Regularity results for bounded minimizers

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**THE PROBLEM**

Regularity results for local bounded minimizers of integral functionals of the type

$$
\mathcal{F}(v, \Omega) = \int_{\Omega} f(x, Dv) \, dx \quad \Omega \subset \mathbb{R}^n
$$

in case

- *unconstrained* problem
- *constrained* problem
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In both cases the integrand \( f \)
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- $\xi \rightarrow f(x, \xi)$  $p$-growth
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in case

- *unconstrained* problem
- *constrained* problem

In both cases the integrand \( f \)

- \( \xi \to f(x, \xi) \) \( p \)-growth

- can be *discontinuous* with respect to the \( x \)-variable.

M. Caselli, A. Gentile, R. G. *Regularity results for solutions to obstacle problems with Sobolev coefficients* J. Differential Equations 269 (2020)
**Assumptions**

Let us consider

\[ F(v, \Omega) = \int_{\Omega} f(x, Dv) \, dx \]  

(\text{F})

\( \Omega \) open bounded set in \( \mathbb{R}^n, n > 2 \)

- \( v : \Omega \rightarrow \mathbb{R}^N \quad N \geq 2 \)

- \( f : \Omega \times \mathbb{R}^{n \times N} \rightarrow \mathbb{R} \) is a Carathéodory mapping satisfying
**Assumptions w.r.t $\xi$-variable**

there exist $p \geq 2$ and positive constants $L, \ell, \nu > 0$ s.t.

\[
\frac{1}{L} |\xi|^p \leq f(x, \xi) \leq L(1 + |\xi|^p). \tag{F1}
\]

\[
\langle D_\xi f(x, \xi) - D_\xi f(x, \eta), \xi - \eta \rangle \geq \nu (1 + |\xi|^2 + |\eta|^2) \frac{p-2}{2} |\xi - \eta|^2 \tag{F2}
\]

\[
|D_\xi f(x, \xi) - D_\xi f(x, \eta)| \leq \ell (1 + |\xi|^2 + |\eta|^2) \frac{p-2}{2} |\xi - \eta| \tag{F3}
\]

for all $\xi, \eta \in \mathbb{R}^{n \times N}$ and for almost every $x \in \Omega$. 
ASSUMPTIONS W.R.T $x$-VARIABLE

There exists $g(x) \in L^\sigma(\Omega)$, $\sigma > 1$ s.t.

$$|D_{\xi}f(x, \xi) - D_{\xi}f(y, \xi)| \leq (|g(x)| + |g(y)|)|x - y|(1 + |\xi|^2)^{\frac{p-1}{2}} \quad (F4)$$

for a.e. $x, y \in \Omega$ and for all $\xi \in \mathbb{R}^{n \times N}$.
ASSUMPTIONS W.R.T $x$-VARIABLE

There exists $g(x) \in L^\sigma(\Omega), \sigma > 1$ s.t.

$$|D_\xi f(x, \xi) - D_\xi f(y, \xi)| \leq (|g(x)| + |g(y)|)|x - y|(1 + |\xi|^2)^{\frac{p-1}{2}} \quad (F4)$$

for a.e. $x, y \in \Omega$ and for all $\xi \in \mathbb{R}^{n \times N}$.

Assumption (F4) with $g \in L^\sigma_{\text{loc}}(\Omega)$ implies that

$$x \rightarrow D_\xi f(x, \xi) \in W^{1,\sigma}_{\text{loc}}(\Omega, \mathbb{R}^{n \times N})$$

(see Hajlasz, Potential Anal. 5 (1996))

MODEL CASE

\[ \int_{\Omega} a(x) (1 + |Du|^2)^{p/2} \, dx \quad \text{with} \quad a(x) \in L^\infty \cap W^{1,\sigma}(\Omega) \]

\( p \geq 2 \) and \( \sigma > 1 \)
Model case

\[ \int_{\Omega} a(x)(1 + |Du|^2)^{p/2} \, dx \quad \text{with} \quad a(x) \in L^\infty \cap W^{1,\sigma}(\Omega) \]

\( p \geq 2 \) and \( \sigma > 1 \)

Question:
How does the regularity of \( a(x) \) transfer to \( Du \)?
Unconstrained case
ABOUT THE ASSUMPTION ON $x$-VARIABLE

