq C \left\{ \int_{B_R} [1 + f(x, Du)] \, dx \right\}^\beta \]

Holds for every \(0 < \rho < R\) and with positive constants \(C\) and \(\beta\) depending on \(n, r, p, q, \nu, L, R, \rho\) and on the local bounds for \(\|D\psi\|_{W^{1, r}}\) and \(\|h\|_{L^r}\).
The iteration process


Final result

\[ \| Du \|_{L^\infty(B_\rho)} \leq C \left\{ \int_{B_R} \left[ 1 + f(x, Du) \right] \, dx \right\}^\beta \]

Holds for every \( 0 < \rho < R \) and with positive constants \( C \) and \( \beta \) depending on \( n, r, p, q, \nu, L, R, \rho \) and on the local bounds for \( \| D\psi \|_{W^{1,r}} \) and \( \| h \|_{L^r} \).
APPROXIMATION IN CASE OF OCCURRENCE OF LAVRENTIEV PHENOMENON
For each \( u \in \mathcal{K}_\psi(\Omega) \), there exists a sequence \( u_k \in \mathcal{K}^*_\psi(\Omega) \) such that \( u_k \rightharpoonup u \) weakly in \( W^{1,p}(\Omega) \) and
\[
\overline{F}(u) = \lim_{k \to +\infty} F(u_k)
\]
This Lemma’s proof is based on a diagonal argument with sequences of elements in the class of solutions \( \mathcal{K}^*_\psi(\Omega) \).
**Convergence Lemma**

---

**Lemma**

For each \( u \in \mathcal{K}_\psi(\Omega) \), there exists a sequence \( u_k \in \mathcal{K}^*_\psi(\Omega) \) such that \( u_k \rightharpoonup u \) weakly in \( W^{1,p}(\Omega) \) and

\[
\overline{F}(u) = \lim_{k \to +\infty} F(u_k)
\]

This Lemma’s proof is based on a diagonal argument with sequences of elements in the class of solutions \( \mathcal{K}^*_\psi(\Omega) \).
**Lemma**

For each \( u \in \mathcal{K}_\psi(\Omega) \), there exists a sequence \( u_k \in \mathcal{K}^*_\psi(\Omega) \) such that \( u_k \rightharpoonup u \) weakly in \( W^{1,p}(\Omega) \) and

\[
\overline{F}(u) = \lim_{k \to +\infty} F(u_k)
\]

This Lemma’s proof is based on a diagonal argument with sequences of elements in the class of solutions \( \mathcal{K}^*_\psi(\Omega) \).
Regularization of $f$

**Theorem**

Let $f$ be satisfying the growth conditions and strictly convex at infinity and $f_{\xi\xi}$ and $f_{\xi x}$ be two Carathéodory functions, satisfying ellipticity and growing conditions. Then there exists a sequence of $C^2$—functions

$$f^{lk} : \Omega \times \mathbb{R}^n \to [0, +\infty)$$

with $f^{lk}$ convex in the last variable and strictly convex at infinity, such that $f^{lk}$ converges to $f$ as $l \to \infty$ and $k \to \infty$ for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^n$ and uniformly in $\Omega_0 \times K$, where $\Omega_0 \subset \Omega$ and $K$ being a compact set of $\mathbb{R}^n$. Moreover the functions $f^{lk}$ satisfy the hypothesis with constants which are independent on $k$ and satisfy the additional hypothesis necessary to conclude our proof with constants which are dependent only on $k$.

Let $f$ be satisfying the growth conditions and strictly convex at infinity and $f_{\xi\xi}$ and $f_{\xi x}$ be two Carathéodory functions, satisfying ellipticity and growing conditions. Then there exists a sequence of $C^2$ functions

$$f^{lk}: \Omega \times \mathbb{R}^n \rightarrow [0, +\infty)$$

with $f^{lk}$ convex in the last variable and strictly convex at infinity, such that $f^{lk}$ converges to $f$ as $l \rightarrow \infty$ and $k \rightarrow \infty$ for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^n$ and uniformly in $\Omega_0 \times K$, where $\Omega_0 \subseteq \Omega$ and $K$ being a compact set of $\mathbb{R}^n$. Moreover the functions $f^{lk}$ satisfy the hypothesis with constants which are independent on $k$ and satisfy the additional hypothesis necessary to conclude our proof with constants which are dependent only on $k$.

