On the existence of integrable solutions to elliptic and parabolic systems with linear growth - applications in (visco)-elasticity theory

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Monday’s Nonstandard Seminar

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The talk is based on the following results

- L. Beck, M. Bulíček, F. Gmeineder: **On existence of $W^{1,1}$ solutions to variational problems with linear growth**, to appear at Annali della Scuola Normale Superiore (Pisa) 2020
Minimal surface equation

- $\Omega \subset \mathbb{R}^d$ is an open bounded smooth set and $U_0 : \Omega \to \mathbb{R}$ is smooth.
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- We look for $U \in W^{1,1}(\Omega)$

$$\text{div} \left( \frac{\nabla U}{(1 + |\nabla U|^2)^{\frac{1}{2}}} \right) = 0 \quad \text{in } \Omega, \quad U = U_0 \quad \text{on } \partial \Omega.$$
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  \]
- It is equivalent to find $U \in W^{1,1}(\Omega)$, which minimizes
  \[
  \min_{U - U_0 \in W_0^{1,1}(\Omega)} \int (1 + |\nabla U|^2)^{\frac{1}{2}}.
  \]

If $\Omega$ is convex (or generally has nonnegative mean curvature) then there always exists (smooth) solution.

If $\Omega$ has negative mean curvature (or is nonconvex in 2D) then there always exists $U_0$ for which the solution does not exist.
Minimal surface like systems

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- It is equivalent to find $U \in W^{1,1}(\Omega)$, which minimizes

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\min_{U - U_0 \in W^{1,1}_0(\Omega)} \int (1 + |\nabla U|^2)^{\frac{1}{2}}.
\]

Answers:
- If $\Omega$ is convex (or generally has nonnegative mean curvature) then there always exists (smooth) solution.
- If $\Omega$ has negative mean curvature (or is nonconvex in 2D) then there always exists $U_0$ for which the solution does not exist.
Minimal surface equation - BV setting - relaxed formulation

- Minimize the relaxed functional over the space $BV(\Omega)$, i.e., find $U \in BV(\overline{\Omega})$ with $\overline{\Omega} \subset \tilde{\Omega}$ that minimizes

$$\min_{U \in BV(\tilde{\Omega}); U = U_0 \text{ in } \tilde{\Omega} \setminus \Omega} \int (1 + |(\nabla U)^r|^2)^{\frac{1}{2}} + |(\nabla U)^s|(\overline{\Omega}),$$

where $(\nabla U)^r$ is the regular (absolutely continuous w.r.t. Lebesgue measure) part of $\nabla U$ (which is a Radon measure), and $(\nabla U)^s$ is the singular part.
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Answers:

- Weak lower semicontinuity $\implies$ minimizer always exists.
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**Answers:**

- Weak lower semicontinuity $\implies$ minimizer always exists.
- De Giorgi $\implies$ $U \in C^{1,\alpha}_{loc}(\Omega)$ and in fact we have the “half” relaxed formulation:

$$
\min_{U \in W^{1,1}(\Omega)} \int (1 + |\nabla U|^2)^{\frac{1}{2}} + \int_{\partial \Omega} |U - U_0|
$$
Generalized problem

**DATA:** $\Omega \subset \mathbb{R}^d$ open smooth bounded (connected), $a \in (0, \infty)$, $U_0 \in C^\infty(\overline{\Omega})$, $G \in C^\infty(\overline{\Omega}; \mathbb{R}^d)$, and $\Gamma_D, \Gamma_N \subset \partial \Omega$ are smooth open (in (d-1) sense) disjoint parts of the boundary whose union is of the full measure of $\partial \Omega$

**GOAL:** To find for $U \in W^{1,1}(\Omega)$ such that

$$\text{div} \left( \frac{\nabla U}{(1 + |\nabla U|^a)^{1/3}} \right) = \text{div} \, G \quad \text{in} \, \Omega,$$

$$U = U_0 \quad \text{on} \, \Gamma_D,$$

$$\frac{\nabla U}{(1 + |\nabla U|^a)^{1/3}} \cdot n = G \cdot n \quad \text{on} \, \Gamma_N.$$
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- Necessary compatibility condition
  $$\|G\|_\infty \leq 1$$

- Safe data condition
  $$\|G\|_\infty < 1.$$
Concepts of solutions

- Weak solution: we look for $U$, such that $U - U_0 \in W^{1,1}_{\Gamma_D}(\Omega)$ and for all $\varphi \in W^{1,1}_{\Gamma_D}(\Omega)$

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\int_{\Omega} \frac{\nabla U}{(1 + |\nabla U|^a)^{\frac{1}{2}}} \cdot \nabla \varphi = \int_{\Omega} G \cdot \nabla \varphi
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It is equivalent to find $U \in W^{1,1}(\Omega)$ such that $U = U_0$ on $\Gamma_D$ which minimizes

\[
\min_{U \in W^{1,1}(\Omega)} \int_{\hat{\Omega}} F(|\nabla U|) - G \cdot \nabla U,
\]

where $F(s) := \hat{s}^0 t \left(1 + t^\alpha\right)^{\frac{1}{3}}$.