Classical Theory

- $x \mapsto D_\xi f(x, \xi) \in \text{Lip}(\Omega)$

i.e. there exists a constant $K > 0$

$$|D_\xi f(x, \xi) - D_\xi f(y, \xi)| \leq K|x - y|(1 + |\xi|^2)^{p-1 \over 2}$$

\[\downarrow\]

$$(1 + |Du|^2)^{p-2 \over 4} Du \in W^{1,2}$$
SOBOLEV ASSUMPTION

More recent Developments

- \( x \mapsto D_\xi f(x, \xi) \in W^{1,n} \)

i.e. there exists a non negative function \( g \in L^n \) such that

\[
|D_\xi f(x, \xi) - D_\xi f(y, \xi)| \leq (|g(x)| + |g(y)|)|x - y|(1 + |\xi|^2)^{\frac{p-1}{2}}
\]

\( \downarrow \)

Higher differentiability results with integer order
$W^{1,n}$ assumption: Higher differentiability
results with integer order
$W^{1,n}$ ASSUMPTION: HIGHER DIFFERENTIABILITY RESULTS WITH INTEGER ORDER

Beltrami Equations

  \( n = 2 \) and \( A(x, \xi) = A(x) \cdot \xi \) with \( \det A = 1 \)

  in connections with planar mappings with finite distortion
$W^{1,n}$ assumption: Higher differentiability results with integer order

Beltrami Equations
  ($n = 2$ and $A(x, \xi) = A(x) \cdot \xi$ with $\det A = 1$)
  in connections with planar mappings with finite distortion

Systems and integral functionals
  $p = n = 2, \quad 2 \leq p < n$
  variable exponents
- G. - NoDEA (2016) Orlicz – Sobolev coefficients
$W^{1,n}$ assumption: Higher differentiability results with integer order

- Cupini, Giannetti, G. & Passarelli di Napoli - J. Differential Equation (2018) **convexity only at $\infty$**
- Cupini, Marcellini, Mascolo & Passarelli di Napoli, Preprint (2021) **degenerate ellipticity**
FURTHER RESULTS IN CASE OF SOBOLEV COEFFICIENTS

- Eleuteri, Marcellini & Mascolo
  - Ann. Mat. Pura Appl. (2016),
- De Filippis & Mingione, Preprint (2020)
- Cupini, Marcellini, Mascolo & Passarelli di Napoli, Preprint (2021)
$W^{1,n}$
$W^{1,n} \hookrightarrow VMO$
\( W^{1,n} \hookrightarrow \text{VMO} \)

- Bögelein, J. Differential Equation (2012)
- Di Fazio, Fanciullo & Zamboni, Algebra i Analiz (2013)
- Goodrich & Ragusa, Nonlinear Anal (2019)
- Goodrich, Scilla & Stroffolini, Preprint (2021)
\( W^{1,n} \hookrightarrow \text{VMO} \)

- Bögelein, J. Differential Equation (2012)
- Di Fazio, Fanciullo & Zamboni, Algebra i Analiz (2013)
- Goodrich & Ragusa, Nonlinear Anal (2019)
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Question:
What happens if we weaken the assumption on $g$?
A PRIORI BOUNDED MINIMIZERS

Theorem. [G.- Passarelli di Napoli (2019)]

Let \( f : \Omega \times \mathbb{R}^{n \times N} \rightarrow \mathbb{R} \) be an integrand satisfying the assumptions (F1)–(F4) for a function \( g \in L_{\text{loc}}^{p+2}(\Omega) \). If \( u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N) \) is a local minimizer of the functional \( \mathcal{F} \), then

\[
(1 + |Du|^2)^{\frac{p-2}{4}} Du \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^{n \times N})
\]

Moreover, for every balls \( B_R \subset B_{2R} \subset \Omega \), we have that

\[
\int_{B_R} \left| D \left( (1 + |Du|^2)^{\frac{p-2}{4}} Du \right) \right|^2 dx \\
\quad \leq c \int_{B_{2R}} (1 + |Du|^2)^{\frac{p}{2}} dx + c \int_{B_{2R}} |g(x)|^{p+2} dx,
\]

where \( c = c(\|u\|_\infty, R, p, n, N, L, \nu) \).
Remarks

\[ g \in L^{p+2} \]
Remarks

1. assumption on the summability of the function $g(x)$ that is independent of the dimension $n$. 

$$g \in L^{p+2}$$
Remarks

\[ g \in L^{p+2} \]

1. assumption on the summability of the function \( g(x) \) that is independent of the dimension \( n \).

