Unbounded supersolutions with generalized Orlicz growth

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We say that \( \varphi : \Omega \times [0, \infty) \to [0, \infty] \) is a \textit{weak \( \Phi \)-function}, and write \( \varphi \in \Phi_w(\Omega) \), if the following conditions hold:

- For every measurable function \( f : \Omega \to \mathbb{R} \) the function \( x \mapsto \varphi(x, f(x)) \) is measurable and for every \( x \in \Omega \) the function \( t \mapsto \varphi(x, t) \) is non-decreasing.
- \( \varphi(x, 0) = \lim_{t \to 0^+} \varphi(x, t) = 0 \) and \( \lim_{t \to \infty} \varphi(x, t) = \infty \) for every \( x \in \Omega \).
- The function \( t \mapsto \frac{\varphi(x,t)}{t} \) is \( L \)-almost increasing on \((0, \infty)\) with \( L \) independent of \( x \).
Some special cases of $\Phi$-functions:

- $\varphi(x, t) = t^p$ the classical Lebesgue space
- $\varphi(x, t) = \varphi(t)$ the Orlicz space
- $\varphi(x, t) = t^{p(x)} a(x)$ the variable exponent Lebesgue space
- $\varphi(x, t) = t^{p(x)} \log(e + t)$
- $\varphi(x, t) = t^p + a(x) t^q$ the double phase case
We assume that $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following $\varphi$-growth conditions:

$$\nu \varphi(x, |\xi|) \leq f(x, \xi) \cdot \xi \quad \text{and} \quad |f(x, \xi)| |\xi| \leq \Lambda \varphi(x, |\xi|)$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, and fixed but arbitrary constants $0 < \nu \leq \Lambda$. We are interested in local (weak) supersolutions:

**Definition 1**

A function $u \in W^{1,\varphi}_{\text{loc}}(\Omega)$ is a supersolution if

$$\int_{\Omega} f(x, \nabla u) \cdot \nabla h \, dx \geq 0,$$

for all non-negative $h \in W^{1,\varphi}(\Omega)$ with compact support in $\Omega$. 
If $\varphi$ is differentiable wrt second variable, then our assumptions covers also the equation

$$
\int_{\Omega} \frac{\varphi'(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla h \geq 0,
$$

for all non-negative $h \in W_{0}^{1,\varphi}(\Omega)$.

Instead of supersolutions, you can think local superminimizers: Every open set $D \subseteq \Omega$ and for every non-negative $v \in W^{1,\varphi}(\Omega)$ with a compact support in $D$, we have

$$
\int_{D} F(x, |\nabla u|) \, dx \leq \int_{D} F(x, |\nabla (u + v)|) \, dx.
$$

Here $F(x, t) \approx \varphi(x, t)$. 
Special case: $\varphi(x, t) = t^p$.

The standard $p$-Laplace equation $-\text{div}(|\nabla u|^{p-2}\nabla u) = 0$, $1 < p < \infty$. The non-negative weak supersolutions satisfies the weak Harnack inequality

$$\left(\int_{2B} u^s \, dx\right)^{\frac{1}{s}} \lesssim \text{ess inf}_B u,$$

where

- the constant is independent of $u$,
- $0 < s < \frac{n}{n-p}(p-1)$ when $p < n$, and $s \in (0, \infty)$ when $p \geq n$.

Trudinger (1967)
Special case: Orlicz $\varphi(x, t) = \varphi(t)$.

**Theorem 2 (Arriagada–Huentutripay (2018))**

Assume that $1 < p \leq \frac{t\psi(t)}{\varphi(t)} \leq q < \infty$ and $\varphi(t) = \int_0^t \psi(t) \, dt$. Let $u \geq 0$ be bounded supersolution. Then

\[
\left( \int_B u^s \, dx \right)^{\frac{1}{s}} \lesssim \text{ess inf}_B u + \text{diam}(B).
\]


There have to be some results for corresponding minimizers.
Special case: variable exponent $\varphi(x, t) = t^{p(x)}$.