"Half" relaxed formulation: to find $U \in W^{1,1}(\Omega)$ that minimizes

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Relaxed formulation: to find $U_0 \in BV(\tilde{\Omega})$ being equal to $U_0$ outside $\Omega$ that minimizes

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\min_{U \in BV(\hat{\Omega})} \int_{\hat{\Omega}} \left(\frac{\nabla U}{r}\right) + |\nabla U|_{\Omega \setminus \Gamma_N} - \langle G, \nabla U \rangle.
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**Concepts of solutions**

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- Relaxed formulation: to find $U_0 \in BV(\tilde{\Omega})$ being equal to $U_0$ outside $\Omega$ that minimizes

$$\min_{U \in BV} \int_{\Omega} F((|\nabla U|^r)) + |\nabla U^s|(\tilde{\Omega} \setminus \Gamma_N) - \langle G, \nabla U \rangle.$$
Answers & open problems

- \( a \in (0, \infty) \) and \( \Omega \) (uniformly) convex - unique weak solution always exists
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- \( a \in (0, \infty) \) and general \( \Omega \) - \( \text{BV} \) solution exists and the regular part \( (\nabla U)' \) is unique
**Answers & open problems**

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- \( a \in (0, \infty) \) and general \( \Omega \) - BV solution exists and the regular part \((\nabla U)^r\) is unique
- \( a \in (0, 2) \) and general \( \Omega \) - “half” relaxed formulation is enough - \( U \in C^{0,1}_{loc} \) (Bildhauer & Fuchs, ....) (works even for systems)

**Problem 1:** \( a > 1 \) - the weak solution does not exist in general (standard counterexample on the annulus for constant data and Dirichlet problem)

**Problem 2:** \( a \in (0, 1] \) is there always a weak solution? - the standard counterexample on the annulus does not work here

**Problem 3:** Can we find a range of \( a \)’s and a class of nonconvex domains for which the weak solution always exists?
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- Problem 3: Can we find a range of $a$’s and a class of nonconvex domains for which the weak solution always exists?
Theorem

- **Problem 3:** \( a \in (0, 2) \) and \( \Gamma_D = \bigcup_{i=1}^{N} \Gamma_i \) such that either \( \Gamma_i \) is uniformly convex and \( U_0 \) is smooth on \( \Gamma_i \) or \( \Gamma_i \) is flat and \( U_0 \) is constant there \( \Rightarrow \) there exists a weak solution (B, Málek, Rajagopal, Walton).

- **Problem 2:** \( a \in (0, 1] \) and \( \Gamma_N = \emptyset = \Rightarrow \) there is always a weak solution (Beck, B, Maringová).

- **Problem 1:** \( a > 0 \), \( \Omega \) **simply connected** and \( \Gamma_D = \emptyset \) \( \Rightarrow \) there is always a weak solution. (Beck, B, Gmeineder)
General result 1 - regularity up to the boundary

Theorem (Beck, Bulíček, Maringová)

Let $F \in C^2(0, \infty)$ be increasing strictly convex fulfilling

$$
\lim_{s \to \infty} \frac{F(s)}{s} = \lim_{s \to \infty} F'(s) = K, \quad 0 < \lim_{s \to \infty} \frac{F''(2s)}{F''(s)} < \infty.
$$

Then the following is equivalent

- For any $\Omega \in C^{1,1}$ and any $u_0 \in C^{1,1}(\overline{\Omega})$ there exists unique $u \in W^{1,\infty}(\Omega)$ fulfilling

$$
\int_{\Omega} F(|\nabla u|) \leq \int_{\Omega} F(|\nabla u_0 + \nabla \varphi|) \quad \text{for all } \varphi \in W^{1,1}_0(\Omega).
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- $\int_1^\infty sF''(s) = \infty.$

The second condition is equivalent to the fact that

$F^* \in C^1$.
General result I - regularity up to the boundary

**Theorem (Beck, Bulíček, Maringová II)**

Let $F \in C^2(0, \infty)$ be increasing strictly convex fulfilling

$$\lim_{s \to \infty} \frac{F(s)}{s} = \lim_{s \to \infty} F'(s) = K.$$

If $\int_1^{\infty} sF''(s) < \infty$ then for any smooth domain satisfying the inner ball condition (in 2d any nonconvex domain) there exists a smooth function $U_0$ such that the minimizer does not belong to $W^{1,1}(\Omega) \cap C(\overline{\Omega})$. 
General result II - no BV needed

Theorem (Beck, Bulíček, Gmeineder)