The existence Theorem

Assume that $f$ satisfies the hypothesis we stated before. The Dirichlet problem

$$\min \{ \overline{F}(u) : u \in K_\psi(\Omega) \}$$

with $\overline{F}$ defined above and $u_0 \in W^{1,q}(\Omega)$, has at least one locally Lipschitz continuous solution.
Variational problems

\[
\inf \left\{ \int_\Omega f^{lk}(x, Du) \, dx : u \in \mathcal{K}_\psi^*(\Omega) \right\}
\]

with \( f^{lk}(x, \xi) = f^l(x, \xi) + \frac{1}{k} \left(1 + |\xi|^2\right)^{\frac{q}{2}} \)

There exists a solution \( u^{lk} \in \mathcal{K}_\psi^*(\Omega) \) with \( u_0 \in W^{1,q}(\Omega) \).
Moreover, we can consider \( u_0 \in \mathcal{K}_\psi^*(\Omega) \).

Remark

Let us notice that, by replacing \( u_0 \) by \( \tilde{u}_0 = \max\{u_0, \psi\} \), we may assume that the boundary value function \( u_0 \) satisfies \( u_0 \geq \psi \) in \( \Omega \). Moreover assumptions \( f(x, Du) \in L^1_{\text{loc}}(\Omega) \) and \( f(x, Du_0) \in L^1_{\text{loc}}(\Omega) \) imply \( f(x, D\tilde{u}_0) \in L^1_{\text{loc}}(\Omega) \).
Variational problems

\[
\inf \left\{ \int_{\Omega} f^{lk}(x, Du) \, dx : u \in K_\psi^*(\Omega) \right\}
\]

with \( f^{lk}(x, \xi) = f^l(x, \xi) + \frac{1}{k} (1 + |\xi|^2)^{\frac{q}{2}} \)

There exists a solution \( u^{lk} \in K_\psi^*(\Omega) \) with \( u_0 \in W^{1,q}(\Omega) \).

Moreover, we can consider \( u_0 \in K_\psi^*(\Omega) \).

Remark

Let us notice that, by replacing \( u_0 \) by \( \tilde{u}_0 = \max \{ u_0, \psi \} \), we may assume that the boundary value function \( u_0 \) satisfies \( u_0 \geq \psi \) in \( \Omega \). Moreover assumptions \( f(x, Du) \in L^1_{\text{loc}}(\Omega) \) and \( f(x, Du_0) \in L^1_{\text{loc}}(\Omega) \) imply \( f(x, D\tilde{u}_0) \in L^1_{\text{loc}}(\Omega) \).
Proof of the existence Theorem

Variational problems

\[ \inf \left\{ \int_{\Omega} f^{\ell k}(x, Du) \, dx : u \in K^*(\Omega) \right\} \]

with \( f^{\ell k}(x, \xi) = f^l(x, \xi) + \frac{1}{k} (1 + |\xi|^2)^{\frac{q}{2}} \)

There exists a solution \( u^{\ell k} \in K^*(\Omega) \) with \( u_0 \in W^{1,q}(\Omega) \).

Moreover, we can consider \( u_0 \in K^*(\Omega) \).

Remark

Let us notice that, by replacing \( u_0 \) by \( \tilde{u}_0 = \max \{ u_0, \psi \} \), we may assume that the boundary value function \( u_0 \) satisfies \( u_0 \geq \psi \) in \( \Omega \). Moreover assumptions \( f(x, Du) \in L^1_{\text{loc}}(\Omega) \) and \( f(x, Du_0) \in L^1_{\text{loc}}(\Omega) \) imply \( f(x, D\tilde{u}_0) \in L^1_{\text{loc}}(\Omega) \).
Proof of the existence Theorem

By the growth conditions, the minimality of $u^{lk}$ and the previous remark

$$\int_{\Omega} |Du^{lk}|^p \, dx \leq \int_{\Omega} f^l(x, Du_0) \, dx + \frac{1}{k} \int_{\Omega} (1 + |Du_0|^2)^{\frac{q}{2}} \, dx$$

$$\lim_{l \to +\infty} \int_{\Omega} |Du^{lk}|^p \, dx \leq \int_{\Omega} f(x, Du_0) \, dx + \frac{1}{k} \int_{\Omega} (1 + |Du_0|^2)^{\frac{q}{2}} \, dx$$

