Regularity issues for orthotropic functionals

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References

The research presented is part of a project started in 2011, mainly in collaboration with Pierre Bousquet (Toulouse)


Outline

1. The functionals

2. Isotropic VS. Orthotropic

3. Lipschitz regularity

4. Beyond standard growth
Orthotropic functionals

We want to consider local minimizers of a functional with orthotropic structure

\[ \sum_{i=1}^{N} \int f_i(u_{x_i}) \, dx \quad f_i \text{ convex}, \quad u_{x_i} = \frac{\partial u}{\partial x_i} \]

Example

By taking \( f_i(t) = t^2/2 \), we get

\[ \sum_{i=1}^{N} \frac{1}{2} \int |u_{x_i}|^2 \, dx = \frac{1}{2} \int |\nabla u|^2 \, dx \quad \text{Dirichlet integral} \]

A well-known functional without orthotropic structure

For \( p \neq 2 \), the classical

\[ \frac{1}{p} \int |\nabla u|^p \, dx \quad p-\text{Dirichlet integral} \]

does not fall in this class
Leading example

Orthotropic $p$–Dirichlet integral

$$
\sum_{i=1}^{N} \frac{1}{p} \int |u_{x_i}|^p \, dx
$$

This is a natural generalization of the Dirichlet integral

The orthotropic $p$–Laplacian operator

Local minimizers are weak solutions of the Euler-Lagrange equation

$$
\sum_{i=1}^{N} \left( |u_{x_i}|^{p-2} \, u_{x_i} \right)_{x_i} = 0
$$

Remark

This equation looks similar to the more familiar one

$$
\sum_{i=1}^{N} \left( |\nabla u|^{p-2} \, u_{x_i} \right)_{x_i} = 0
$$

but they are quite different
So similar, yet so different

Let us set

$$\mathcal{I}(z) = |z|^p \quad \text{and} \quad \mathcal{O}(z) = \sum_{i=1}^{N} |z_i|^p$$

Similarities: growth

Both of them are strictly convex, with $p$–growth, i.e.

$$\mathcal{O}(z) \simeq |z|^p = \mathcal{I}(z)$$

For basic regularity (i.e. $L^\infty$ and $C^{0,\alpha}$ estimates, Harnack inequalities, Gehring-type gradient integrability etc.)

$$\sum_{i=1}^{N} (|\nabla u|^{p-2} u_{x_i})_{x_i} \quad \text{and} \quad \sum_{i=1}^{N} (|u_{x_i}|^{p-2} u_{x_i})_{x_i}$$

can be treated in exactly the same manner and there is nothing new (see Chapters 6 & 7 of Giusti’s book)
Differences: ellipticity \((p \geq 2)\)

- **isotropic**
  \[
  \langle D^2 I(z) \xi, \xi \rangle \simeq |z|^{p-2} |\xi|^2
  \]
  least eigenvalue of \(D^2 I(z)\) becomes 0 only at \(z = 0\)

- **orthotropic**
  \[
  \langle D^2 O(z) \xi, \xi \rangle \simeq \sum_{i=1}^{N} |z_i|^{p-2} |\xi_i|^2
  \]
  least eigenvalue of \(D^2 O(z)\) becomes 0 each time \(z_i = 0\)

For higher regularity (i.e. Lipschitz and \(C^{1,\alpha}\)) these are completely different

- In this talk I will be interested in \textbf{Lipschitz regularity}
Some variations on the theme

Our motivation for this orthotropic functional was a problem in Optimal Transport, but once we opened the hell’s gates....

1. General norms

\[ \int \| \nabla u \|^p \, dx \]

where \( \| \cdot \| \) is any norm

The relevant \( p \)-Laplacian behaves like the isotropic one only when \( \text{the norm } \| \cdot \| \text{ is uniformly convex} \), otherwise it is a completely different story

2. Orthotropic & non-standard growth

\[ \sum_{i=1}^{N} \int |u_{x_i}|^{p_i} \, dx, \quad 1 < p_1 \leq p_2 \leq \ldots p_N < +\infty \]

For gradient regularity, this is one of the nastiest functionals (introduced by the Soviet school already in the 70s and independently by Marcellini in Western Countries)
A handful of (old) references

1. Orthotropic $p-$Laplacian has been considered for example in
   - Lions’ book “*Quelques méthodes de résolution etc.*” (1969)

They tackle the **existence issue** for its parabolic version

$$\sum_{i=1}^{N} \left( |u_{x_i}|^{p-2} u_{x_i} \right)_{x_i} = u_t$$

2. For **higher regularity** (i.e. Lipschitz & $C^{1,\alpha}$), this equation has been overlooked or neglected, apart for


They proved Lipschitz regularity for $p \geq 4$, without using energy methods, but Bernstein’s one
From now on, I will manipulate solutions as if they were $C^2$.

I will focus on formally obtaining *a priori estimates*.