Let $F \in \mathcal{C}^2(0, \infty)$ be increasing strictly convex fulfilling

$$\lim_{s \to \infty} \frac{F(s)}{s} = \lim_{s \to \infty} F'(s) = K.$$ 

Let $\Omega$ be simply connected domain, $\Gamma_D = \emptyset$ and $G$ satisfy the safe data condition. Then there exists a unique (up to a constant) weak solution.
General result II - no BV needed

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The method works even for elliptic systems, having not variational nor radial structure. But we require both of these structures at least asymptotically.
Minimal surface like systems

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There is no improvement of integrability of $\nabla u$!!!
If you are not interested in continuum mechanics then thank you for your attention!
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If you are not interested in calculus of variations but you are interested in continuum mechanics, please wake up!
Limiting strain

Linearized nonlinear elasticity

We consider the elastic deformation of the body $\Omega \subset \mathbb{R}^d$ with $\Gamma_D \cap \Gamma_N = \emptyset$ and $\overline{\Gamma_D \cup \Gamma_N} = \partial \Omega$ described by

\begin{align*}
- \text{div} \mathbf{T} &= \mathbf{f} \quad \text{in } \Omega, \\
\mathbf{u} &= \mathbf{u}_0 \quad \text{on } \Gamma_D, \quad \text{and} \quad \mathbf{T}n &= \mathbf{g} \quad \text{on } \Gamma_N.
\end{align*}

(El)

where $\mathbf{u}$ is displacement, $\mathbf{T}$ the Cauchy stress, $\mathbf{f}$ the external body forces, $\mathbf{g}$ the external surface forces and $\varepsilon$ is the linearized strain tensor, i.e.,

$$
\varepsilon = \varepsilon(\mathbf{u}) := \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)
$$
Linearized nonlinear elasticity

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- \operatorname{div} T &= f \quad \text{in } \Omega, \\
u &= u_0 \quad \text{on } \Gamma_D, \quad \text{and} \quad Tn &= g \quad \text{on } \Gamma_N.
\end{align*}$$

(El)

where $u$ is displacement, $T$ the Cauchy stress, $f$ the external body forces, $g$ the external surface forces and $\varepsilon$ is the linearized strain tensor, i.e.,

$$\varepsilon = \varepsilon(u) := \frac{1}{2} \left( \nabla u + (\nabla u)^T \right)$$

- The implicit relation between the Cauchy stress and the strain

$$G(T, \varepsilon) = 0$$
Linearized nonlinear elasticity

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$$\mathbf{u} = \mathbf{u}_0 \quad \text{on} \ \Gamma_D,$$

and

$$\mathbf{T} n = g \quad \text{on} \ \Gamma_N.$$  \hspace{1cm} (E1)

where $\mathbf{u}$ is displacement, $\mathbf{T}$ the Cauchy stress, $f$ the external body forces, $g$ the external surface forces and $\varepsilon$ is the linearized strain tensor, i.e.,

$$\varepsilon = \varepsilon(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$$

- The implicit relation between the Cauchy stress and the strain

$$\mathbf{G} = \mathbf{G}(\mathbf{T}, \varepsilon) = 0$$

- The key assumption in linearized elasticity

$$|\varepsilon| \ll 1.$$  \hspace{1cm} (A)
Motivation for symmetric $p$-Laplace like operator for $p = 1$ or $p = \infty$

- $p = 1$: the plasticity model, i.e.,

  $$T \sim \frac{\varepsilon}{|\varepsilon|} \quad \text{for } |\varepsilon| \gg 1$$

- $p = \infty$: the limiting strain model, i.e.,

  $$\varepsilon \sim \frac{T}{|T|} \quad \text{for } |T| \gg 1.$$
Limiting strain model

The standard linear models immediately may lead to the contradiction:
Limiting strain model

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- Consider $\Omega$ a domain with non-convex corner at $x_0$, $\Gamma = \partial \Omega$, $u_0 = 0$ and $G$ of the form $T = \varepsilon$.
The standard linear models immediately may lead to the contradiction:

- Consider $\Omega$ a domain with non-convex corner at $x_0$, $\Gamma = \partial \Omega$, $u_0 = 0$ and $G$ of the form $T = \varepsilon$.

- There exists a smooth $f$ such that the solution $(T, \varepsilon)$ fulfils

$$|T(x)| = |\varepsilon(x)| \xrightarrow{x \to x_0} \infty.$$
Limiting strain model

The standard linear models immediately may lead to the contradiction:

- Consider $\Omega$ a domain with non-convex corner at $x_0$, $\Gamma = \partial \Omega$, $u_0 = 0$ and $G$ of the form $T = \varepsilon$.