By the previous Theorem, the functions $f^{lk}$ satisfy the hypothesis, so we can apply the a-priori estimate on $u^{lk}$ and obtain for all $B \subseteq \Omega$ that

$$\|Du^{lk}\|_{L^\infty(B)} \leq C \left\{ \int_{\Omega} [1 + f(x, Du_0) + \frac{1}{k} (1 + |Du_0|^2)^{\frac{q}{2}}] \, dx \right\}^{\frac{\gamma}{p}}$$

where $C, \gamma$ depend on all the parameters except for $l, k.$

Therefore there exist $u^k \in K_\psi(\Omega),$ for all $k \in \mathbb{N},$ such that

$$u^{lk} \xrightarrow{l \to \infty} u^k \text{ weakly in } W^{1,p}(\Omega)$$

$$u^{lk} \xrightarrow{l \to \infty} u^k \text{ weakly star in } W^{1,\infty}_{\text{loc}}(\Omega)$$
Proof of the existence Theorem

By the growth conditions, the minimality of $u^{l_k}$ and the previous remark

$$
\int_{\Omega} |D u^{l_k}|^p \, dx \leq \int_{\Omega} f^l(x, D u_0) \, dx + \frac{1}{k} \int_{\Omega} (1 + |D u_0|^2)^{\frac{q}{2}} \, dx
$$

$$
\lim_{l \to +\infty} \int_{\Omega} |D u^{l_k}|^p \, dx \leq \int_{\Omega} f(x, D u_0) \, dx + \frac{1}{k} \int_{\Omega} (1 + |D u_0|^2)^{\frac{q}{2}} \, dx
$$

By the previous Theorem, the functions $f^{l_k}$ satisfy the hypothesis, so we can apply the a-priori estimate on $u^{l_k}$ and obtain for all $B \Subset \Omega$ that

$$
\|D u^{l_k}\|_{L^\infty(B)} \leq C \left\{ \int_{\Omega} \left[ 1 + f(x, D u_0) + \frac{1}{k} (1 + |D u_0|^2)^{\frac{q}{2}} \right] \, dx \right\}^{\frac{\gamma}{p}}
$$

where $C, \gamma$ depend on all the parameters except for $l, k$.

Therefore there exist $u^k \in K_\psi(\Omega)$, for all $k \in \mathbb{N}$, such that

$$
\begin{align*}
&u^{l_k} \xrightarrow{l \to \infty} u^k \text{ weakly in } W^{1,p}(\Omega) \\
&u^{l_k} \xrightarrow{l \to \infty} u^k \text{ weakly star in } W^{1,\infty}_{\text{loc}}(\Omega)
\end{align*}
$$
**Proof of the existence Theorem**

By the growth conditions, the minimality of $u^{lk}$ and the previous remark

$$
\int_{\Omega} |Du^{lk}|^p \, dx \leq \int_{\Omega} f^l(x, Du_0) \, dx + \frac{1}{k} \int_{\Omega} (1 + |Du_0|^2)^{\frac{q}{2}} \, dx
$$

$$
\lim_{l \to +\infty} \int_{\Omega} |Du^{lk}|^p \, dx \leq \int_{\Omega} f(x, Du_0) \, dx + \frac{1}{k} \int_{\Omega} (1 + |Du_0|^2)^{\frac{q}{2}} \, dx
$$

By the previous Theorem, the functions $f^{lk}$ satisfy the hypothesis, so we can apply the a priori estimate on $u^{lk}$ and obtain for all $B \subset \Omega$ that

$$
\|Du^{lk}\|_{L^\infty(B)} \leq C \left\{ \int_{\Omega} [1 + f(x, Du_0) + \frac{1}{k} (1 + |Du_0|^2)^{\frac{q}{2}}] \, dx \right\}^{\frac{\gamma}{p}}
$$

where $C, \gamma$ depend on all the parameters except for $l, k$.

Therefore there exist $u^k \in K_{\psi}(\Omega)$, for all $k \in \mathbb{N}$, such that

$$
\begin{align*}
\lim_{l \to \infty} u^{lk} & \to u^k \text{ weakly in } W^{1,p}(\Omega) \\
\lim_{l \to \infty} u^{lk} & \to u^k \text{ weakly star in } W^{1,\infty}_{loc}(\Omega)
\end{align*}
$$
Proof of the existence Theorem

By the growth conditions, the minimality of $u^{l^k}$ and the previous remark

$$
\int_{\Omega} |Du^{l^k}|^p \, dx \leq \int_{\Omega} f^l(x, Du_0) \, dx + \frac{1}{k} \int_{\Omega} \left(1 + |Du_0|^2\right)^{\frac{q}{2}} \, dx
$$

$$
\lim_{l \to +\infty} \int_{\Omega} |Du^{l^k}|^p \, dx \leq \int_{\Omega} f(x, Du_0) \, dx + \frac{1}{k} \int_{\Omega} \left(1 + |Du_0|^2\right)^{\frac{q}{2}} \, dx
$$

