On regularity properties of $p(x)$-harmonic functions

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Introduction

In this talk we are concerned with regularity of solutions to

$$\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = 0, \quad x \in D \subset \mathbb{R}^n, \qquad (1)$$

where the exponent $p(\cdot)$ is an $L^\infty(D)$ function satisfying

$$1 < \alpha \leq p(x) \leq \beta < +\infty \qquad (2)$$

for almost all $x \in D$.

Solutions of (1) are $p(x)$-*harmonic functions.* They are (local) minimizers of

$$\int \frac{|\nabla u|^{p(x)}}{p(x)}\, dx.$$

Our aim is to investigate regularity properties of $p(x)$-harmonic functions under minimal assumptions on the regularity of $p(x)$.

The research in the area of equations with variable exponent of nonlinearity was initiated by V.V. Zhikov in the 1980s:

Zhikov V.V. Questions of convergence, duality, and averaging for functionals of the calculus of variations // Math USSR-Izv. 1984. V. 23, No. 2. P. 243–276. (translated from Izv. Akad. Nauk. SSSR Ser. Mat. 1983. V. 47, No. 5. P. 961–998. Russian).

Zhikov V.V. Averaging of functionals of the calculus of variations and elasticity theory // Math USSR-Izv. 1987. V. 29, No. 1. P. 33–66. (translated from Izv. Akad. Nauk SSSR Ser. Mat. 1986. V. 50, No. 4. P. 675–710. Russian).

V.V. Zhikov discovered that in this situation the so called Lavrentiev phenomenon may arise.

Lavrentiev's phenomenon

Let

$$E[u] = F[u] - \int_D g \cdot \nabla u \, dx, \quad F[u] = \int_D f(x, \nabla u) \, dx, \quad (g \in (L^\infty(D))^n,$$

where $f(x, \xi)$ is measurable in x for all ξ, convex in ξ for almost all $x \in D$ and satisfies

$$c_1 |\xi|^\alpha - c_0 \le f(x, \xi) \le c_2 |\xi|^\beta + c_0, \quad c_1, c_2 > 0, \quad c_0 \ge 0.$$

An important example is

$$f(x, \xi) = \frac{|\xi|^{p(x)}}{p(x)}. \tag{3}$$

Zhikov's example: the infimum of E over $u \in W_0^{1,\alpha}(D)$ can be strictly smaller than the infimum of E over $W_0^{1,\beta}(D)$.

Lavrentiev's phenomenon for the Dirichlet problem

Let

$$F[u] = \int_D \frac{|\nabla u|^{p(x)}}{p(x)}\,dx,$$

$$E_1 = \min_{u \in S_1} F[u], \quad E_2 = \min_{u \in S_2} F[u],$$

where

$$S_1 = \{u \in W^{1,\alpha}(D) \,:\, u = \psi \quad \text{on} \quad \partial D\},$$
$$S_2 = \{u \in W^{1,\infty}(D) \,:\, u = \psi \quad \text{on} \quad \partial D\},$$

One can choose $p(\cdot)$ and $\psi \in C^\infty(\partial D)$ so that

$$E_1 < E_2.$$

Different Sobolev spaces

Let D be a bounded Lipschitz domain. We introduce the natural Sobolev space W associated with the model Lagrangian (3) (that is, $f(x,\xi) = |\xi|^{p(x)}/p(x)$):

$$W = \{u \in W_0^{1,1}(D) \ : \ |\nabla u|^{p(x)} \in L^1(D)\},$$
$$\|u\|_{W_0^{1,p(\cdot)}(D)} = \|\nabla u\|_{L^{p(\cdot)}(D)}.$$

We remind that the Luxemburg norm is defined by

$$\|f\|_{L^{p(\cdot)}(D)} = \inf\left\{\lambda > 0 \ : \ \int_D |f\lambda^{-1}|^{p(x)}\,dx \leq 1\right\}.$$

It is not hard to see that $W \subset W_0^{1,\alpha}$.

Let H be the closure of $C_0^\infty(D)$ in W. Clearly, $H \subset W$. If the codimension of H in W is greater than 1 there can be intermediate spaces, $H \subseteq V \subseteq W$.

Solutions of different type

For the model Lagrangian (3) the minimization problem

$$E[u] \to \min, \quad u \in V,$$

has a unique solution $u \in V$ which satisfies

$$\int_D |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx = \int_D g \cdot \nabla \varphi \, dx. \tag{4}$$

for all $\varphi \in V$. Such a solution can also be constructed by the monotone operator theory.

On the other hand, $u \in W$ is a weak solution to

$$\operatorname{div} \left(|\nabla u|^{p(x)-2} \nabla u \right) = \operatorname{div} g \tag{5}$$

if (4) holds for all $\varphi \in C_0^\infty(D)$.