- There exists a smooth $f$ such that the solution $(T, \varepsilon)$ fulfils

$$|T(x)| = |\varepsilon(x)| \xrightarrow{x \to x_0} \infty.$$  

$\Rightarrow$ contradicts the assumption of the model $(A) \Rightarrow$ not valid model at least in the neighborhood of $x_0$. 
Simplified setting - potential structure

We look for \((u, T)\) such that \(u = u_0\) on \(\Gamma_D\) and \(Tn = g\) on \(\Gamma_N\) and fulfilling

\[
\begin{align*}
- \text{div } T &= f, \\
\varepsilon(u) &= \varepsilon^*(T).
\end{align*}
\]

\[\Leftrightarrow\]

\[
\left\{ - \text{div } T^*(\varepsilon(u)) = f \right. \]

in \(\Omega\) with

\[
\varepsilon^*(T) := \frac{T}{(1 + |T|^a)^{\frac{1}{a}}} \quad \text{and} \quad T^*(W) := (\varepsilon^*)^{-1}(W) := \frac{W}{\left(1 - |W|^a\right)^{\frac{1}{a}}}
\]

for all \(T \in \mathbb{R}_{sym}^{d \times d}\) and \(W \in \mathbb{R}_{sym}^{d \times d}\) satisfying \(|W| < 1\).
First, we introduce the space of functions having bounded the symmetric gradient

\[ E := \{ u \in W^{1,1}(\Omega)^d; \, \varepsilon(u) \in L^\infty(\Omega)^{d \times d} \}. \]

and assume at least \( u_0 \in E, \, f \in L^2(\Omega)^d \) and \( g \in L^1(\Gamma_N)^d \).
Simplified setting - potential structure

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\[ E := \{ u \in W^{1,1}(\Omega)^d ; \varepsilon(u) \in L^\infty(\Omega)^{d \times d} \}. \]

and assume at least \( u_0 \in E, f \in L^2(\Omega)^d \) and \( g \in L^1(\Gamma_N)^d \).

- the set of admissible displacements

\[ V := \{ u \in E : u - u_0 \in W^{1,1}_{\Gamma_D}(\Omega)^d \}. \]

- the set of admissible stresses

\[ S := \left\{ T \in L^1(\Omega)^{d \times d}_{\text{sym}} : \forall v \in E \cap W^{1,1}_{\Gamma_D} \int_\Omega T \cdot \varepsilon(v) = \int_\Omega f \cdot v + \int_{\Gamma_N} g \cdot v \right\} \]

**Weak solution:** Find \((u, T) \in V \times S\) such that \(\varepsilon(u) = \varepsilon^*(T)\) a.e. in \(\Omega\).
Potential structure - primary formulation

Find potential $F : \mathbb{R}^{d \times d}_{\text{sym}} \to \mathbb{R}_+$ such that $F(0) = 0$ and

$$\frac{\partial F(W)}{\partial W} = T^*(W) \quad \text{if } |W| < 1,$$

$$F(W) = \infty \quad \text{if } |W| > 1.$$  

Primary (variational) formulation: Find $u \in \mathcal{V}$ such that for all $v \in \mathcal{V}$

$$\int_{\Omega} F(\varepsilon(u)) - f \cdot u - \int_{\Gamma_N} g \cdot u \leq \int_{\Omega} F(\varepsilon(v)) - f \cdot v - \int_{\Gamma_N} g \cdot v$$

Lemma

Let $\|\varepsilon(u_0)\|_\infty < 1$ (the safety strain condition). Then there exists a unique $u$ solving the primary formulation. Moreover there exists $T \in L^1(\Omega)^{d \times d}$ such that $\varepsilon(u) = \varepsilon^*(T)$ and for all $v \in \mathcal{V}$ such that $T^*(\varepsilon(v)) \in L^1$ there holds

$$\int_{\Omega} T \cdot \varepsilon(u - v) \leq \int_{\Omega} f \cdot (u - v) + \int_{\Gamma_N} g \cdot (u - v)$$

In addition, if there is a weak solution then it also solves the primary formulation. Similarly, if $u$ satisfies the safety strain condition, then $(u, T)$ is a weak solution.
Variational approach and BV setting

Potential structure - dual formulation

Find potential $F^* : \mathbb{R}^{d \times d}_{\text{sym}} \to \mathbb{R}_+$ such that $F(0) = 0$ and (note here that $F(W) \sim |W|$ at infinity

$$\frac{\partial F^*(W)}{\partial W} = \varepsilon^*(W).$$

Dual (variational) formulation: Find $T \in S$ such that for all $W \in S$

$$\int_{\Omega} F^*(T) - T \cdot \varepsilon(u_0) \leq \int_{\Omega} F^*(W) - W \cdot \varepsilon(u_0)$$

Lemma

The existence of a weak solution is equivalent to the existence of a minimizer to the dual problem. Moreover, if $\|\varepsilon(u_0)\|_\infty < 1$ (the safety strain condition) then there exists a finite infimum of the dual formulation which may be attained by $\overline{T} \in \mathcal{M}(\overline{\Omega})^{d \times d}_{\text{sym}}$. 
Variational approach and BV setting

