Monday's Nonstandard Seminar 2020/2021

The Alexandrov Theorem in Minkowski spaces

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Stawomir Kolasinski

joint work with Antonio De Rosa
and Mario Santilli
For any set $A \subseteq \mathbb{R}^n$ we define
\[
\text{ess Int } A = \mathbb{R}^n \setminus \{ x : \mathbb{H}^n (L^\infty (\mathbb{R}^n \setminus A), x) = 0 \}
\]
\[
\text{essential interior of } A
\]
and
\[
\text{ess Bdry } A = \mathbb{R}^n \setminus \left( \text{ess Int } A \cup \text{ess Int } (\mathbb{R}^n \setminus A) \right)
\]
\[
\text{essential boundary of } A
\]

$A \subseteq \mathbb{R}^n$ is a Caccioppoli set iff $H^{n-1} (K \setminus \text{ess Bdry } A) < \infty$

for any compact set $K \subseteq \mathbb{R}^n$.

For these sets there is a proper notion of the exterior normal $n (A, b)$ defined for $H^{n-1}$ almost all $b \in \text{ess Bdry } A$;

namely,
\[
S \exists u = n (A, b) \iff \begin{cases}
\mathbb{H}^n (L^\infty (\mathbb{R}^n \setminus \{ x : (x - b) \cdot u > 0 \} \cap A, b)) = 0 \\
\mathbb{H}^n (L^\infty (\mathbb{R}^n \setminus \{ x : (x - b) \cdot u < 0 \} \cap A, b)) = 0
\end{cases}
\]

Assume $F : \mathbb{R}^n \to \mathbb{R}$ is a norm on $\mathbb{R}^n$

which is $\ast$ smooth on $\mathbb{R}^n \setminus \{ 0 \}$ [we need $C^2$]

$\ast$ uniformly convex, i.e., $q(x) = F(x) - \frac{\varepsilon}{2} |x|^2$ is convex for $x \in \mathbb{R}^n$.

The $F$-perimeter of a Caccioppoli set $A$ is defined as
\[
\mathcal{P}_F (A) = \int_{\text{ess Bdry } A} F (n (A, x)) \, dH^{n-1} (x)
\]

Remark. In case $F(x) = |x|_1$ we have
\[
\mathcal{P}_F (A) = H^{n-1} (\text{ess Bdry } A)
\]

Q. Is it true that $\mathcal{P}_F (A) = H^{n-1} (\text{ess Bdry } A)$, where $q(x, y) = F(x - y)$?
The isoperimetric problem

**Competitors:**

\[ \mathcal{A} = \mathbb{R}^n \cap \{ A : H^{n-1}(\text{ess}\, \text{Bdry} \, A) < \infty, \, L^n(A) = 1 \} \]

**Functional:** \[ \mathcal{P}_F |_{\mathcal{A}} \]

- these are, up to translation, Wulff shapes, i.e.,

\[ B_*^F(a,r) = \mathbb{R}^n \cap \{ x : F^*(x-a) \leq r \} \]

where \( F^*(y) = \sup \{ x \cdot y : F(x) = 1 \} \) is the dual norm.

What about other critical points?

Considering deformations preserving the volume of \( A \) \( [L^n(A) = 1] \) one derives the "E-L eqns."

\( A \) is a critical point of \( \mathcal{P}_F |_{\mathcal{E}} \) iff

\[ \text{the mean } F\text{-curvature vector } h_F(B, b) \]

is a constant multiple of \( n(A, b) \) for \( H^{n-1} \) a.a. \( b \in B \),

where "\( F\text{-div}_B g(x) \)"

\[ \star \int_B \text{trace} \left( P_A(x) \circ Dg(x) \right) \, d\mu(x) = - \int B h_F(B, x) \cdot g(x) \, d\mu(x) \]

* \( \mu = F \circ m(A, \cdot)(H^{n-1} \setminus B) \)

* \( P_A(x) \circ P_A(x) = P_A(x) \in \text{Ham}(\mathbb{R}^n, \mathbb{R}^n) \) for \( \mu \)-a.a. \( x \)

* \( \text{im } P_A(x) = \text{span} \{ n(A, x) \} \perp = \text{Tan}^{n-1}(\mu, x) \)

* \( \ker P_A(x) = \text{span} \{ \text{grad } F(n(A, x)) \} \)
In case $F$ is Euclidean, i.e., $F(x) = \sqrt{x \cdot x'}$, we have $\text{grad } F(x) = \frac{x}{|x|}$, and $P_A(x)$ is an orthogonal projection for $\mu$-a.e. $x$, and $\text{H}(B, \cdot)$ is the classical mean curvature vector.