It is natural to say that $u \in V$ is a V-solution to (5) if (4) holds for any $\varphi \in V$.

V-solutions are called variational solutions.

Variational solutions are unique due to monotonicity. Any variational solution is a weak solution but there are weak solutions that are not variational solutions.

A weak solution is a variational solution iff

$$\int_D |\nabla u|^{p(x)}\, dx = \int_D g \nabla u\, dx.$$

That is, u is an admissible test function in (4) and the corresponding V is $H \oplus \{u\}$.

For Zhikov's classical chessboard exponent p the codimension of H in W is 1. If $\min_W E < \min_H E$ then W-solution is discontinuous at 0, H-solution is continuous in $\overline{D}$.

Same effects occur for other type of problems.

When Lavrentiev's phenomenon is absent

Density of smooth functions in the variable exponent Sobolev space guarantees the absence of the Lavrentiev phenomenon.

In Zhikov's example the exponent p is discontinuous and has saddle-point structure:

$$p(x_1, x_2) = \begin{cases} \alpha < 2 & \text{if} \quad x_1 x_2 > 0. \\ \beta > 2 & \text{if} \quad x_1 x_2 < 0, \end{cases} \qquad \Rightarrow H \neq W.$$

In the same 1986 paper Zhikov observed if that if the two constant phases $p(x) = \alpha$ and $p(x) = \beta$ are separated by a smooth hypersurface then smooth functions are dense in the corresponding variable exponent Sobolev space:

$$p(x_1, x_2) = \begin{cases} \alpha & \text{if} \quad x_2 > 0. \\ \beta & \text{if} \quad x_2 < 0, \end{cases} \qquad \Rightarrow H = W.$$

One simple condition

Edmunds D.E., Rakosnik J. Density of smooth functions in $W^{k,p(x)}(\Omega)$ // Proc. Roy. Soc. London A. 1992. V. 437. P. 229–236.

Let for all $x \in D$ there exist $r(x) > 0$ and an open cone $C(x)$ with vertex at the origin such that $B_{r(x)}(x) + C(x) \subset D$ and

$$p(z + y) \geq p(y) \quad \forall y \in B_{r(x)}(x), \quad z \in C(x).$$

Then $C^\infty(D) \cap W^{k,p(x)}(D)$ is dense in $W^{k,p(x)}(D)$.

For $D = B_1(0) \subset \mathbb{R}^2$ if $p(\cdot)$ takes three constant values, p_1, p_2 and p_3, separated by three rays emanating from the origin, then $H = W$ (there is a direction of growth of p).

Zhikov's Log condition

Zhikov V.V. On Lavrentiev's phenomenon // Russian J. Math. Phys. 1995. V. 3, No. 2. P. 249–269:

Let the exponent $p(\cdot)$ satisfy

$$|p(x) - p(y)| \leq \frac{L}{\ln|x-y|^{-1}}, \quad |x-y| < \frac{1}{2}. \tag{6}$$

Then $H = W$, i.e. smooth functions are dense in variable exponent Sobolev space.

In the same paper Zhikov refined his previous example showing that Log-condition can not be significantly improved and Lavrentiev's phenomenon can occur even for continuous $p(\cdot)$.

Definitions for the $p(x)$-Laplace equation

Let $W(D) = \{u \in W^{1,1}(D) \,:\, |\nabla u|^{p(x)} \in L^1(D\}$. We say that u_ε converges to u in $W(D)$ if

$$\int_D |u_\varepsilon - u|\,dx + \int_D |\nabla u_\varepsilon - \nabla u|^{p(x)}\,dx \to 0$$

The space $W_0(D)$ is the closure in $W(D)$ of functions compactly supported in D.

The space $H(D)$ is the closure of $C^\infty(D)$ in $W(D)$.

The space $H_0(D)$ is the closure of $C_0^\infty(D)$ functions in $W(D)$.

Clearly, $H_0(D) \subset W_0(D)$, $H(D) \subset W(D)$.

A function $u \in W(D)$ is a W-solution to (1) if

$$\int_D |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \, dx = 0 \tag{7}$$

for all $\varphi \in W_0(D)$. A function $u \in H(D)$ is an H-solution if (7) holds for all $\varphi \in C_0^\infty(D)$.

A function $u \in W(D)$ ($u \in H(D)$) is a W-supersolution (H-supersolution) if

$$\int_D |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx \geq 0$$

for any nonnegative $\varphi \in W_0(D)$ ($\varphi \in H_0(D)$).