Potential structure - relaxed dual formulation

the relaxed set of admissible stresses

\[ S^m := \left\{ T \in \mathcal{M}(\Omega)^{d \times d}_{\text{sym}} : \forall \nu \in C^1_{\Gamma_D}(\Omega)^d, \int_{\Omega} T \cdot \varepsilon(\nu) = \int_{\Omega} f \cdot \nu + \int_{\Gamma_N} g \cdot \nu \right\} \]

Dual (variational) relaxed formulation: For \( u_0 \in C^1(\Omega)^d \), find \( T \in S^m \) such that for all \( W \in S^m \)

\[
\int_{\Omega} F^*(T^r) + (W^r - T^r) \cdot \varepsilon(u_0) + |T^s|(\Omega) + \langle W^s - T^s, \varepsilon(u_0) \rangle \leq \int_{\Omega} F^*(W^r) + |W^s|(\Omega)
\]

where \( T = T^r + T^s \) and \( T^r \) is a regular part (i.e., absolutely continuous w.r.t. Lebesgue measure) and \( T^s \) is a singular part (i.e., supported on the set of zero Lebesgue measure).

Lemma

Let \( \|\varepsilon(u_0)\|_{\infty} < 1 \). Then there exists a minimizer to relaxed dual formulation. Moreover, the regular part \( T^r \) is unique and satisfies \( \varepsilon(u) = \varepsilon^*(T^r) \), where \( u \) is the (unique) minimizer to primary formulation. In addition, if \( T^s_1 \) and \( T^s_2 \) are two singular parts then for all \( \nu \in C^1_{\Gamma_D}(\Omega)^d \)

\[
|T^s_1|(\Omega) - \langle T^s_1, \varepsilon(u_0) \rangle = |T^s_2|(\Omega) - \langle T^s_2, \varepsilon(u_0) \rangle \text{ and } \langle T^s_1 - T^s_2, \nabla \nu \rangle = 0
\]
Conclusion

We solved the problem completely. Natural setting is the relaxed dual formulation. The displacement is unique. The regular part of the Cauchy stress is unique. There is non-uniquely given singular part of the Cauchy stress. Where is the singular measure supported? Is it really there? How do you explain that the regular part did not solve the balance equation? Is there some crack/damage possible region? Is there any influence of the shape of $\Omega$ or the parameter $a$? etc. etc.
Conclusion

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Where is the singular measure supported? Is it really there? How do you explain that the regular part did not solve the balance equation? Is there some crack/damage possible region? Is there any influence of the shape of $\Omega$ or the parameter $a$? etc. etc.
Limiting strain model - anti-plane stress

We consider the following special geometry

and we look for the solution in the following form:

\[ u = u(x_1, x_2) = (0, 0, u(x_1, x_2)), \quad g(x) = (0, 0, g(x_1, x_2)), \]

and

\[ T(x) = \begin{pmatrix} 0 & 0 & T_{13}(x_1, x_2) \\ 0 & 0 & T_{23}(x_1, x_2) \\ T_{13}(x_1, x_2) & T_{23}(x_1, x_2) & 0 \end{pmatrix}. \] (1)
Equivalent reformulation-simply connected domain

Find $U : \Omega \to \mathbb{R}$ - the Airy stress function such that

$$T_{13} = \frac{1}{\sqrt{2}} U_{x_2} \quad \text{and} \quad T_{23} = -\frac{1}{\sqrt{2}} U_{x_1}.$$ 

$\implies \text{div} \mathbf{T} = 0$ is fulfilled.
Find $U : \Omega \rightarrow \mathbb{R}$ - the Airy stress function such that

$$T_{13} = \frac{1}{\sqrt{2}} U_{x_2} \quad \text{and} \quad T_{23} = -\frac{1}{\sqrt{2}} U_{x_1}.$$ 

$\implies \text{div } \mathbf{T} = 0$ is fulfilled.

$U$ must satisfy $(\varepsilon(u) = \frac{\mathbf{T}}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}})$

$$\text{div} \left( \frac{\nabla U}{(1 + |\nabla U|^a)^{\frac{1}{a}}} \right) = 0 \quad \text{in } \Omega,$$

$$U_{x_2} n_1 - U_{x_1} n_2 = \sqrt{2} g \quad \text{on } \partial \Omega.$$
Equivalent reformulation-simply connected domain

Find $U : \Omega \to \mathbb{R}$ - the Airy stress function such that

$$T_{13} = \frac{1}{\sqrt{2}} U_{x_2} \quad \text{and} \quad T_{23} = -\frac{1}{\sqrt{2}} U_{x_1}.$$ 

$\implies$ $\text{div} \, T = 0$ is fulfilled.

