Monday's Nonstandard

Seminar 2020/2021

The Alexandrov Theorem in Minkowski spaces

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joint work with Antonio De Rosa and Mario Santilli

① Nonstandard seminar 29. III 2021

For any set $A \subseteq \mathbb{R}^n$ we define

$$\underbrace{\operatorname{ess\,Int} A}_{\text{essential interior of } A} = \mathbb{R}^n \cap \{x : \Theta^n(\mathcal{L}^n \llcorner (\mathbb{R}^n \sim A), x) = 0\}$$

and

$$\underbrace{\operatorname{ess\,Bdry} A}_{\text{essential boundary of } A} = \mathbb{R}^n \sim \left(\operatorname{ess\,Int} A \cup \operatorname{ess\,Int}(\mathbb{R}^n \sim A)\right)$$

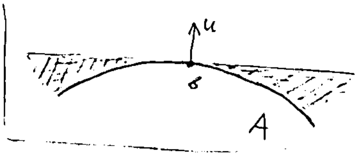
see 4.5.12 in Federer 1969

$A \subseteq \mathbb{R}^n$ is a <u>Caccioppoli set</u> iff $\mathcal{H}^{n-1}(K \cap \operatorname{ess\,Bdry} A) < \infty$ for any compact set $K \subseteq \mathbb{R}^n$.

For these sets there is a proper notion of the <u>exterior normal $n(A, b)$</u> defined for $\mathcal{H}^{n-1}$ almost all $b \in \operatorname{ess\,Bdry} A$; namely,

$$S^{n-1} \ni u = n(A,b) \iff \begin{cases} \Theta^n(\mathcal{L}^n \llcorner \{x : (x-b)\bullet u > 0\} \cap A, b) = 0 \\ \Theta^n(\mathcal{L}^n \llcorner \{x : (x-b)\bullet u < 0\} \sim A, b) = 0 . \end{cases}$$

Assume $F : \mathbb{R}^n \to \mathbb{R}$ is a norm on $\mathbb{R}^n$ which is
* smooth on $\mathbb{R}^n \sim \{0\}$ [we need $C^{2,\alpha}$]
* uniformly convex, i.e., $g(x) = F(x) - \frac{\alpha}{2}|x|$ is convex for $x \in \mathbb{R}^n$.



The <u>F-perimeter</u> of a Caccioppoli set A is defined as

$$\mathcal{P}_F(A) = \int_{\operatorname{ess\,Bdry} A} F(n(A,x))\, d\mathcal{H}^{n-1}(x)$$

<u>Remark</u>. In case $F(x) = |x|$ we have $\mathcal{P}_F(A) = \mathcal{H}^{n-1}(\operatorname{ess\,Bdry} A)$

<u>Q</u>. Is it true that $\mathcal{P}_F(A) = \mathcal{H}^{n-1}_\rho(\operatorname{ess\,Bdry} A)$, where $\rho(x,y) = F(x-y)$?

The isoperimetric problem

Competitors:

$$\mathcal{A} = 2^{\mathbb{R}^n} \cap \{ A : \mathcal{H}^{n-1}(\operatorname{ess\,Bdry} A) < \infty,\ \mathcal{L}^n(A) = 1 \}$$

Functional: $\mathcal{P}_F | \mathcal{A}$

(even for non-smooth norms F)

Minimisers characterised by Jean Taylor, 1972 - 1975.
- these are, up to translation, Wulff shapes, i.e.,

$$\mathbb{B}^{F^*}(a, r) = \mathbb{R}^n \cap \{ x : F^*(x-a) \le r \}$$

where $F^*(y) = \sup \{ x \bullet y : F(x) = 1 \}$ is the dual norm.

$$\boxed{\text{What about other critical points?}}$$

Considering deformations preserving the volume of A $[\mathcal{L}^n(A) = 1]$ one derives the "E-L eqns.":

$B = \operatorname{ess\,Bdry} A$

A is a critical point of $\mathcal{P}_F | \mathcal{E}$ iff

F-CMC $\left\{ \begin{array}{l} \text{the } \underline{\text{mean } F\text{-curvature vector } \mathbb{h}_F(B, b)} \\ \text{is a constant multiple of } n(A, b) \text{ for } \mathcal{H}^{n-1} \text{ a.a. } b \in B. \end{array} \right.$

where

smooth vector fields with compact spt.