Regularity of solutions under Log-condition

The majority of known results for regularity of $p(x)$-harmonic functions and generalizations assume Zhikov's log-condition

$$|p(x) - p(y)| \leq L \left(\ln \frac{1}{|x-y|} \right)^{-1}, \quad x, y \in D, \quad |x-y| \leq 1/2.$$

Alkhutov Yu. A. The Harnack inequality and the Hoölder property of solutions of nonlinear elliptic equations with a nonstandard growth condition // Differ. Uravn. 1997. V. 33, No. 12. P. 1651–1660. (English transl.: Differ. Equ. 1997. V. 33. No. 12. P. 1653–1663):

The Hölder continuity and Harnack inequality under Log-condition: for a bounded nonnegative $p(x)$-harmonic function in the ball $B_{4R}(x_0)$ there holds

$$\sup_{B_R(x_0)} u \leq C(n, \alpha, \beta, L, \|u\|_\infty) \left(\inf_{B_R(x_0)} + R \right).$$

Gradient estimates

Assuming Log-condition, Zhikov obtained Meyers type estimates for the gradient of a solution.

Zhikov V.V. Meyer-type estimates for solving the nonlinear Stokes system // Differ. Uravn. 1997. V. 33, No. 1. P. 107–114. (English transl.: Differ. Equ. 1997. V. 33. No. 1. P. 108-115).

Gradient estimates were later generalized and sharpened by A. Coscia, E. Acerbi, G. Mingione, L. Diening, etc.

In particular, if the exponent $p(\cdot)$ is Hölder continuous, then the gradient of a $p(x)$-harmonic function is also Hölder continuous.

Coscia A., Mingione G. Hölder continuity of the gradient of $p(x)$-harmonic mappings // C. R. Acad. Sci. Paris. 1999. V. 328, P. 363–368.

Acerbi E., Mingione G. Regularity Results for a Class of Functionals with Non-Standard Growth // Arch. Rational Mech. Anal. 2001. V. 156. P. 121–140.

Acerbi E., Mingione G. Regularity results for a class of quasiconvex functionals with nonstandard growth // Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4). 2001. V. 30. P. 311-339.

Acerbi E., Mingione G. Regularity Results for Stationary Electro-Rheological Fluids // Arch. Rational Mech. Anal. 2002. V. 164. P. 213–259.

Acerbi E., Mingione G. Gradient estimates for the $p(x)$-Laplacean system // J. Reine Angew. Math. 2005. V. 584. P. 117–148.

Diening L., Schwarzsacher S. Global gradient estimates for the p(·)-Laplacian // Nonlinear Analysis. 2014. V. 106. P. 70–85.

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Gradient estimates: limiting case

Bögelein V., Habermann J. Gradient estimates via non standard potentials and continuity // Annales Academiae Scientiarum Fennicae Mathematica. 2010. V. 35. P. 641–678

Ok J. Gradient continuity for $p(\cdot)$-Laplace systems // Nonlinear Analysis. 2016. V. 141. P. 139–166.

Ok J. C^1-regularity for minima of functionals with $p(x)$-growth // J. Fixed Point Theory Appl. 2017. V. 19. P. 2697–2731.

All these advanced results require the modulus of continuity of the exponent p to be (slightly) better than log-Hölder.

Jihoon Ok, 2016:

$$\int_0 \omega(r) \log\left(\frac{1}{r}\right) \frac{dr}{r} < \infty$$

implies that solutions are C^1 (even with $L^{n,1}$ RHS and Dini weights).

Relaxing Log-condition

Zhikov V.V. On density of smooth functions in Sobolev–Orlich spaces // Zap. Nauchn. Sem. POMI. 2004. V. 310. P. 67–81.

Smooth functions are dense in the Sobolev-Orlicz space (i.e. $H = W$) provided that

$$\int_0 t^{n\omega(t)/\alpha} \frac{dt}{t} = \infty,$$

where $\omega(\cdot)$ is the modulus of continuity of p. For example,

$$\omega(t) = k \frac{\ln \ln t^{-1}}{\ln t^{-1}}, \quad t < e^{-1}, \tag{8}$$

will do provided that $k < \alpha/n$. An example shows that the restriction on k here is essential.

Zhikov V. V., Pastukhova S. E. Improved integrability of the gradients of solutions of elliptic equations with variable nonlinearity exponent // Mat. Sb. 2008. V. 199. No. 12. P. 19–52. (English transl.: Sb. Math. 2008. V. 199. N. 12. P. 1751–1782).

The higher integrability of solutions still holds if the Logarithmic condition is replaced by (8). Let D be a bounded Lipschitz domain in $\mathbb{R}^n$. If $u \in W_0^{1,p(x)}(D)$ is a W-solution (or H-solution) to

$$\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \mathrm{div}\, g, \quad u = 0 \quad \text{on} \quad \partial D,$$

then

$$\int_D |\nabla u|^p \ln^\delta(2 + |\nabla u|)\, dx \le C \int_D |g|^{p'} \ln^\delta(2 + |g|)\, dx$$

where posititve constants C and δ depend only on D, α, n, k, and $\|g\|_{\alpha'}$.