$U$ must satisfy $(\varepsilon(u) = \frac{T}{1 + |T|^a})$

$$\text{div} \left( \frac{\nabla U}{(1 + |\nabla U|^a)^{\frac{1}{a}}} \right) = 0 \quad \text{in} \ \Omega,$$

$$U_{x_2} \mathbf{n}_1 - U_{x_1} \mathbf{n}_2 = \sqrt{2} g \quad \text{on} \ \partial \Omega.$$

Dirichlet problem, indeed assume that $\partial \Omega$ is parameterized by $\gamma(s) = (\gamma_1(s), \gamma_2(s))$. Then

$$U(\gamma(s_0)) = a_0 + \sqrt{2} \int_0^{s_0} g(\gamma(s)) \sqrt{(\gamma_1'(s))^2 + (\gamma_2'(s))^2} \, ds =: U_0(x).$$
Consequences for $U$

- We look for $U \in W^{1,1}(\Omega)$

$$\text{div} \left( \frac{\nabla U}{(1 + |\nabla U|^a)^{\frac{1}{a}}} \right) = 0 \quad \text{in } \Omega, \quad U = U_0 \quad \text{on } \partial \Omega.$$
Consequences for $U$

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- Calculus of variations and BV setting are back! Please wake up!
All presented results were based either on the scalar structure or on the radial structure.

There is NO theory for nonlinear systems having the radial-like structure with respect to the symmetric gradient, which would be better than the theory for general elliptic systems having no structure.
Consider $\varepsilon^*(\mathbf{T}) = \mathbf{T} / (1 + |\mathbf{T}|^a)^{\frac{1}{a}}$:

**Theorem (General result for $a > 0$)**

Let $a > 0$ and $\mathbf{u}_0$ satisfy the safety strain condition. Then there exists a unique triple $(\mathbf{u}, \mathbf{T}, \tilde{g}) \in \mathcal{V} \times L^1(\Omega)^d \times (C_0^1(\Gamma_N))^*$ such that for all $\mathbf{v} \in C^1_{\Gamma_D}(\Omega)$

$$
\varepsilon(\mathbf{u}) = \varepsilon^*(\mathbf{T})
$$

$$
\int_{\Omega} \mathbf{T} \cdot \varepsilon(\mathbf{u} - \mathbf{w}) \leq \int_{\Omega} \mathbf{f} \cdot (\mathbf{u} - \mathbf{w}) + \int_{\Gamma_N} \mathbf{g} \cdot (\mathbf{u} - \mathbf{w})
$$

$$
\mathbf{u} = \mathbf{u}_0 \text{ on } \Gamma_D,
$$

where $\mathbf{w} \in \mathcal{V}$ is arbitrary such that there exists $\tilde{\mathbf{T}} \in L^1$ fulfilling $\varepsilon(\mathbf{w}) = \varepsilon^*(\tilde{\mathbf{T}})$. 
Result for particular model and general geometry

Consider $\varepsilon^*(\mathbf{T}) = \mathbf{T}/(1 + |\mathbf{T}|^a)^{1/3}$:

**Theorem (General result for $a > 0$)**

Let $a > 0$ and $\mathbf{u}_0$ satisfy the safety strain condition. Then there exists a unique triple $(\mathbf{u}, \mathbf{T}, \tilde{g}) \in \mathcal{V} \times L^1(\Omega)^d \times \left(\mathcal{C}_0^1(\Gamma_N)\right)^*$ such that for all $\mathbf{v} \in \mathcal{C}_1(\Gamma_D(\Omega))$

$$\varepsilon(\mathbf{u}) = \varepsilon^*(\mathbf{T})$$

$$\int_{\Omega} \mathbf{T} \cdot \varepsilon(\mathbf{u} - \mathbf{w}) \leq \int_{\Omega} \mathbf{f} \cdot (\mathbf{u} - \mathbf{w}) + \int_{\Gamma_N} \mathbf{g} \cdot (\mathbf{u} - \mathbf{w})$$

$$\mathbf{u} = \mathbf{u}_0 \text{ on } \Gamma_D,$$

where $w \in \mathcal{V}$ is arbitrary such that there exists $\tilde{T} \in L^1$ fulfilling $\varepsilon(\mathbf{w}) = \varepsilon^*(\tilde{T})$. Moreover,

$$\int_{\Omega} \mathbf{T} \cdot \nabla \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} + \langle \mathbf{g} - \tilde{g}, \mathbf{v} \rangle_{\Gamma_N}$$
Assumptions for general model

**Assumptions on $\varepsilon^*$:** Denote $A(T) := \frac{\partial \varepsilon^*(T)}{\partial T}$.

- $\varepsilon^*$ is coercive, i.e.,
  $$\varepsilon^*(T) \cdot T \geq C_1 |T| - C_2$$

- $\varepsilon^*$ is $h$-elliptic, i.e., there exists nonincreasing function $h$ such that for all $W \neq 0$
  $$0 < h(|T|)|W|^2 \leq (W, W)_{A(T)} \leq \frac{|W|^2}{1 + |T|},$$
  where
  $$(W, W)_{A(T)} := \sum A_{\mu j}^{\nu i}(T) W^{\nu i} W^{\mu j}, \quad A_{\mu j}^{\nu i}(T) := \frac{\partial (\varepsilon^*)^{\nu i}(T)}{\partial T^{\mu j}}.$$