**Theorem 3 (Lukkari (2010))**

Assume that $p$ is log-Hölder continuous and $1 < p^- \leq p^+ < \infty$. Let $t > 0$, $0 < s < \frac{n}{n-1}(p^- - 1)$, and let $u \geq 0$ be supersolution. Then

$$
\left( \int_{2B} u^s \, dx \right)^{\frac{1}{s}} \lesssim \text{ess inf}_B u + \text{diam}(B),
$$

where the constant depends on $L^t_{t}(4B)$-norm of $u$.

- Bounded supersolutions and $0 < s < \frac{n}{n-1}(p_0 - 1)$, Alkhutov–Krasheninnikova (2004).
Special case: variable exponent $\varphi(x, t) = t^{p(x)}$.

- "$+ \text{diam}(B)$" is not needed if $p \in C^1$, Julin (2015)
- It is not known if "$+ \text{diam}(B)$" is necessary or not.
- In the Harnack’s inequality the constant cannot be independent of $u$, example in H–Kinnunen–Lukkari (2007)
Special case: double phase $\varphi(x, t) = t^p + a(x)t^q$.

**Theorem 4 (Baroni–Colombo–Mingione (2015))**

Let $a \in C^{0, \alpha}$, $\alpha \geq \frac{n}{p}(q - p)$. Let $u \geq 0$ be bounded supersolution. Then there exists $s > 0$ such that

$$\left(\int_B u^s \, dx\right)^{\frac{1}{s}} \lesssim \text{ess inf}_B u.$$  

Here the constant depends on $\|u\|_{\infty}$. 
Other related results:

- \( \varphi(x, t) = t^{p(x)} \) and general structural conditions, Latvala–Toivanen (2017)

- \( \varphi(x, t) = t^{p(x)} \) and \( p \) makes a jump at a hyperplane, Alkhutov–Surnachev (2019)

- \( \varphi(x, t) = t^{p(x)} \) and \( p \) is piecewise constant, Alkhutov–Surnachev (2019, 2020)

- \( \varphi(x, t) = t^{p(x)} \log(e + t) \), Ok (2018)

- Generalized double phase functional, Byen–Oh (2020)
Let \( p, q, s > 0 \) and let \( \omega : \Omega \times [0, \infty) \rightarrow [0, \infty) \) be almost increasing. We say that \( \varphi : \Omega \times [0, \infty) \rightarrow [0, \infty) \) satisfies

(A0) if there exists \( \beta \in (0, 1] \) such that \( \beta \leq \varphi^{-1}(x, 1) \leq \frac{1}{\beta} \) for a.e. \( x \in \Omega \),

(A1-ω) if there exists \( \beta \in (0, 1] \) such that, for every ball \( B \) and a.e. \( x, y \in B \cap \Omega \),

\[
\varphi(x, \beta t) \leq \varphi(y, t) \quad \text{when} \quad \omega_B^{-}(t) \in \left[ 1, \frac{1}{|B|} \right];
\]

(A1-s) if it satisfies (A1-ω) for \( \omega(x, t) := t^s \);

(A1) if it satisfies (A1-ϕ);

(alnc)_p if \( t \mapsto \frac{\varphi(x,t)}{t^p} \) is \( L_p \)-almost increasing in \((0, \infty)\) for some \( L_p \geq 1 \) and a.e. \( x \in \Omega \);

(aDec)_q if \( t \mapsto \frac{\varphi(x,t)}{t^q} \) is \( L_q \)-almost decreasing in \((0, \infty)\) for some \( L_q \geq 1 \) and a.e. \( x \in \Omega \).
\[ \varphi(x, t) := \begin{cases} (A0) & \text{true} \\ (A1) & \text{true} \\ (A1-s) & \text{true} \\ (aInc) & \nabla_2 \\ (aDec) & \Delta_2 \end{cases} \]