Krasheninnikova O. V. Continuity at a Point for Solutions to Elliptic Equations with a Nonstandard Growth Condition // Tr. Mat. Inst. Steklova. 2002. V. 236. P. 204–211. (English transl. Proc. Steklov Inst. Math. 2002. V. 236. P. 193–200).

If the exponent $p(\cdot)$ satisfies Log-condition at a given point then H-and W-solutions are Hölder continuous at this point.

Let u be a $p(x)$-harmonic function in $B_R^{x_0}$ and

$$|p(x) - p(x_0)| \le L \left(\ln \frac{1}{|x - x_0|} \right)^{-1}.$$

Then for $x \in B_{R/2}^{x_0}$ there holds

$$|u(x) - u(x_0)| \le C(n, \alpha, \beta, L, \|u\|_\infty) \left(\frac{|x - x_0|}{R} \right)^{\gamma}, \quad \gamma = \gamma(n, \alpha, \beta, L, \|u\|_\infty).$$

Alkhutov Yu.A., Krasheninnikova O.V. On the Continuity of Solutions to Elliptic Equations with Variable Order of Nonlinearity // Tr. Mat. Inst. Steklova. 2008. V. 261. P. 7–15. (transl. in Proc. Steklov Inst. Math. 2008. V. 261. P. 1–10).

Let

$$|p(x) - p(x_0)| \le L \frac{\ln\ln\ln|x - x_0|^{-1}}{\ln|x - x_0|^{-1}}, \quad |x - x_0| < \frac{1}{27}, \tag{9}$$

where $L < \alpha/(n+1)$. Then all W-solutions and all H-solutions of the $p(x)$-Laplace equation are continuous at x_0.

There exists $\rho_0 = \rho_0(n, \alpha, \beta, \|u\|_\infty, L)$ such that

$$\operatorname*{ess\,osc}_{B_r(x_0)} u \le 2\|u\|_\infty \left(\ln\frac{\rho}{r}\right)^{-1/4} \operatorname*{ess\,osc}_{B_{\rho_0}(x_0)} u + \rho, \quad r < \rho/4 < \rho_0.$$

Different relaxation of log-condition

Alkhutov Yu.A., Surnachev M.D. Hölder continuity and Harnack's inequality for $p(x)$-harmonic functions // Tr. Mat. Inst. Steklova. 2020. V. 308. P. 7–27. (transl.: Proc. Steklov Inst. Math. 2020. V. 308. P. 1–21).

Let $B^{x_0}_{R_0} \subset D$, $R_0 \in (0, 1/2)$, and for a measurable $E \subset D$ there holds

$$|p(x) - p_0| \le \frac{L}{\ln|x - x_0|^{-1}}, \quad x \in B^{x_0}_{R_0} \setminus E,$$

where $p_0 \in [\alpha, \beta]$, and

$$|B^{x_0}_r \cap E| \le C_E r^{n+2\gamma n}, \quad 0 < r \le R_0,$$

where

$$\gamma = (\beta - \alpha) \max\left\{1, \frac{1}{\alpha - 1}\right\}.$$

Theorem. *Under these conditions for* H*- and* W*-solutions to the* $p(x)$*-Laplace equation there holds*

$$\operatorname*{ess\,osc}_{B_r^{x_0}} u \le C \left(\frac{r}{R_0}\right)^{\nu} \left(\operatorname*{ess\,osc}_{B_{R_0}^{x_0}} u + R\right)$$

where the constants C *and* ν *depend only on* n, α, β, L, C_E, $\|u\|_\infty$.

The condition on the set E is satisfied for instance if E is the solid of revolution

$$|x' - x_0'| \le C|x_n - (x_0)_n|^{\delta}, \quad x = (x', x_n), \quad \delta = 1 + \frac{2\gamma n}{n-1}.$$

On the set E itself no continuity is assumed, just

$$1 < \alpha \le p(x) \le \beta < \infty, \quad x \in E.$$

Theorem. *For any bounded nonnegative H- or W-supersolution of the $p(x)$-Laplace equation in $B_{4R}^{x_0}$, $0 < R \leq R_0/4$, there holds*

$$\left(\int_{B_{2R}^{x_0}} (u+R)^q \, dx \right)^{1/q} \leq C \operatorname*{ess\,inf}_{B_R^{x_0}} (u+R)$$

where $0 < q < n(p_0 - 1)/(n-1)$ and the positive constant $C = C(n, \alpha, \beta, L, C_E, \|u\|_\infty)$.

Theorem. *For any bounded nonnegative H- or W-solution of the $p(x)$-Laplace equation in $B_{4R}^{x_0}$, $0 < R \leq R_0/4$, there holds*

$$\operatorname*{ess\,sup}_{B_R^{x_0}} u \leq C \operatorname*{ess\,inf}_{B_R^{x_0}} (u+R)$$

where the positive constant $C = C(n, \alpha, \beta, L, C_E, \|u\|_\infty)$.