- $A$ is asymptotically symmetric, i.e.,
  $$\frac{|A^s(T) - A(T)|^2}{h(|T|)} \leq \frac{C_2}{1 + |T|}.$$  

- Either $h$ does not decrease faster than $|T|^{-1-2/d}$ or $\varepsilon^*$ is asymptotically radial, i.e., there exists a function $g$ such that $g(|T|) \leq C(1 + |T|)$ fulfilling
  $$\frac{|g(|T|)\varepsilon^*(T) - T|^2}{h(|T|)} \leq C_2(1 + |T|^3).$$
Assumptions on data:
- \( f \in L^2 \)
- \( g \in L^1 \)
- \( u_0 \) satisfies safety strain condition, i.e., there exists a compact set \( K \subset \mathcal{E}^*(\mathbb{R}^d_{sym}) \) such that for almost all \( x \in \Omega \)
  \[
  \mathcal{E}(u_0(x)) \in K
  \]
Result for limiting strain models

Theorem (General result)

There exists a unique triple \((u, T, \tilde{g}) \in W^{1,1}(\Omega)^d \times L^1(\Omega)^{d \times d} \times (C^1_0(\Gamma_d))^*\) such that

\[
\int_{\Omega} T \cdot \varepsilon(v) = \int_{\Omega} f \cdot v + \langle g - \tilde{g}, v \rangle_{\Gamma_N}
\]

\[
\varepsilon(u) = D(T) \in L^\infty(\Omega; \mathbb{R}^{d \times d})
\]

\[
\int_{\Omega} T \cdot \varepsilon(u - w) \leq \int_{\Omega} f \cdot (u - w) + \int_{\Gamma_N} g \cdot (u - w)
\]

\[
u = u_0 \text{ on } \Gamma_D,
\]

are satisfied for all \(v \in C^1_{\Gamma_D}(\bar{\Omega})\) and all \(w \in W^{1,\infty}(\Omega)\), where \(w\) is an arbitrary function being equal to \(u_0\) on \(\Gamma_D\) and for which there exists \(\tilde{T} \in L^1(\Omega)^{d \times d}_{sym}\) fulfilling \(\varepsilon(w) = \varepsilon^*(\tilde{T})\).
The first result for the symmetric gradient, where the *radial* setting plays the crucial role.
The same result obviously holds also for the full gradient case.
For *any* $C^1$ strictly monotone operator being asymptotically symmetric and radial we avoided the presence of the singular part in the interior!
At least in 2D and simply connected domains, we can convert this setting to the minimal surface-like problems and get the same result.
The method does not use the improved integrability result (which even may not be true)!
The same theory for minimal surface-like problems and general geometries. **Sharp identification** of the cases when the theory can be built up to the boundary.
Scheme of the proof

We find a mollified problem for which we have a solution and then go to the limit. The approximation is of the form

$$\varepsilon_n^*(T) := \varepsilon^*(T) + n^{-1} \frac{T}{(1 + |T|)^{1 - \frac{1}{n}}}.$$  

- The first a priori estimate

$$\int_{\Omega} |T^n| \leq C, \quad \|\varepsilon(u^n)\|_n \leq C.$$

- $T^n \rightharpoonup^* \bar{T}$ in $\mathcal{M}(\Omega)^{d \times d}_{\text{sym}}$,

$$\varepsilon(u^n) \rightharpoonup \varepsilon(u) \quad \text{in } L^q(\Omega)^{d \times d}_{\text{sym}}, \text{ for all } q < \infty.$$

and $\bar{T}$ solves the equation but we do not know that $\varepsilon(u) = \varepsilon^*(\bar{T})$.  

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Scheme

- First we show that

\[ T^n \rightarrow T \quad \text{a.e. in } \Omega, \]

where \( T \in L^1(\Omega)^{d \times d}_{\text{sym}} \) but we do not know that \( T = \overline{T} \).

- Then due to the continuity of \( \varepsilon^* \) we have

\[ \varepsilon(u) = \varepsilon^*(T) \quad \text{a.e. in } \Omega. \]

- The Fatou lemma and monotonicity justify the limit passage in

\[ \int_\Omega T \cdot \varepsilon(u - w) \leq \int_\Omega f \cdot (u - w) + \int_{\Gamma_N} g \cdot (u - w). \]

- The final step is to show that

\[ - \text{div } T = f. \]
Point-wise convergence of $T^n$

- The final uniform bound

\[
\int_\Omega \frac{\tau^2|\nabla T^n|^2}{(1 + |T^n|)^{a+1}} \leq C \sum_k \int_\Omega (\tau \partial_k T^n, \tau \partial_k T^n) A^n(T^n) \leq C.
\]

- we are able to deduce that

\[ T^n \to T \text{ a.e. in } \Omega \]