Alexandrov, 1956 - 1962

If $M \subseteq \mathbb{R}^n$ is an embedded compact hypersurface with CMC, then $M$ is an $(n-1)$-sphere.

Montiel & Ros, 1991

If $M \subseteq \mathbb{R}^n$ is an embedded compact hypersurface with constant $H_r$ for some $r \in \{1, 2, \ldots, n-1\}$, then $M$ is an $(n-1)$-sphere.

\[ H_r - \text{the } r\text{-th elementary symmetric polynomial in the principal curvatures of } M. \]
\[ H_r(x) = \text{H}(M, x) \cdot V_M(x) \]

He, Li, Ha, Ge, 2009

adaptation of the Montiel-Ros argument

If $M \subseteq \mathbb{R}^n$ is an embedded compact hypersurface with constant $H_r^F$ for some $r \in \{1, 2, \ldots, n-1\}$, is an $(n-1)$-dimensional $F^*$-sphere, i.e., the boundary of some Wulff shape.

Delgadino & Maggi, 2019

Montiel-Ros argument
Schätzle's maximum principle

Among Caccioppoli sets of finite volume, finite unions of balls with equal radii are the unique critical points of the Euclidean isoperimetric problem.
Theorem (De Rosa, K., Santilli, 2020)

\( \alpha \in (0,1) \), \( F \) a uniformly convex norm of class \( C^{2,\alpha} \).

Let \( E \) a Caccioppoli set such that

\[
\mathcal{H}^{m-1}(B_{\text{dry}} E \sim \text{ess } B_{\text{dry}} E) = 0
\]

\[
\mathcal{H}^{m-1}(\text{ess } B_{\text{dry}} E) < \infty \quad B = \text{ess } B_{\text{dry}} E,
\]

\[-F(m(E,x)) |h_F(B,x) \cdot m(E,x)| = H(x) \quad \text{for } x \in B,\]

\( H \) is locally \( C^{0,\alpha} \) on the \( C^{1,\alpha} \)-regular part of \( B \),

\( c \in (0,\infty) \), \( 0 < H(x) < c \) for \( x \in B \).

Then

\[
L^m(E) \leq \frac{m-1}{m} \int_B \frac{d\mathcal{H}^{m-1}}{H} \quad \text{(H-K inequality)}
\]

Moreover, equality holds iff there is a finite union of Wulff shapes \( \Omega \) with radii \( \sqrt[\frac{m-1}{c}] \) such that

\[
L^m((\Omega \sim E) \cup (E \sim \Omega)) = 0.
\]

Remark. Santilli proved earlier a similar result for \( F \) Euclidean without employing Schätzle’s maximum principle and without assuming \( H \) is \( C^{0,\alpha} \).

Sketch of the proof

I. The case when \( B \) is smooth

Remark. \( F(m(E,x)) \cdot h_F(B,x) = \text{trace } D(\text{grad } F \cdot m(E,\cdot))(x) \)

Def. Eigenvalues of the map \( D(\text{grad } F \cdot m(E,\cdot))(x)|_{\text{Tan}(B,x)} \)

are called the anisotropic principal curvatures of \( B \) and denoted

\[
\kappa_{B,F,i}^m(x) \leq \cdots \leq \kappa_{B,F,m-1}^m(x)
\]

Remark. In case \( W = \mathbb{R}^m \setminus \{x : F^*(x) \leq r \} \) is a Wulff shape

we have \( \kappa_{B,F,i}^m(x) = \frac{1}{r} \) for \( i \in \{1, \ldots, m-1\} \) and \( x \in \partial W \).
Lemma 5.1. If $M$ is a $C^{1,\alpha}$-hypersurface in $\mathbb{R}^m$ with $K^F_i(x) = K^F_j(x)$ for $i, j \in \{1, \ldots, m-1\}$, then $M = \partial \Omega$ for some Wulff shape $\Omega$.