Dirichlet problem with variational data

Let $f \in C^\infty(\overline{D})$. We can set two Dirichlet problems

$$Lu = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = 0 \quad \text{in} \quad D, \quad u - f \in W_0(D). \tag{10}$$

and

$$\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = 0 \quad \text{in} \quad D, \quad u - f \in H_0(D). \tag{11}$$

Solutions to (10), (11) can be constructed by minimizing the functional

$$\mathcal{F}[v] = \int_D \frac{|\nabla(v+f)|^{p(x)}}{p(x)}\,dx \tag{12}$$

over $v \in W_0(D)$ or $v \in H_0(D)$. For the minimizer v of this problem $u = v + f$. A solution to (10) (or (11)) satisfies

$$\int_D |\nabla u|^{p(x)-2}\nabla u \nabla\varphi\,dx = 0$$

for all $\varphi \in W_0(D)$ ($\varphi \in H_0(D)$, respectively).

Dirichlet problem with continuous boundary function

Let $f \in C(\partial D)$. Extending f to $\mathbb{R}^n$ and approximating f by $f_k \in C^\infty(\overline{D})$, constructing corresponding solutions u_k to (10) or (11) (that is, u_k is a sequence of W-solutions or H-solutions), and passing to the limit, we can construct a generalized solution u_f to the Dirichlet problem

$$Lu_f = 0 \quad \text{in} \quad D, \quad u_f = f \quad \text{on} \quad \partial D. \tag{13}$$

This solution belongs to $W(D')$ ($H(D')$, resp.) for any subdomain $D' \Subset D$, and satisfies $Lu = 0$ in the sense that

$$\int_D |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \, dx = 0$$

for all $\varphi \in W_0(D)$ ($\varphi \in H_0(D)$, respectively), compactly supported in D. We call this solution a generalized W-solution (H-solution, resp.) to (13). A generalized W-solution (H-solution) is uniquely defined by the maximum principle.

Regular boundary points

Definition. A boundary point $x_0 \in \partial D$ is regular iff for any $f \in C(\partial D)$ the corresponding generalized solution u_f of (13) satisfies

$$\lim_{D \ni x \to x_0} u_f(x) = f(x_0).$$

$\boxed{L = \triangle}$ — H. Lebesgue (irregular points, Comptes Rendus Soc. Math. de France. 1913), N. Wiener (criterion, J. Math. Phys. 1924).

$\boxed{L = \mathrm{div}(a\nabla u)}$ — W. Littman, G. Stampacchia, and H.F. Weinberger, Ann. Scuola Norm. Sup. Pisa 1963.

$\boxed{L = \triangle_p}$ — V.G. Mazya (sufficient condition, Vestn. Leningr. Gos. Univ. 1970), R. Gariepy and W.P. Ziemer (general equations, Arch. Rational Mech. Anal. 1977), T. Kilpelainen and J. Maly (necessity part, Acta Math. 1994).

Wiener's criterion for the p(x)-Laplacian

Alkhutov Yu.A., Krasheninnikova O.V. Continuity at boundary points of solutions of quasilinear elliptic equations with a non-standard growth condition // Izv. RAN. Ser. Mat. 2004. V. 68. No. 6. P. 3–60. (English transl.: Izv. Math. 2004. V. 68. No. 6. P. 1063–1117).

Wiener's criterion under global log-condition:

$$|p(x) - p(y)| \le L\left(\ln \frac{1}{|x-y|}\right)^{-1}, \quad |x-y| < 1/e, \quad x \in D.$$

The $p(x)$-capacity of a compact set $K \Subset B_R^{x_0}$ with respect to the ball $B_R^{x_0}$ is the number

$$C_p(K, B_R^{x_0}) = \inf\left\{\int_{B_R^{x_0}} \frac{|\nabla\varphi|^{p(x)}}{p(x)}\,dx \ : \ \varphi \in C_0^\infty(B_R^{x_0}),\ \varphi \ge 1 \quad \text{on} \quad K\right\}.$$

For a boundary point x_0 let $p_0 = p(x_0)$ and extend p by p_0 outside D.

Define

$$\gamma(t) = \left(C_p(\overline{B}_t^{x_0} \setminus D, B_{2t}^{x_0}) \right)^{1/(p_0-1)}.$$

Theorem. *The boundary point* x_0 *is regular if and only if*

$$\int_0 \gamma(t) t^{-1} = +\infty.$$

Alkhutov Yu.A., Surnachev M.D. Regularity of a boundary point for the p(x)-Laplacian // J. Math. Sci. 2018. V. 232. N. 3. P. 206–231.