- Renormalized solution - for any $g \in D(\mathbb{R})$ and any $v \in D(\Omega)^d$

\[
\int_\Omega T \cdot (g(|T|)\nabla v) - \int_\Omega g(|T|)f \cdot v = -\int_\Omega T \cdot (v \otimes \nabla g(|T|))
\]

- Our goal is to let $g \nearrow 1$. In the first two terms it is easy. The last term causes troubles.
We look for \((u, T)\) fulfilling \((Q := (0, T) \times \Omega)\)

\[
\begin{align*}
\partial_{tt}^2 u - \text{div} \, T &= f \\
\varepsilon(u) &= \varepsilon^*(T) \\
u &= u_0 \\
T n &= g
\end{align*}
\]

in \(Q\),
in \(Q\),
on \(\Gamma_D \subset (0, T) \times \partial \Omega \cap \{0\} \times \Omega\),
on \(\Gamma_N := (0, T) \times \partial \Omega \setminus \Gamma_D\)

with

\[
\varepsilon^*(T) := \frac{T}{(1 + |T|^a)^{\frac{1}{a}}}
\]

- nonlinear hyperbolic system of second order
- case is lost - except one dimensional setting
We look for \((u, T)\) fulfilling \((Q := (0, T) \times \Omega)\)

\[
\begin{align*}
\partial_{tt}^2 u - \text{div} T &= f \\
\varepsilon(u) + \varepsilon(\partial_t u) &= \varepsilon^*(T) \\
u &= u_0 \\
T n &= g
\end{align*}
\]

in \(Q\),
in \(Q\),
on \(\Gamma_D \subset (0, T) \times \partial \Omega \cap \{0\} \times \Omega\),
on \(\Gamma_N := (0, T) \times \partial \Omega \setminus \Gamma_D\)

with

\[
\varepsilon^*(T) := \frac{T}{(1 + |T|^a)^{\frac{1}{a}}}
\]

- nonlinear parabolic - hyperbolic system of second order
- a hope for the existence of solution
Limiting strain - viscoelastic

Consider

\[ \varepsilon^*(T) \sim \frac{T}{(1 + |T|^a)^{\frac{1}{a}}}. \]

Assume that the existence of \( \psi^* \) fulfilling (it has linear growth)

\[ \frac{\partial \psi^*(T)}{\partial T} = \varepsilon^*(T) \]

convex conjugate (corresponds to the Helmholtz free energy)

\[ \psi(\varepsilon) := \sup_{W} (\varepsilon \cdot W - \psi^*(W)) \]

\[ \psi(\varepsilon) = \infty \text{ if } |\varepsilon| > 1. \]

Theorem (Bulíček, Patel, Şengül, Süli (2020))

Let \( \Gamma_N = \emptyset \). Then for any reasonable data there exists a weak solution.
A priori estimates

- Multiply

\[ \partial_{tt}^2 u - \text{div} \, T = f \]

by \( \partial_t u \) and integrate over \( \Omega \) (e.g. periodic data).

- Using that \( \partial_T \psi^* = (\partial_T \psi)^{-1} \)

\[
\int_{\Omega} f \cdot \partial_t u = \frac{d}{dt} \int_{\Omega} \frac{|\partial_t u|^2}{2} + \int_{\Omega} T \cdot \partial_t \varepsilon(u) \\
= \frac{d}{dt} \left( \int_{\Omega} \frac{|\partial_t u|^2}{2} + \psi(\varepsilon(u)) \right) + \int_{\Omega} (T - \partial_T \psi(\varepsilon(u))) \cdot \partial_t \varepsilon(u) \\
= \frac{d}{dt} \left( \int_{\Omega} \frac{|\partial_t u|^2}{2} + \psi(\varepsilon(u)) \right) + \int_{\Omega} (T - \partial_T \psi(\varepsilon(u))) \cdot (\partial_T \psi^*(T) - \varepsilon(u)) \\
= \frac{d}{dt} \left( \int_{\Omega} \frac{|\partial_t u|^2}{2} + \psi(\varepsilon(u)) \right) + \int_{\Omega} (T - \partial_T \psi(\varepsilon(u))) \cdot (\partial_T \psi^*(T) - \partial_T \psi^*(\partial_T \psi(\varepsilon(u))))
\]

energy \hspace{2cm} dissipation \geq 0
A priori estimates

- First a priori estimate
  \[ |\varepsilon(u)| \leq 1, \quad \partial_t u \in L^\infty(L^2), \quad T \in L^1(Q) \]

- Second a priori estimates - test by \( \Delta u, \Delta \partial_t u \) and \( \partial_{tt} u \)
  \[ \varepsilon(u) \in L^\infty(L^2), \quad T \in L^\infty(L^1), \quad \partial_{tt}^2 u \in L^2(Q), \quad \int_Q (\nabla T, \nabla T)_{A(T)} < \infty \]

- Starting point done - time for renormalization, etc...