$\forall g \in \mathcal{X}(\mathbb{R}^n)$

$$* \int_B \overbrace{\operatorname{trace}(P_A(x) \circ Dg(x))}^{\text{"}F\text{-}\operatorname{div}_B g(x)\text{"}} \, d\mu(x) = - \int \mathbb{h}_F(B, x) \bullet g(x) \, d\mu(x)$$

$$* \quad \mu = F \circ n(A, \cdot)(\mathcal{H}^{n-1} \llcorner B)$$

$$* \left\{ \begin{array}{ll} P_A(x) \circ P_A(x) = P_A(x) \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^n) & \text{for } \mu\text{-a.a. } x \\ \operatorname{im} P_A(x) = \operatorname{span}\{n(A, x)\}^{\perp} = \operatorname{Tan}^{n-1}(\mu, x) & \\ \ker P_A(x) = \operatorname{span}\{\operatorname{grad} F(n(A, x))\} & \end{array} \right.$$

In case F is Euclidean, i.e, $F(x) = \sqrt{x \bullet x}$ ③

we have $\operatorname{grad} F(x) = \frac{x}{|x|}$, and $P_A(x)$ is an orthogonal projection for μ-a.a. x, and $\mathbb{h}_F(B,\cdot)$ is the classical mean curvature vector.

<u>Alexandrov, 1956 - 1962</u> (using maximum principle for elliptic PDEs)

If $M \subseteq \mathbb{R}^n$ is an embedded compact hypersurface with CMC, then M an $(n-1)$-sphere.

<u>Montiel & Ros, 1991</u> (using the Heintze - Karcher ineq.)

If $M \subseteq \mathbb{R}^n$ is an embedded compact hypersurface with constant H_r for some $r \in \{1, 2, \dots, n-1\}$, then M is an $(n-1)$-sphere.

[H_r - the r-th elementary symmetric polynomial in the principal curvatures of M.

$H_1(x) = \mathbb{h}(M, x) \bullet \nu_M(x)$]

<u>He, Li, Ma, Ge, 2009</u> (adaptation of the Montiel-Ros argument)

If $M \subseteq \mathbb{R}^n$ is an embedded compact hypersurface with constant H_r^F for some $r \in \{1, 2, \dots, n-1\}$, is an $(n-1)$-dimensional F^*- sphere, i.e., the boundary of some Wulff shape.

<u>Delgadino & Maggi, 2019</u> (Montiel - Ros argument Schätzle's maximum principle)

Among Caccioppoli sets of finite volume, finite unions of balls with equal radii are the unique critical points of the Euclidean isoperimetric problem.

Theorem (De Rosa, K., Santilli, 2020)

$\alpha \in (0,1)$, F a uniformly convex norm of class $C^{2,\alpha}$
E a Caccioppoli set such that

$$\mathcal{H}^{n-1}(\operatorname{Bdry} E \sim \operatorname{ess}\operatorname{Bdry} E) = 0$$

$$\mathcal{H}^{n-1}(\operatorname{ess}\operatorname{Bdry} E) < \infty, \quad B = \operatorname{ess}\operatorname{Bdry} E,$$

$$-F(n(E,x))\, \mathbf{h}_F(B,x) \bullet n(E,x) = H(x) \quad \text{for } x \in B,$$

H is locally $\mathcal{C}^{0,\alpha}$ on the $\mathcal{C}^{1,\alpha}$-regular part of B,
$c \in (0,\infty)$, $0 < H(x) < c$ for $x \in B$

Then

$$\mathcal{L}^n(E) \le \frac{n-1}{n} \int_B \frac{d\mathcal{H}^{n-1}}{H}.$$

H-K ineq.