Everything can then be rigorously justified by approximations.
1. The functionals

2. Isotropic VS. Orthotropic

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4. Beyond standard growth
One step back: isotropic case

Consider a local weak solution of the standard $p-$Laplacian

$$- \sum_{i=1}^{N} (|\nabla u|^{p-2} u_{x_i})_{x_i} = 0$$

How to prove that $\nabla u \in L_{\text{loc}}^\infty$?

Equation for the gradient

We still use the notation $I(z) = |z|^p$, then the equation rewrites

$$- \text{div} \nabla I(\nabla u) = 0$$

Differentiate the equation in direction $x_k$, we get that $u_{x_k}$ solves

$$- \text{div} (D^2 I(\nabla u) \nabla u_{x_k}) = 0$$

We can think of this as degenerate linear equation, with coefficients matrix $D^2 I(\nabla u)$
Subsolutions
For every $f : \mathbb{R} \to \mathbb{R}$ convex
\[
\int \langle D^2\mathcal{I}(\nabla u) \nabla f(u_{x_k}), \nabla \varphi \rangle \leq 0 \quad \text{for every } \varphi \geq 0
\]
that is, $f(u_{x_k})$ is a subsolution of the linearized equation
\[
-\operatorname{div}(D^2\mathcal{I}(\nabla u) \nabla \psi) = 0
\]

Caccioppoli for the gradient
Take the test function $\varphi = \eta^2 f(u_{x_k})$, then we get

Caccioppoli inequality for convex functions of $u_{x_k}$
\[
\int \langle D^2\mathcal{I}(\nabla u) \nabla f(u_{x_k}), \nabla f(u_{x_k}) \rangle \eta^2 \lesssim \int \langle D^2\mathcal{I}(\nabla u) \nabla \eta, \nabla \eta \rangle f(u_{x_k})^2
\]
One step back: isotropic case III

We are in troubles, since

\[ D^2\mathcal{I}(\nabla u) \simeq |\nabla u|^{p-2} \]

thus Caccioppoli for the gradient is **apparently useless** when \( \nabla u = 0 \)

Absorption trick

We can bypass this problem by absorbing \( D^2\mathcal{I}(\nabla u) \) into the subsolution. More precisely, find suitable convex functions \( f \) and \( F \) such that

\[
\langle D^2\mathcal{I}(\nabla u) \nabla f(u_{x_k}), \nabla f(u_{x_k}) \rangle \simeq |\nabla u|^{p-2} |\nabla f(u_{x_k})|^2
\geq |u_{x_k}|^{p-2} |\nabla f(u_{x_k})|^2
= |\nabla F(u_{x_k})|^2
\]

Ok....but which kind of \( f \) and \( F \) work?
Power functions! Take $f(u_{x_k}) = |u_{x_k}|^\beta$, then $F$ is still a power

By using this trick, from “Caccioppoli for the gradient” we get

$$\int_{B_\varrho} \left| \nabla |u_{x_k}|^{\beta + \frac{p-2}{2}} \right|^2 \lesssim \int_{B_R} |\nabla u|^{2\beta + p - 2}$$

and combining with Sobolev inequality

$$\left( \int_{B_\varrho} |u_{x_k}|^{(2\beta + p - 2)\frac{2^*}{2}} \right)^{\frac{2}{2^*}} \lesssim \int_{B_R} |\nabla u|^{2\beta + p - 2}$$

iterative scheme of reverse Hölder inequalities ($2^*/2 > 1$)

Moser’s iteration
Start with $\beta = 1$ and iterate infinitely many times
Now move forward

For the orthotropic case, we try to mimick the same strategy

**Equation for the gradient**

We have a look at the equation solved by $u_{x_k}$

By differentiating the equation with respect to $x_k$, we get

$$\sum_{i=1}^{N} \int |u_{x_i}|^{p-2} (u_{x_k})_{x_i} \varphi_{x_i} = 0$$

a **linear degenerate** elliptic equation with **diagonal** coefficient matrix

$$D^2 \mathcal{O}(\nabla u) = \begin{bmatrix} |u_{x_1}|^{p-2} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ |u_{x_N}|^{p-2} & \cdots & \cdots & \cdots \end{bmatrix}$$

The **least eigenvalue is 0** each time a component of $\nabla u$ vanishes
Subsolutions
For every \( f : \mathbb{R} \to \mathbb{R} \) convex

\[
\sum_{i=1}^{N} \int |u_{x_i}|^{p-2} \left( f(u_{x_k}) \right)_{x_i} \varphi_{x_i} \leq 0 \quad \text{for every } \varphi \geq 0
\]

that is \( f(u_{x_k}) \) is a subsolution of the linearized equation

\[
\text{div}(D^2O(\nabla u) \nabla \psi) = \sum_{i=1}^{N} \left( |u_{x_i}|^{p-2} \psi_{x_i} \right)_{x_i} = 0
\]