2. this is a weaker assumption with respect to previous papers when \( 2 \leq p < n - 2 \).
Proof of the Theorem

Step 1: The approximation. We construct the approximating problems:

Fix a compact set $\Omega' \subseteq \Omega$, and for a smooth kernel $\phi \in C^\infty_c(B_1(0))$ with $\phi \geq 0$ and $\int_{B_1(0)} \phi = 1$, let us consider the corresponding family of mollifiers $(\phi_\varepsilon)_{\varepsilon > 0}$. Put

$$g_\varepsilon = g * \phi_\varepsilon$$

and

$$f_\varepsilon(x, \xi) = \int_{B_1} \phi(\omega)f(x + \varepsilon\omega, \xi) \, d\omega$$

on $\Omega'$, for each positive $\varepsilon < \text{dist} (\Omega', \Omega)$. 
Fix a real number \( a \geq ||u||_{L^\infty(\Omega')} \) and, for \( m > \frac{p}{2} \), let \( u_{\varepsilon,m} \) be a minimizer to the functional

\[
\mathcal{F}_{\varepsilon,m}(v, \Omega') = \int_{\Omega'} \left( f_{\varepsilon}(x, Dv) + (|v| - a)^{2m} \right)
\]

Proof of the Theorem

Step 2: Uniform higher differentiability estimates (by using interpolation inequality)

\[ \tau_{s,h} u_{\varepsilon,m}(x) = u_{\varepsilon,m}(x + h e_s) - u_{\varepsilon,m}(x) \]

Choosing \( \varphi = \tau_{s,-h}(\rho^{p+2}\tau_{s,h}u_{\varepsilon,m}) \) as test function in the Euler–Lagrange system associated to the functional \( \mathcal{F}_{\varepsilon,m}(v, \Omega') \) and using the assumptions and some properties of the difference quotients we obtain

\[
\int_{B_{2R}} |\tau_{s,h}(\rho^{\frac{p+2}{2}} V(Du_{\varepsilon,m}))|^2 \leq c|h|^2 \int_{B_{2R}} \rho^{p+2}(g_{\varepsilon}(x) + g_{\varepsilon}(x + h))^2 (1 + |Du_{\varepsilon,m}|^2)^{\frac{p}{2}}
\]

\[ + c \frac{|h|^2}{R^2} \int_{B_{3R}} (1 + |Du_{\varepsilon,m}|^2)^{\frac{p}{2}}. \]
PROOF OF THE THEOREM

By a suitable interpolation inequality we have

\[ Du_{\varepsilon,m} \in L^{\frac{m}{m+1}}(p+2) \]
**Proof of the Theorem**

By a suitable interpolation inequality we have

$$Du_{\varepsilon,m} \in L^{\frac{m}{m+1} (p+2)}$$

we can use Hölder’s inequality with exponents \( \frac{m}{m+1} \frac{p+2}{p} \) and \( \frac{m(p+2)}{2m-p} \) to get

$$\int_{B_{2R}} |\tau_{s,h}(\rho^{\frac{p+2}{2}} V(Du_{\varepsilon,m}))|^2$$

$$\leq c |h|^2 \left( \int_{B_{2R}} \rho^{p+2} (g_{\varepsilon}(x) + g_{\varepsilon}(x + h)) \frac{2m(p+2)}{2m-p} \right)^{\frac{2m-p}{m(p+2)}}$$

$$\cdot \left( \int_{B_{2R}} \rho^{p+2} (1 + |Du_{\varepsilon,m}|^2) \frac{m}{m+1} \frac{(p+2)}{2} \right)^{\frac{m+1}{m} \frac{p}{p+2}}$$

$$+ c \frac{|h|^2}{R^2} \int_{B_{3R}} (1 + |Du_{\varepsilon,m}|^2)^{\frac{p}{2}}$$
PROOF OF THE THEOREM