Moreover, equality holds iff there is a finite union of Wulff shapes Ω with radii $\geq \frac{n-1}{c}$ such that $\mathcal{L}^n((\Omega \sim E) \cup (E \sim \Omega)) = 0$.

Remark. Santilli proved earlier a similar result for F Euclidean without employing Schätzle's maximum principle and without assuming H is $\mathcal{C}^{0,\alpha}$.

Sketch of the proof

I The case when B is smooth

Remark $F(n(E,x)) \cdot \mathbf{h}_F(B,x) = \operatorname{trace} D(\operatorname{grad} F \circ n(E,\cdot))(x)$

Def. Eigenvalues of the map $D(\operatorname{grad} F \circ n(E,\cdot))(x)|\operatorname{Tan}(B,x)$ are called the anisotropic principal curvatures of B and denoted

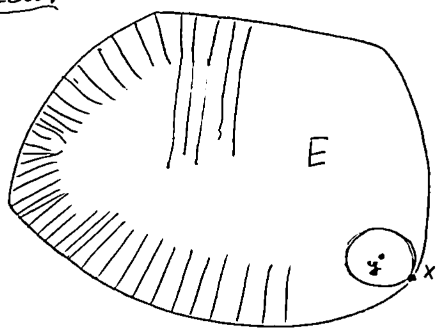
$$\kappa^F_{B,1}(x) \le \cdots \le \kappa^F_{B,n-1}(x)$$

Remark In case $W = \mathbb{R}^n \cap \{x : F^*(x) \le r\}$ is a Wulff shape we have $\kappa^F_{\partial W, i}(x) = \frac{1}{r}$ for $i \in \{1, \dots n-1\}$ and $x \in \partial W$.

Lemma If M is a $\mathcal{C}^{1,1}$-hypersurface in $\mathbb{R}^n$ with $\kappa_i^F(x) = \kappa_j^F(x)$ for $i,j \in \{1 \dots n-1\}$, $x \in M$, then $M = \partial W$ for some Wulff shape W.

Def.
$$\delta_E^F(x) = \inf\{F^*(x-y) : y \in E\} \quad \text{- distance function}$$
$$\operatorname{Unp}^F(E) = \mathbb{R}^n \cap \{x : \exists!\, y \in E \;\; \delta_E^F(x) = F^*(x-y)\} \quad \text{- unique nearest point}$$
$$\xi_E^F(x) = y \iff x \in \operatorname{Unp}^F(E) \;\&\; \delta_E^F(x) = F^*(x-y) \quad \text{- nearest point projection}$$
$$\underbrace{N^F(E)}_{F\text{-normal bundle of } E} = E \times S^{n-1} \cap \{(a,u) : \delta_E^F(a+su) = s \text{ for some } s > 0\}$$

$$\underbrace{n^F(E,x)}_{F\text{-normal vector}} = \operatorname{grad} F(n(E,x))$$



$C = \mathbb{R}^n \sim E$, $B = \operatorname{Bdry} E$

$y \in \operatorname{Unp}^F(C)$

Observe

$$0 \le \frac{H(x)}{n-1} \le -\kappa_{C,1}^F \le \frac{1}{\delta_C^F(y)}$$
for $y \in \operatorname{Unp}^F(E)$

Set
$$Z = B \times \mathbb{R} \cap \left\{(x,t) : 0 < t \le -\frac{1}{\kappa_{C,1}^F(x)}\right\}$$
$$\zeta : Z \to \mathbb{R}^n, \quad \zeta(x,t) = x + t\, n^F(C,x)$$

Easy computation gives
$$J_n \zeta(x,t) = \| \Lambda_n (\mathcal{H}^n \llcorner Z, n) \operatorname{ap} D\zeta(x,t) \|$$
$$= F(n(C,x)) \prod_{i=1}^{n-1} \left(1 + t\, \kappa_{B,i}^F(x)\right) \quad \text{for } (x,t) \in Z$$