By the previous Theorem, the functions $f^{l^k}$ satisfy the hypothesis, so we can apply the a-priori estimate on $u^{l^k}$ and obtain for all $B \subseteq \Omega$ that

$$
\|Du^{l^k}\|_{L^\infty(B)} \leq C \left\{ \int_{\Omega} \left[1 + f(x, Du_0) + \frac{1}{k} \left(1 + |Du_0|^2\right)^{\frac{q}{2}} \right] \, dx \right\}^{\frac{\gamma}{p}}
$$

where $C, \gamma$ depend on all the parameters except for $l, k$.

Therefore there exist $u^k \in K_\psi(\Omega)$, for all $k \in \mathbb{N}$, such that

$$
u^{l^k} \overset{l \to \infty}{\rightharpoonup} u^k \text{ weakly in } W^{1,p}(\Omega)
$$

$$
u^{l^k} \overset{l \to \infty}{\rightharpoonup} u^k \text{ weakly star in } W^{1,\infty}_{loc}(\Omega)
$$
Proof of the existence Theorem

Following the previous estimates we also have

$$\|Du^k\|_{L^p(\Omega)} \leq \int_{\Omega} f(x, Du_0) \, dx + \int_{\Omega} (1 + |Du_0|^2)^{\frac{q}{2}} \, dx$$

$$\|Du^k\|_{L^\infty(B)} \leq C \left\{ \int_{\Omega} \left[ 1 + f(x, Du_0) + \frac{1}{k} (1 + |Du_0|^2)^{\frac{q}{2}} \right] \, dx \right\}^{\frac{\gamma}{p}}$$

So there exists, up to subsequences, $\bar{u} \in K_\psi(\Omega)$ such that

$$u^k \rightharpoonup \bar{u} \text{ weakly in } W^{1,p}(\Omega)$$

$$u^k \rightharpoonup \star \bar{u} \text{ weakly star in } W^{1,\infty}_{\text{loc}}(\Omega)$$


Strong convergence

$$u^k \rightarrow \bar{u} \text{ in } W^{1,p}_{0}(\Omega) + u_0, \quad \bar{u} \in K_\psi(\Omega)$$
Proof of the existence Theorem

Following the previous estimates we also have

\[
\|D u^k\|_{L^p(\Omega)} \leq \int_{\Omega} f(x, Du_0) \, dx + \int_{\Omega} (1 + |Du_0|^2)^{\frac{q}{2}} \, dx
\]

\[
\|D u^k\|_{L^\infty(B)} \leq C \left\{ \int_{\Omega} \left[ 1 + f(x, Du_0) + \frac{1}{k} \left( 1 + |Du_0|^2 \right)^{\frac{q}{2}} \right] \, dx \right\}^{\frac{\gamma}{p}}
\]

So there exists, up to subsequences, \( \bar{u} \in K_\psi(\Omega) \) such that

\[
u^k \rightharpoonup \bar{u} \text{ weakly in } W^{1,p}(\Omega)
\]

\[
u^k \rightharpoonup \bar{u} \text{ weakly star in } W^{1,\infty}_{loc}(\Omega)
\]


\textbf{Strong convergence}

\[
u^k \to \bar{u} \text{ in } W^{1,p}_0(\Omega) + u_0, \quad \bar{u} \in K_\psi(\Omega)
\]
Proof of the existence Theorem

Following the previous estimates we also have

\[ \| Du^k \|_{L^p(\Omega)} \leq \int_{\Omega} f(x, Du_0) \, dx + \int_{\Omega} \left(1 + |Du_0|^2 \right)^{\frac{q}{2}} \, dx \]

\[ \| Du^k \|_{L^\infty(B)} \leq C \left\{ \int_{\Omega} \left[1 + f(x, Du_0) + \frac{1}{k} \left(1 + |Du_0|^2 \right)^{\frac{q}{2}} \right] \, dx \right\}^{\frac{\gamma}{p}} \]

So there exists, up to subsequences, \( \bar{u} \in K_\psi(\Omega) \) such that

\[ u^k \overset{k \to \infty}{\to} \bar{u} \text{ weakly in } W^{1,p}(\Omega) \]

\[ u^k \overset{k \to \infty}{\rightharpoonup} \bar{u} \text{ weakly star in } W^{1,\infty}_{loc}(\Omega) \]