Step 3: we show that such estimates are preserved in passing to the limit.
SYSTEMS UNDER SUITABLE STRUCTURE ASSUMPTIONS

We consider elliptic systems of the form

$$\text{div} A(x, Du) = \sum_{i=1}^{n} D_{x_i} \left( \sum_{j=1}^{n} a_{ij}(x, Du) u_{x_j}^\alpha \right) = 0, \ 1 \leq \alpha \leq N, \ \text{in} \ \Omega \subset \mathbb{R}^n \ (\ast)$$

satisfying

$$A(x, 0) = 0 \quad (A0)$$

$$\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq \alpha |\xi - \eta|^2 (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \quad (A1)$$

$$|A(x, \xi) - A(x, \eta)| \leq \beta |\xi - \eta| (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \quad (A2)$$

There exists a nonnegative function $g \in L^p_{loc}(\Omega)$, such that

$$|A(x, \xi) - A(y, \xi)| \leq (g(x) + g(y)) |x - y| \left(1 + |\xi|^2 + \frac{2}{p-2} \right)^{\frac{p-2}{2}}; \quad (A3)$$

for every $\xi \in \mathbb{R}^n \times \mathbb{N}$ and for almost every $x, y \in \Omega$.
SYSTEMS UNDER SUITABLE STRUCTURE ASSUMPTIONS

We consider elliptic systems of the form

$$\text{div} A(x, Du) = \sum_{i=1}^{n} D_{x_i} \left( \sum_{j=1}^{n} a_{ij}(x, Du) u_\alpha^{x_j} \right) = 0, \quad 1 \leq \alpha \leq N, \text{ in } \Omega \subset \mathbb{R}^n \quad (*)$$

satisfying

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There exists a nonnegative function $g \in L_{\text{loc}}^{p+2}(\Omega)$, such that

$$|A(x, \xi) - A(y, \xi)| \leq (g(x) + g(y))|x - y| (1 + |\xi|^2)^{\frac{p-1}{2}} \quad \text{(A3)}$$

for every $\xi \in \mathbb{R}^{n \times N}$ and for almost every $x, y \in \Omega$. 
Theorem. [G.- Passarelli di Napoli (2019)]

Let $A : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n}$ be a Carathéodory function satisfying the assumptions (A0)–(A3). If $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a local solution of the system (*) , then

$$(1 + |Du|^2)^{\frac{p-2}{4}} Du \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^{n \times N})$$

Moreover, for every ball $B_r \subseteq \Omega$

$$\int_{B_{r/4}} (1 + |Du|^2)^{\frac{p-2}{2}} |D^2u|^2 \, dx \leq \frac{c}{r^2} \int_{B_r} (1 + |Du|^2)^{\frac{p}{2}} \, dx$$

$$\frac{c}{r^n} \|u\|_{L^p(B_{2r})}^p \left( \int_{B_r} (1 + g(x))^{p+2} \, dx \right),$$

for a constant $c = c(\alpha, \beta, p, n)$. 
**Proof of the Theorem**

**Step 1** A priori estimate

- difference quotient method
- local boundedness of the solutions $u \in \mathcal{W}^{1,p}_{\text{loc}}(\Omega)$ of the system and following estimate

\[
\sup_{B_{\frac{R}{2}}(x_0)} |u| \leq c \left\{ \int_{B_R(x_0)} (|u| + 1)^{p^*} \, dx \right\}^{\frac{1}{p^*}}
\]


(see also Leonetti Boll. Un. Mat. Ital. (1991))

- interpolation inequality
**Proof of the Theorem**

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- difference quotient method
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(see also Leonetti Boll. Un. Mat. Ital. (1991))

- interpolation inequality

**Step 2** Approximation procedure
**Remark**

If it is assumed a priori

\[ u \in L^q, \quad \text{with} \quad q > \frac{np}{n - p - 2} \quad \text{(instead of} \quad u \in L^\infty) \]

the interpolation inequality gives

\[ Du \in L^{\frac{q}{q+2}(p+2)} \quad \text{(instead of} \quad Du \in L^{p+2}) \]
**Remark**

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the interpolation inequality gives

\[ Du \in L^{\frac{q}{q+2}(p+2)} \quad (\text{instead of} \quad Du \in L^{p+2}) \]

Such higher integrability allow us to obtain the same higher differentiability result assuming \( g \in L^{\frac{q}{q-p}(p+2)} \).