We also know that

* δ_C^F is 1-Lipschitz $\Rightarrow$ differentiable $\mathcal{L}^n$ a.e.
* $E \cap \{x : D\delta_C^F(x) \text{ exists}\} \subseteq \operatorname{Unp}^F(C)$

$$\Rightarrow 0 = \mathcal{L}^n(E \sim \operatorname{Unp}^F(C)) = \mathcal{L}^n\left(E \sim (\xi_C^F)^{-1}[B]\right)$$
$$= \mathcal{L}^n(E \sim \zeta(Z))$$

The Montiel - Ros argument

$$\mathcal{L}^n(E) = \mathcal{L}^n(\zeta(Z)) \le \int_{\zeta(Z)} \mathcal{H}^0(\zeta^{-1}(y))\, d\mathcal{L}^n(y) \overset{\text{Area formula}}{=} \int_Z J_n \zeta \, d\mathcal{H}^n$$

$$\le \int_B F(n(c,x)) \int_0^{-1/k^F_{c,1}(x)} \left(\frac{1}{n-1} \sum_{i=1}^{n-1} (1 + t\, k^F_{c,i}(x)) \right)^{n-1} dt\, d\mathcal{H}^{n-1}$$

geometric mean $\le$ arithmetic mean

$$= \frac{n-1}{n} \int_B \frac{F(n(c,x))}{H(x)}\, d\mathcal{H}^{n-1}(x).$$

In case of equality we must have $k^F_{c,i}(x) = k^F_{c,j}(x)$ for $i,j \in \{1,\dots,n-1\}$ and $\mathcal{H}^{n-1}$ a.a. $x \in B$

geometric mean = arithmetic mean

$\Rightarrow$ B is totally F-umbilical

$\Rightarrow$ B is a Wulff shape.

II B might not be smooth but assume

$\circledast$ $\quad \mathcal{H}^{n-1}(\mathrm{Bdry}\, E \sim \mathrm{ess\,Bdry}\, E) = 0$

e.g. Lipschitz domains

Def. A satisfies the Lusin (N) condition in U iff

$$S \subseteq A \cap U,\ \mathcal{H}^{n-1}(S) = 0 \quad \Rightarrow \mathcal{H}^{n-1}(N^F(A)|S) = 0$$

Lemma 1 B satisfies the Lusin (N) condition

[Proof uses a result of De Philippis, De Rosa, and Hirsch from 2019 on area blow up set for bounded mean curvature submanifolds.

Lemma 2 $\mathcal{H}^{n-1}$ almost all points of B are $\mathcal{C}^{2,\alpha}$-regular points, i.e., in some neighbourhood of such a point B coincides with $\mathcal{C}^{2,\alpha}$ manifold.

[Proof uses the Allard Regularity Theorem from 1986 plus some classical elliptic PDE theory

Set Q = the set of $\mathcal{C}^{2,\alpha}$ points of B.

Using Lemma 1 & Lemma 2 we show that

$$\mathcal{H}^{n-1}(B \sim Q) = 0$$

$$\mathcal{H}^{n-1}\left(\{x : \delta_c^F(x) = r\}\right) \sim (\zeta_c^F)^{-1}[Q]\Big) = 0$$

for $r \in (0, \infty)$

Then coarea formula gives

$$\mathcal{L}^n\left(E \sim (\zeta_c^F)^{-1}[Q]\right) = 0$$

and we can use the Montiel-Ros argument and get H-K ineq.

But this is <u>not</u> the end!

We prove the <u>anisotropic Steiner formula</u> to conclude that <u>Q has positive F-reach $\geq r_0$</u>

The level-sets $\{x : \delta_c^F(x) = r\} = S^F(c, r)$ for $r \in (0, r_0)$ are always $\mathcal{C}^{1,1}$ submanifolds of $\mathbb{R}^n$.

Moreover, one can compute principal curvatures of $S^F(c, r)$ in terms of principal curvatures of Q.

This gives that $S^F(c, r)$ are totally umbilical and $\mathcal{C}^{1,1}$ regular; hence, Wulff shapes.

Passing to the limit $r \to 0^+$ concludes the proof.

<u>Open question:</u> Is there a similar characterisation if F is only continuous?