Proof of the existence Theorem

Following the previous estimates we also have

\[ \| Du^k \|_{L^p(\Omega)} \leq \int_{\Omega} f(x, Du) \, dx + \int_{\Omega} (1 + |Du_0|^2)^{\frac{q}{2}} \, dx \]

\[ \| Du^k \|_{L^\infty(B)} \leq C \left\{ \int_{\Omega} \left[ 1 + f(x, Du_0) + \frac{1}{k} (1 + |Du_0|^2)^{\frac{q}{2}} \right] \, dx \right\}^{\gamma \over p} \]

So there exists, up to subsequences, \( \bar{u} \in K_\psi(\Omega) \) such that

\[ u^k \xrightarrow{k \to \infty} \bar{u} \text{ weakly in } W^{1,p}(\Omega) \]

\[ u^k \xrightarrow{k \to \infty} \bar{u} \text{ weakly star in } W^{1,\infty}_{\text{loc}}(\Omega) \]


Strong convergence

\[ u^k \to \bar{u} \text{ in } W^{1,p}_0(\Omega) + u_0, \quad \bar{u} \in K_\psi(\Omega) \]
Proof of the existence Theorem

For any fixed $k \in \mathbb{N}$, using the uniform convergence of $f^l$ to $f$ in $\Omega_0 \times K$ (for any $K$ compact subset of $\mathbb{R}^n$) and the minimality of $u^{lk}$, we get for all $w \in K^*_\psi(\Omega)$

$$
\int_{\Omega_0} f(x, Du^k) \, dx \leq \liminf_{l \to \infty} \int_{\Omega} f^l(x, Dw) \, dx + \frac{1}{k} \int_{\Omega} (1 + |Dw|^2)^{\frac{q}{2}} \, dx
$$

Then, for $\Omega_0 \to \Omega$

$$
\int_{\Omega} f(x, Du^k) \, dx \leq \int_{\Omega} f(x, Dw) \, dx + \frac{1}{k} \int_{\Omega} (1 + |Dw|^2)^{\frac{q}{2}} \, dx
$$

By the relaxed functional’s definition, we have

$$
\overline{F}(\bar{u}) \leq \liminf_{k \to \infty} \int_{\Omega} f(x, Du^k) \, dx \leq \int_{\Omega} f(x, Dw) \, dx \quad \forall w \in K^*_\psi(\Omega)
$$
Proof of the existence Theorem

For any fixed $k \in \mathbb{N}$, using the uniform convergence of $f^l$ to $f$ in $\Omega_0 \times K$ (for any $K$ compact subset of $\mathbb{R}^n$) and the minimality of $u^l_k$, we get for all $w \in K^*_\psi(\Omega)$

$$\int_{\Omega_0} f(x, D u^k) \, dx \leq \liminf_{l \to \infty} \int_{\Omega} f^l(x, D w) \, dx + \frac{1}{k} \int_{\Omega} (1 + |D w|^2)^{q/2} \, dx$$

Then, for $\Omega_0 \to \Omega$

$$\int_{\Omega} f(x, D u^k) \, dx \leq \int_{\Omega} f(x, D w) \, dx + \frac{1}{k} \int_{\Omega} (1 + |D w|^2)^{q/2} \, dx$$

By the relaxed functional’s definition, we have

$$F(\bar{u}) \leq \liminf_{k \to \infty} \int_{\Omega} f(x, D u^k) \, dx \leq \int_{\Omega} f(x, D w) \, dx \quad \forall w \in K^*_\psi(\Omega)$$
Proof of the existence Theorem

For any fixed $k \in \mathbb{N}$, using the uniform convergence of $f^l$ to $f$ in $\Omega_0 \times K$ (for any $K$ compact subset of $\mathbb{R}^n$) and the minimality of $u^{l_k}$, we get for all $w \in K^*_\psi(\Omega)$

$$\int_{\Omega_0} f(x, Du^k) \, dx \leq \liminf_{l \to \infty} \int_{\Omega} f^l(x, Dw) \, dx + \frac{1}{k} \int_{\Omega} (1 + |Dw|^2)^{\frac{q}{2}} \, dx$$

Then, for $\Omega_0 \to \Omega$

$$\int_{\Omega} f(x, Du^k) \, dx \leq \int_{\Omega} f(x, Dw) \, dx + \frac{1}{k} \int_{\Omega} (1 + |Dw|^2)^{\frac{q}{2}} \, dx$$