We’d like to point out that for \( p < n - 2 \) it results \( \frac{q}{q-p}(p + 2) < n \).
Constrained case
Obstacle Problem

We consider the following obstacle problem

\[
\min \left\{ \int_{\Omega} f(x, Dv(x)) : v \in K_\psi(\Omega) \right\},
\]

(1)

where \( \Omega \subset \mathbb{R}^n \) is a bounded open set,

- \( \psi : \Omega \mapsto [-\infty, +\infty) \) belonging to \( W^{1,p}_{\text{loc}} \) is the obstacle,

- \( K_\psi(\Omega) = \{ v \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}) : v \geq \psi \text{ a.e. in } \Omega \} \) is the class of the admissible functions.
OBSTACLE PROBLEMS AND VARIATIONAL FORMULATION

We observe that

\[ u \in W^{1,p}_{\text{loc}}(\Omega) \text{ is a solution to the obstacle problem in } \mathcal{K}_\psi \]

\[ \uparrow \]

\[ u \in \mathcal{K}_\psi(\Omega) \text{ is a solution to the variational inequality} \]

\[ \int_\Omega \langle A(x,Du), D(\varphi - u) \rangle \, dx \geq 0 \quad \forall \varphi \in \mathcal{K}_\psi(\Omega), \]

where \( A(x, \xi) = D_\xi f(x, \xi) \).
It is well known that:

*the regularity of solutions to the obstacle problems depends on the regularity of the obstacle itself*
Analysis of the **extra differentiability** of the solutions of the obstacle problems

\[ \int_{\Omega} \langle A(x, Du(x)), D(\varphi(x) - u(x)) \rangle \, dx \geq 0 \quad \forall \varphi \in K_\psi(\Omega), \]

assuming that the gradient of the obstacle \( D\psi \) has some differentiability property
ASSUMPTIONS

Let us fix $\psi \in W_{\text{loc}}^{1,p}(\Omega)$ and consider

$$\int_{\Omega} \langle A(x, Du), D(\varphi - u) \rangle \, dx \geq 0,$$

for every $\varphi \in K_{\psi}(\Omega) = \{ v \in W_{\text{loc}}^{1,p}(\Omega, \mathbb{R}) : v \geq \psi \text{ a.e. in } \Omega \}$

There exist constants $\nu, L > 0$ and an exponent $p \geq 2$ such that

$$\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq \nu |\xi - \eta|^2 (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \tag{A1}$$

$$|A(x, \xi) - A(x, \eta)| \leq L |\xi - \eta| (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \tag{A2}$$
**ASSUMPTIONS**

Let us fix $\psi \in W^{1,p}_{\text{loc}}(\Omega)$ and consider

$$
\int_{\Omega} \langle A(x, Du), D(\varphi - u) \rangle \, dx \geq 0,
$$

(\star \star)

for every $\varphi \in K_\psi(\Omega) = \{ v \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}) : v \geq \psi \text{ a.e. in } \Omega \}$

There exist constants $\nu, L > 0$ and an exponent $p \geq 2$ such that

$$
\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq \nu |\xi - \eta|^2 (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}
$$

(A1)

$$
|A(x, \xi) - A(x, \eta)| \leq L |\xi - \eta| (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}
$$

(A2)

There exists a nonnegative function $g \in L^{p+2}_{\text{loc}}(\Omega)$, such that

$$
|A(x, \xi) - A(y, \xi)| \leq (g(x) + g(y)) |x - y| (1 + |\xi|^2)^{\frac{p-1}{2}};
$$

(A3)

for all $\xi, \eta \in \mathbb{R}^n$ and for almost every $x, y \in \Omega$. 
Remark

The regularity of the solutions to the obstacle problem (**) is strictly connected to the regularity of the solutions to PDE’s of the form

$$\text{div}A(x, Du) = \text{div}A(x, D\psi).$$

It is well known that no extra differentiability properties for the solutions of equations of the type

$$\text{div}A(x, Du) = \text{div}G$$

can be expected even if $G$ is smooth, unless some assumption is given on the $x$-dependence of the operator $A$. 
Some Results