Global log-condition relaxed to log-condition at the given boundary point:

$$|p(x) - p(x_0)| \leq L \left(\ln \frac{1}{|x - x_0|} \right)^{-1}, \quad |x - x_0| < 1/e, \quad x \in D.$$

When density of smooth functions in $W(D)$ is not known one has to consider different types of solutions, H- and W-solutions.

The H-capacity (W-capacity) of a compact set $K \Subset B_R^{x_0}$ with respect to the ball $B_R^{x_0}$ is the number

$$C_p(K, B_R^{x_0}) = \inf \int_{B_R^{x_0}} \frac{|\nabla \varphi|^{p(x)}}{p(x)}\, dx$$

where the infimum is taken over the set of $C_0^\infty(B_R^{x_0})$ ($W_0(D)$) functions greater than or equal to one in the neighborhood of K.

When treating H-solutions one has to use H-capacity and for W-solutions one uses W-capacity.

Wiener under relaxed log-condition

Alkhutov Yu.A., Surnachev M.D. Behavior of solutions of the Dirichlet problem for the p(x)-Laplacian at a boundary point // Algebra i Analiz. 2019. V. 31. N. 2. P. 88–117. (transl.: St. Petersburg Math. J. 2020. V. 31. No. 2. P. 251–271.)

Let $x_0 \in \partial D$ and

$$\operatorname*{ess\,osc}_{B_r^{x_0} \cap D} p \le \omega(r), \quad \omega(0) = 0.$$

We assume that the function

$$\theta(r) = r^{-\omega(r)}$$

is nondecreasing on $(0, d]$.

Recall that

$$\gamma(t) = \left(C_p(\overline{B}_t^{x_0} \setminus D, B_{2t}^{x_0})\right)^{\frac{1}{p(x_0)-1}} .$$

Theorem. *If*

$$\int_0 \exp\left(-\theta^{3+2n/\alpha}(t)\right) \gamma(t) t^{-1}\, dt = +\infty$$

then the boundary point x_0 is regular.

Corollary. *Assume that the complement of D contains an open cone with vertex x_0 and*

$$\omega(t) \le k |\ln t|^{-1} \ln\ln|\ln t|, \quad t \in (0, 1/27),$$

where $k \in (0, \alpha/5n)$. Then the boundary point x_0 is regular.

Weak Harnack inequality

The key instrument:

Theorem. *Let* u *be a bounded nonnegative supersolution of* (1) *in* $B_{4R}^{x_0}$*. Then for* $0 < q < n(s-1)/(n-s)$, $s = \operatorname*{ess\,inf}_{B_{4R}^{x_0}} p < n$ *(any* $q > 0$ *for* $s = n$*), there holds*

$$\left(R^{-n} \int_{B_{2R}^{x_0}} (u+R)^q \, dx\right)^{1/q}$$

$$\leq \exp\left(C(n,\alpha,\beta,\|u\|_\infty,q)\theta(4R)^{2(n+s)/2}\right) \operatorname*{ess\,inf}_{B_R^{x_0}}(u+R).$$

Double phase problems

Acerbi E., Fusco N. A transmission problem in the calculus of variations // Calc. Var. Partial Differ. Equ. 1994. V. 2, No. 1. P. 1–16.

Boundedness, Hölder contiuity and higher integrability of the gradient (Meyers type estimates) for local minimizers of

$$F[u] = \int_D \frac{|\nabla u|^{p(x)}}{p(x)}\, dx$$

when the domain D is divided by the hyperplane $\Sigma = \{x_n = 0\}$ into two parts, $D^{(1)} = D \cap \{x_n > 0\}$, $D^{(2)} = D \cap \{x_n < 0\}$, and $p(x) = p_1$ for $x \in D^{(1)}$, $p(x) = p_2$ for $x \in D^{(2)}$, p_1 and p_2 are constant.

A function $u \in W^{1,1}(D)$, $F[u] < \infty$, is a local minimizer if $F[u + \varphi] \le F[u]$ for all $\varphi \in C_0^\infty(D)$.

Alkhutov Yu.A. Hölder continuity of $p(x)$-harmonic functions // Mat. Sb. 2005. V. 196, No. 2. P. 3–28. (English translation: Sb. Math. 2005. V. 196, No. 2. P. 147–171).

Let $x_0 \in \Sigma = \{x_n = 0\}$, and

$$|p(x) - p_1| \leq \frac{L}{\log \frac{1}{|x-x_0|}}, \quad x \in D^{(1)} = D \cap \{x_n > 0\},$$

$$|p(x) - p_2| \leq \frac{L}{\log \frac{1}{|x-x_0|}}, \quad x \in D^{(2)} = D \cap \{x_n < 0\},$$

then both H and W solutions are Hölder continuous at x_0.
The constants p_1, p_2 are limit values of $p(x)$ when x approaches x_0 from different sides of the hyperplane Σ.