By the relaxed functional’s definition, we have

$$\overline{F}(\bar{u}) \leq \liminf_{k \to \infty} \int_{\Omega} f(x, Du^k) \, dx \leq \int_{\Omega} f(x, Dw) \, dx \quad \forall w \in K^*_\psi(\Omega)$$
Proof of the existence Theorem

Let \( v \in K_\psi(\Omega) \). By the Convergence Lemma, there exists \( u_k \in K_\psi^*(\Omega) \) such that \( u_k \rightharpoonup v \) weakly in \( W^{1,p}(\Omega) \) and

\[
\lim_{k \to \infty} \int_\Omega f(x, D u_k) \, dx = \overline{F}(v)
\]

By the last estimate on the relaxed functional

\[
\overline{F}(\overline{u}) \leq \int_\Omega f(x, D u_k) \, dx
\]

and we can conclude that

\[
\overline{F}(\overline{u}) \leq \lim_{k \to \infty} \int_\Omega f(x, D u_k) \, dx = \overline{F}(v) \quad \forall v \in K_\psi(\Omega)
\]

Then \( \overline{u} \in W^{1,\infty}_{loc}(\Omega) \) is a solution to the problem \( \min\{\overline{F}(u) : u \in K_\psi(\Omega)\} \).
Proof of the existence Theorem

Let \( v \in \mathcal{K}_\psi(\Omega) \). By the Convergence Lemma, there exists \( u_k \in \mathcal{K}^*_\psi(\Omega) \) such that \( u_k \rightharpoonup v \) weakly in \( W^{1,p}(\Omega) \) and

\[
\lim_{k \to \infty} \int_{\Omega} f(x, Du_k) \, dx = \overline{F}(v)
\]

By the last estimate on the relaxed functional

\[
\overline{F}(\bar{u}) \leq \int_{\Omega} f(x, Du_k) \, dx
\]

and we can conclude that

\[
\overline{F}(\bar{u}) \leq \lim_{k \to \infty} \int_{\Omega} f(x, Du_k) \, dx = \overline{F}(v) \quad \forall \ v \in \mathcal{K}_\psi(\Omega)
\]

Then \( \bar{u} \in W^{1,\infty}_{\text{loc}}(\Omega) \) is a solution to the problem \( \min\{\overline{F}(u) : u \in \mathcal{K}_\psi(\Omega)\} \).
Proof of the existence Theorem

Let \( v \in K_\psi(\Omega) \). By the Convergence Lemma, there exists \( u_k \in K_\psi^*(\Omega) \) such that \( u_k \rightharpoonup v \) weakly in \( W^{1,p}(\Omega) \) and

\[
\lim_{k \to \infty} \int_\Omega f(x, Du_k) \, dx = \overline{F}(v)
\]

By the last estimate on the relaxed functional

\[
\overline{F}(\bar{u}) \leq \int_\Omega f(x, Du_k) \, dx
\]

and we can conclude that

\[
\overline{F}(\bar{u}) \leq \lim_{k \to \infty} \int_\Omega f(x, Du_k) \, dx = \overline{F}(v) \quad \forall \, v \in K_\psi(\Omega)
\]

Then \( \bar{u} \in W^{1,\infty}_{\text{loc}}(\Omega) \) is a solution to the problem \( \min\{\overline{F}(u) : u \in K_\psi(\Omega)\} \).
Proof of the existence Theorem

Let \( v \in \mathcal{K}_\psi(\Omega) \). By the Convergence Lemma, there exists \( u_k \in \mathcal{K}_\psi^*(\Omega) \) such that \( u_k \rightharpoonup v \) weakly in \( W^{1,p}(\Omega) \) and

\[
\lim_{k \to \infty} \int_{\Omega} f(x, Du_k) \, dx = \overline{F}(v)
\]

By the last estimate on the relaxed functional

\[
\overline{F}(\bar{u}) \leq \int_{\Omega} f(x, Du_k) \, dx
\]

and we can conclude that

\[
\overline{F}(\bar{u}) \leq \lim_{k \to \infty} \int_{\Omega} f(x, Du_k) \, dx = \overline{F}(v) \quad \forall \, v \in \mathcal{K}_\psi(\Omega)
\]

Then \( \bar{u} \in W^{1,\infty}_{\text{loc}}(\Omega) \) is a solution to the problem \( \min \{ \overline{F}(u) : u \in \mathcal{K}_\psi(\Omega) \} \).
THANK YOU FOR THE ATTENTION!