<table>
<thead>
<tr>
<th>\varphi(t)</th>
<th>(A0)</th>
<th>(A1)</th>
<th>(A1-s)</th>
<th>(aInc)</th>
<th>(aDec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ t^p a(x) ]</td>
<td>true</td>
<td>true</td>
<td>true</td>
<td>[ \nabla_2 ]</td>
<td>[ \Delta_2 ]</td>
</tr>
<tr>
<td>[ t^p \log(e + t) ]</td>
<td>[ a \approx 1 ]</td>
<td>[ p \in C^{\log} ]</td>
<td>[ p \in C^{\log} ]</td>
<td>[ p^- &gt; 1 ]</td>
<td>[ p^+ &lt; \infty ]</td>
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<tr>
<td>[ t^p + a(x)t^q ]</td>
<td>[ a \in L^\infty ]</td>
<td>[ a \in C^{0, \frac{n}{p}(q-p)} ]</td>
<td>[ a \in C^{0, \frac{n}{s}(q-p)} ]</td>
<td>[ p &gt; 1 ]</td>
<td>[ q &lt; \infty ]</td>
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**Table:** Assumptions in some special cases
Theorem 5 (Benyaiche-H-Hästö-Karppinen (accepted))

Suppose \( \varphi \) satisfies (A0), (aInc)\( p \) and (aDec)\( q \), \( 1 < p \leq q < \infty \).
Let \( u \geq 0 \) be a supersolution. Assume one of the following:

1. \( \varphi \) satisfies (A1-\( s_* \)) and \( \|u\|_{L^s(B_{2R})} \leq d \), where \( s_* := \frac{ns}{n+s} \) and \( s \in [q-p, \infty] \).
2. \( \varphi \) satisfies (A1) and \( \|u\|_{W^{1,\varphi}(B_{2R})} \leq d \).

Then there exist positive constants \( \ell_0 \) and \( C \) such that

\[
\left( \int_{B_{2R}} (u + R)^{\ell_0} \, dx \right)^{\frac{1}{\ell_0}} \leq C (\text{ess inf}_{B_{R}} u + R).
\]

If (1) holds with \( s > \max\{\frac{n}{p}, 1\}(q-p) \) or if (2) holds with \( p^* > q \), then the weak Harnack inequality holds for any \( \ell_0 < \ell(p) \), where \( \ell(p) = \frac{n}{n-p}(p-1) \) if \( p < n \), and \( \ell(p) = \infty \) if \( p \geq n \).
Other results on generalized Orlicz spaces:

- Bounded supersolutions, Benyaiche–Khlifi (2020).
Proposition 6 (Benyaiche–H–Hästö–Karppinen (accepted))

The (A1-${s}_*$) assumption in the previous theorem is sharp, since for any $s' < s_*$ if, instead of (1), $\varphi$ satisfies (A1-$s'$) and $\|u\|_{L^s(B_{2R})} \leq d$, then the weak Harnack inequality need not hold.
Let $\varphi \in \Phi_w(\mathbb{R})$ be defined by $\varphi(x, 0) := 0$ and
\[
\varphi'(x, t) := \max\{t^{p-1}, a(x)t^{q-1}\},
\]
so that $\varphi(x, t) \approx \max\{t^p, a(x)t^q\} \approx t^p + a(x)t^q$.
Let $u$ be a solution of $(\varphi'(x, |u'|) \frac{u'}{|u'|})' = 0$ on the interval $(a, b)$. We assume that $\lim_{x \to a^+} u(x) < \lim_{x \to b^-} u(x)$, so $u$ is increasing and $\frac{u'}{|u'|} = 1$. Then the differential equation reduces to $\varphi'(x, u') \equiv c$, i.e.
\[
u'(x) = \begin{cases} \frac{1}{c^{p-1}}, & \text{when } c^{-\frac{q-p}{p-1}} \geq a(x), \\ (c/a(x))^{\frac{1}{q-1}}, & \text{otherwise}. \end{cases}
\]
We further assume that $a(x) := \max\{-x, 0\}^\alpha$. Since $a$ is decreasing, we obtain that
\[
u'(x) = \begin{cases} \frac{1}{c^{p-1}}, & \text{when } x \geq -x_0, \\ (c|x|^{-\alpha})^{\frac{1}{q-1}}, & \text{when } x < -x_0, \end{cases} \quad \text{for } x_0 := c^{-\frac{1}{\alpha}} \frac{q-p}{p-1}.
Figure: Solution for $c = 1.01, 1.1, 1.2, 1.3, 1.4$ in $[-1, 1]$. The parameters are $p = 1.1$, $q = 2$ and $\alpha = 0.5$. The right boundary values have been partly cut away but they are in the range $[2, 32]$. The point indicates $x_0$. 