Harnack's inequality for double phase problems

Alkhutov Yu.A., Surnachev M.D. On a Harnack inequality for the elliptic (p,q)-Laplacian // Dokl. Math. 2016. V. 94, No. 2. P. 569–573. (translated from Doklady Akademii Nauk. 2016. V. 470, No. 6, P. 651–655).

Let $x = (x', x_n)$,

$$p(x) = \begin{cases} p_1, & x_n > 0, \\ p_2, & x_n < 0, \end{cases} \qquad p_2 > p_1.$$

For a nonnegative solution in $B_{4R}(x_0)$, $x_0 \in \Sigma$, there holds

$$\sup_{Q_R(x_0)} u \le C(n, p_1, p_2) \left(\inf_{B_R(x_0)} u + R \right), \quad Q_R(x_0) = B_R(x_0) \cap \{x_n < -R/2\}.$$

The classical Harnack inequality is not valid in this case: we can neither replace $Q_R(x_0)$ by $B_R(x_0)$ nor remove R.

Alkhutov Yu.A., Surnachev M.D. A Harnack inequality for a transmission problem with $p(x)$-Laplacian // Applicable Analysis. 2019. V. 98. No. 1/2. P. 332–344.

Constant values p_1 and p_2 replaced by the variable exponent $p(\cdot)$, satisfying the log-condition separately in $D^{(1)} = D \cap \{x_n > 0\}$ and in $D^{(2)} = D \cap \{x_n < 0\}$ and such that $p(x) \geq p(\tilde{x})$ for $x \in D^{(2)}$:

$$|p(x) - p(y)| \leq \frac{L}{\ln |x-y|^{-1}}, \quad |x-y| < \frac{1}{2}, \quad x, y \in D^{(i)}.$$

In this case Harnack's inequality holds in the form

$$\operatorname*{ess\,sup}_{Q_R(x_0)} u \leq C(n, \alpha, \beta, L, \|u\|_\infty) \left(\operatorname*{ess\,inf}_{B_R(x_0)} u + R \right),$$

$$Q_R(x_0) = B_R(x_0) \cap \{x_n < -R/2\}.$$

Alkhutov Yu.A., Surnachev M.D. Harnack's inequality for the p(x)-Laplacian with a two-phase exponent $p(x)$ // J. Math. Sci. 2020. V. 244. No. 2. P. 116–147. (transl. from Tr. Sem. im. I.G. Petrovskogo. 2019. V. 32. P. 8–56).

Let u be a positive bounded W- or H-solution of (1) in $B = B_{8R}(x_0)$, $x_0 \in \Sigma$, $0 < R < 1/32$.

Theorem. *Let* $\operatorname{ess\,osc}_B p \le L/\ln R^{-1}$. *Then*

$$\sup_{B_R(x_0)} u \le C(n, \alpha, \beta, L, \|u\|_\infty) \inf_{B_R(x_0)} (u + R).$$

Theorem. *Let*

$$\operatorname*{ess\,osc}_{B\cap\{x_n>0\}} p \le \frac{L}{\ln R^{-1}}, \quad \operatorname*{ess\,osc}_{B\cap\{x_n<0\}} p \le \frac{L}{\ln R^{-1}}$$

$$\operatorname*{ess\,inf}_{B\cap\{x_n>0\}} p \le \operatorname*{ess\,sup}_{B\cap\{x_n<0\}} p + \frac{L}{\ln R^{-1}}.$$

Then

$$\operatorname*{ess\,sup}_{Q_R(x_0)} u \le C(n,\alpha,\beta,L,\|u\|_\infty)(\operatorname*{ess\,inf}_{B_R(x_0)} u + R).$$

Let $v = \min(u, \tilde{u}) + R^\gamma$, $\gamma \in (0, 1)$, where

$$\tilde{u}(x) = \begin{cases} u(x), & x \in D^{(2)}, \\ u(\tilde{x}), & x \in D^{(1)}. \end{cases}$$

Theorem. *Under the assumptions of the previous theorem,*

$$\left(R^{-n} \int_{B_{2R}(x_0)} v^q \, dx \right)^{1/q} \leq C(n, \alpha, \beta, L, M, q) \operatorname*{ess\,inf}_{B_R(x_0)} v \tag{14}$$

for

$$0 < q < \frac{n(s-1)}{n-1}, \quad s = \operatorname*{ess\,inf}_{B_{8R}(x_0)} p.$$

Under the assumptions of the first theorem, (14) *is valid for* $v = u + R$. *This result holds true if* u *is a* W*- or* H*- supersolution.*

Yet another double phase toy problem

Alkhutov Yu.A., Surnachev M.D. The Boundary Behavior of a Solution to the Dirichlet Problem for the p-Laplacian with Weight Uniformly Degenerate on a Part of Domain with Respect to Small Parameter // J. Math Sci. 2020. V. 250. P. 183–200.