\( x \mapsto A(x, \xi) \in W^{1,r} \quad \text{with} \quad r \geq n \)

- De Filippis & Mingione - (2020)
Theorem. [Caselli – Gentile – G.(2020)]

Let $A(x, \xi)$ satisfy the conditions (A1)–(A4) for an exponent $p \geq 2$ and let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to the obstacle problem. Then, if $\psi \in L^\infty_{\text{loc}}(\Omega)$ the following implication holds

$$D\psi \in W^{1, \frac{p+2}{2}}_{\text{loc}}(\Omega) \Rightarrow \left(\mu^2 + |Du|^2\right)^\frac{p-2}{4} \quad Du \in W^{1,2}_{\text{loc}}(\Omega),$$

Remark: the assumption $\psi \in L^\infty_{\text{loc}}(\Omega)$ is needed to get the boundedness of the solution. Therefore if we deal with a priori bounded minimizers, then the result holds without the hypothesis $\psi \in L^\infty_{\text{loc}}(\Omega)$. (see Caselli – Eleuteri – Passarelli di Napoli, ESAIM - Control. Optim. Calc. Var. (2021))
Theorem. [Caselli – Gentile – G.(2020)]

Let $A(x, \xi)$ satisfy the conditions (A1)-(A4) for an exponent $p \geq 2$ and let $u \in K_\psi(\Omega)$ be a solution to the obstacle problem. Then, if $\psi \in L_{\text{loc}}^\infty(\Omega)$ the following implication holds

$$D\psi \in W_{\text{loc}}^{1, \frac{p+2}{2}}(\Omega) \Rightarrow \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{4}} Du \in W_{\text{loc}}^{1,2}(\Omega),$$

Remark: the assumption $\psi \in L_{\text{loc}}^\infty(\Omega)$ is needed to get the boundedness of the solution. Therefore if we deal with a priori bounded minimizers, then the result holds without the hypothesis $\psi \in L^\infty.$

Proof of the Theorem

- A priori estimate
- Approximation procedure
TEST FUNCTIONS

The main point is the choice of suitable test functions $\varphi$:

1. involving the difference quotient of the solution

2. belonging to the class of the admissible functions $\mathcal{K}_\psi(\Omega)$,

Let us consider $\varphi := u + \tau v$ for a suitable $v \in W_0^{1,p}(\Omega)$ such that

$$u - \psi + \tau v \geq 0 \quad \forall \tau \in [0, 1],$$

(***)

Then $\varphi \in \mathcal{K}_\psi(\Omega)$ for all $\tau \in [0, 1]$, since $\varphi = u + \tau v \geq \psi$. 
Test functions

Let $\eta$ be a cut off function, we consider

$$v_1(x) = \eta^2(x) \left[ (u - \psi)(x + h) - (u - \psi)(x) \right],$$

$v_1$ satisfies (***). Indeed, for a.e. $x \in \Omega$ and for any $\tau \in [0, 1]$,

$$u(x) - \psi(x) + \tau v_1(x) =$$

$$= u(x) - \psi(x) + \tau \eta^2(x) \left[ (u - \psi)(x + h) - (u - \psi)(x) \right]$$

$$= \tau \eta^2(x) (u - \psi)(x + h) + (1 - \tau \eta^2(x))(u - \psi)(x) \geq 0,$$

since $u \in K_\psi(\Omega)$ and $0 \leq \eta \leq 1$. So we can use $\varphi = u + \tau v_1$ as a test function in variational inequality.
**Test functions**

In a similar way, we consider

\[ v_2(x) = \eta^2(x) [(u - \psi)(x - h) - (u - \psi)(x)] , \]

and we have (***) still is satisfied for any \( \tau \in [0, 1] \), since

\[ u(x) - \psi(x) + \tau v_2(x) = \]

\[ = u(x) - \psi(x) + \tau \eta^2(x) [(u - \psi)(x - h) - (u - \psi)(x)] \]

\[ = \tau \eta^2(x)(u - \psi)(x - h) + (1 - \tau \eta^2(x))(u - \psi)(x) \geq 0. \]

So we can use \( \varphi = u + \tau v_2 \) as a test function in variational inequality.
Thanks for your attention!