**Def.**

$S^F_E(x) = \inf \{ F^*(x-y) : y \in E \}$

$U^F_\rho(E) = \mathbb{R}^m \cap \{ x : \exists y \in E \text{ s.t. } S^F_E(x) = F^*(x-y) \}$

$\mathcal{B}^F_E(x) = \{ y : \lambda \in U^F_\rho(E) \text{ and } S^F_E(x) = F^*(x-y) \}$

$N^F(E) = E \times \mathbb{R}^{m-1} \cap \{ (a, u) : S^F_E(a + su) = s \text{ for some } s > 0 \}$

**F-normal bundle of $E$**

$m^F(E, x) = \nabla F(m(E, x))$

**F-normal vector**

$C = \mathbb{R}^m \setminus E$, $B = \text{Ball}(E)$

$y \in U^F_\rho(C)$

**Observe**

$$0 \leq \frac{H(x)}{m-1} \leq -\frac{1}{K^F_{e,1}(x)} \leq \frac{1}{d^F_E(y)}$$

for $y \in U^F_\rho(E)$

Set

$Z = B \times \mathbb{R} \cap \{ (x, t) : 0 \leq t \leq -\frac{1}{K^F_{e,1}(x)} \}$

$\mathcal{S} : Z \to \mathbb{R}^m$, \hspace{1cm} $\mathcal{S}(x, t) = x + tm^F(C, x)$

Easy computation gives

$$\int m \mathcal{S} \mathcal{S}(x, t) = \| \Lambda_m (H^{m-2} \mathcal{S}(x, t)) \|$$

$$= F(m(C, x)) \prod_{i=1}^{m-1} (1 + t W_{B_i}(x))$$

for $(x, t) \in Z$.

We also know that

* $\delta^F_C$ is $\lambda$-Lipschitz $\Rightarrow$ differentiable $\lambda$-a.e.

* $E \cap \{ x : D\delta^F_C(x) \text{ exists} \} \subseteq U^F_\rho(C)$

$$\Rightarrow 0 = \mathcal{L}^m(E \setminus U^F_\rho(C)) = \mathcal{L}^m(E \setminus (B^F_C)^{-1}(B)) = \mathcal{L}^m(E \setminus \alpha(Z))$$
The Hantsel-Ros argument

\[ L^m(E) = L^m(\mathcal{S}(Z)) \leq \int \mathcal{H}^o(\mathcal{S}^{-1}(y)) d\mu^m(y) = \int \int_d d\mathcal{H}^m \]

\[ \leq \int_B F(m(c,x)) \left( \frac{1}{m-1} \sum_{i=0}^{m-1} (1 + t \kappa_{c_i}^F(x)) \right)^{m-1} \mathcal{H}^m(\mathcal{F}(s)) \]

In case of equality we must have \( \kappa_{c_i}^F(x) = \kappa_{c_j}^F(x) \)

for \( i, j \in \{1, \ldots, m-1\} \) and \( \mathcal{H}^{m-1} \text{ a.a. } x \in B \)

\[ \Rightarrow \text{B is totally F-umbilical} \]

\[ \Rightarrow \text{B is a Wulff shape} \]

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II B might not be smooth but assume

\[ \mathcal{H}^{m-1}(Bdy E \sim ess Bdy E) = 0 \]

Def. A satisfies the Lusin (N) condition in \( U \) iff

\[ S \subseteq A \cap U, \mathcal{H}^{m-1}(S) = 0 \Rightarrow \mathcal{H}^{m-1}(N^F(A) \cap S) = 0 \]

Lemma 1 B satisfies the Lusin (N) condition

Proof uses a result of De Philippis, De Rosa, and Hirsch from 2019 on area blow up set for bounded mean curvature submanifolds.

Lemma 2 \( \mathcal{H}^{m-1} \text{ almost all points of B are } C^{2,\alpha} \)-regular points, i.e., in some neighbourhood of such a point B coincides with \( C^{2,\alpha} \) manifold.

Proof uses the Allard Regularity Theorem from 1986 plus some classical elliptic PDE theory.
Set $Q$ = the set of $\mathbb{R}^n$ points of $B$.

Using Lemme 1 & Lemme 2 we show that

\[ \mu^{n-1}(B \sim Q) = 0 \]
\[ \mu^{n-1}(\{ x : S^F_c(x) = r \}) \sim (3F)^{-1} [Q] = 0 \]

for $r \in (0, \infty)$

Then coarse formula gives

\[ L^n(E \sim (3F)^{-1} [Q]) = 0 \]

and we can use the Hontiel-Ros argument.

But this is not the end!

We prove the anisotropic Steiner formula to conclude that $Q$ has positive F-reach, $\geq r_0$.

The level-sets $\{ x : S^F_c(x) = r \} = S^F(C, r)$: for $r \in (0, r_0)$ are always $C^{1,1}$ submanifolds of $\mathbb{R}^n$.

Moreover, one can compute principal curvatures of $S^F(C, r)$ in terms of principal curvatures of $Q$.

This gives that $S^F(C, r)$ are totally umbilical and $C^{1,1}$ regular; hence, Wulff shapes.

Passing to the limit $r \to 0^+$ concludes the proof.

Open question: Is there a similar characterisation if $F$ is only continuous?