Now $p = \text{const}$, $1 < p < \infty$,

$$Lu = \operatorname{div}(\omega_\varepsilon(x)|\nabla u|^{p-2}\nabla u = 0),$$

where

$$\omega_\varepsilon(x) = \begin{cases} \varepsilon, & x_n > 0, \\ 1, & x_n < 0, \end{cases} \qquad \text{and} \quad \varepsilon \in (0, 1].$$

Consider the Dirichlet problem

$$Lu_f = 0 \quad \text{in} \quad D, \quad u_f|_{\partial D} = f \in C(\partial D).$$

Denote $\Sigma = \{x_n = 0\}$. For $x_0 \in \partial D \cap \Sigma$ let

$$\gamma(r) = \left(\frac{C_p((\overline{B}_r^{x_0} \cap \{x_n \leq 0\}) \setminus D, B_{2r}^{x_0})}{r^{n-p}} \right)^{\frac{1}{p-1}},$$

where $C_p(E, \Omega)$ is the standard p-capacity of a compact set E with respect to Ω.

Theorem. *If*

$$\int_0 \gamma(r) r^{-1}\, dr = \infty$$

then the point x_0 is regular and for $0 < r \leq \rho/5 \leq \operatorname{diam} D/4$ there holds

$$\operatorname*{ess\,sup}_{D \cap B_r^{x_0}} |u_f(x_0) - f(x_0)| \leq 2 \operatorname*{osc}_{\partial D \cap B_\rho^{x_0}} f + \operatorname*{osc}_{\partial D} f \cdot \exp\left(-C \int_r^\rho \gamma(t) t^{-1}\, dt \right)$$

where $C = C(n, p)$ is independent of ε.

Special weak Harnack

For a nonnegative supersolution w denote $v = \min(w, \tilde{w})$ where $\tilde{w}$ is the even extension of w from $\{x_n \le 0\}$ to $\{x_n > 0\}$. Then for

$$0 < \beta_0 < p-1, \quad \varepsilon \le \frac{\beta_0}{4} p^{p/(p-1)} (p-1)^{-2}, \quad r \le (p - \beta_0 - 1)\frac{n}{n-1}$$

there holds

$$\inf_{B_R} v \ge C(n,p,\beta_0) \left(R^{-n} \int_{B_{3R}} v^r \, dx \right)^{1/r}.$$

As a corollary, for $\varepsilon \le \varepsilon_0$, $\varepsilon_0 = \varepsilon_0(n,p) > 0$, there holds

$$R^{p-n-1} \int_{B_{2R}} |\nabla v|^{p-1} \, dx + R^{-n} \int_{B_{2R}} v^{p-1} \, dx \le C(n,p) \left(\inf_{B_R} v \right)^{p-1}.$$

Triple phase problem

Alkhutov Yu.A., Surnachev M.D. Harnack inequality for the elliptic p(x)-Laplacian with a three-phase exponent p(x) // Comp. Math. Math. Phys. 2020. V. 60. N. 8. P. 1284–1293.

$$B_R = \{x \in \mathbb{R}^2 : |x| < R\}, \quad D^{(1)} = \{0 < \varphi < \varphi_1\},$$
$$D^{(2)} = \{\varphi_1 < \varphi < \varphi_2\}, \quad D^{(3)} = \{\varphi_2 < \varphi < 2\pi\},$$
$$p(x) = p_i, \quad x \in D^{(i)}, \quad i = 1,2,3, \quad 1 < p_3 < p_2 < p_1,$$
$$\tilde{D}^{(1)} = \{\varphi_1/4 < \varphi < 3\varphi_1/4\}, \quad \varepsilon > 0.$$

Theorem. *For any nonnegative $p(x)$-harmonic function in B_{4R},*

$$\operatorname*{ess\,sup}_{\tilde{D}^{(1)}\cap\{R/2<r<R\}} u \le C(n, p_1, p_2, p_3, \varphi_1, \varphi_2, \varphi_3)\big(\operatorname*{ess\,inf}_{B_R} u + R\big).$$

As a corollary, $p(x)$-Harmonic functions are Hölder continuous in B_R.

Here $H = W$ (smooth functions are dense in the Sobolev-Orlicz space $W^{1,p(x)}(B_R)$): see Edmunds, Rakosnik, or

Fan X.L., Wang S., Zhao D. Density of $C^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with discontinuous exponent $p(x)$ // Math. Nachr. 2006. V. 279, No. 1–2, P. 142–149.

In the latter paper the case of piecewise-constant exponent with multiple phases was treated.

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Thank you!