Caccioppoli inequality for the gradient
Take the test function \( \varphi = \eta^2 f(u_{x_k}) \), then we get

Caccioppoli inequality for convex functions of \( u_{x_k} \)

\[
\sum_{i=1}^{N} \int |u_{x_i}|^{p-2} \left| \left( f(u_{x_k}) \right)_{x_i} \right|^2 \eta^2 \lesssim \sum_{i=1}^{N} \int |u_{x_i}|^{p-2} f(u_{x_k})^2 |\eta_{x_i}|^2
\]
A major obstruction

In the isotropic case Caccioppoli for the gradient gave a control on

\[ |\nabla u|^{p-2} |\nabla f(u_{x_k})|^2 \]

but now it is much worse!

We only control

\[ \sum_{i=1}^{N} |u_{x_i}|^{p-2} |(f(u_{x_k}))_{x_i}|^2 \]

i.e. a weighted gradient of \( f(u_{x_k}) \)...too much degeneracy

No way that the “absorption trick” works as before
1. The functionals

2. Isotropic VS. Orthotropic

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Lipschitz regularity for $p \geq 2$

**Theorem (Bousquet-B.-Leone-Verde)**

Let $p \geq 2$. If $u$ is a local minimizer of

$$\sum_{i=1}^{N} \frac{1}{p} \int |u_{x_i}|^p \, dx$$

then $\nabla u \in L^\infty_{\text{loc}}$ and

$$\|\nabla u\|_{L^\infty(B_{R}/2)} \lesssim \left( \int_{B_{R}} |\nabla u|^p \, dx \right)^{1/p}$$

**Remark**

We want to perform a Moser’s iteration, but we need new ideas in order to exploit **Caccioppoli for the gradient** and circumvent the degeneracy of the weights $|u_{x_i}|^{p-2}$
A technical innovation

We cook up **new Caccioppoli inequalities for the gradient**

**The method**

- as before, take the equation differentiated with respect to $x_k$

\[
\sum_{i=1}^{N} \int |u_{x_i}|^{p-2} (u_{x_k})_{x_i} \varphi_{x_i} = 0
\]

- now insert the **weird test function** ($\alpha \leq \beta$)

\[
\varphi = |u_{x_k}|^{2\alpha-1} |u_{x_j}|^{2\beta} \eta^2
\]

- combine the Caccioppoli inequality so obtained (we call it **weird Caccioppoli**)...

- ... with the Caccioppoli for the gradient (I mean, the one we obtained previously)...

- ... plus a finite iteration on indexes $\alpha$ and $\beta$ with $\alpha + \beta$ fixed (this is the magic & scaring part that nobody wants to see in a talk)
“The dish is ready”

For every $q = 2^m$ we get for every $j, k$

$$\sum_{i=1}^{N} \int |u_{x_i}|^{p-2} u_{x_i x_k}^2 |u_{x_j}|^{2q} \lesssim \int |\nabla u|^{p+2q}$$

Why two indices $j, k$? What we do now?
We can now perform the usual absorption trick on the left-hand side!! In the sum, keep only the term $i = j$

$$\int |u_{x_j}|^{p-2} u_{x_j x_k}^2 |u_{x_j}|^{2q} \simeq \int \left| \left( |u_{x_j}|^{\frac{p}{2} + q} \right)_{x_k} \right|^2$$

and sum over $k$ to reconstruct the full gradient of $|u_{x_j}|^{p/2+p}$!
Conclusion

After all these struggles, we get

**Caccioppoli for power-functions**

\[
\int \left| \nabla |u_{x_j}|^{\frac{p}{2} + q} \right|^2 \lesssim \int |\nabla u|^{p+2q}
\]

We are in the same situation as for the standard \( p \)-Laplacian:

- use Sobolev inequality
- get an iterative Moser’s scheme
- iterate infinitely many times for a diverging sequence \( q_n \)

(I am hiding *sous le tapis* other – lower order yet annoying – technical complications)
Some comments

Related results
The same Lipschitz result has been obtained by means of *viscosity techniques* by Demengel [Adv. Differ. Equ. (2016)]

Right-hand side

- Our result also covers much more degenerate situations and the non-homogeneous case

\[- \sum_{i=1}^{N} \left( |u_{x_i}|^{p-2} u_{x_i} \right)_{x_i} = f\]

under some **non-sharp** assumptions on $f$

- The expected sharp assumption on $f$ to get Lipschitz regularity is $f \in L^q$ with $q > N$ (actually the sharpest assumption should be on the Lorentz scale $f \in L^{N,1}$ as in Beck - Mingione [CPAM (2019)])

- At present, this is still an **open problem**
Other regularity results

Higher differentiability à la Uhlenbeck
Local minimizers are such that

$$|u_{x_i}|^{\frac{p-2}{2}} u_{x_i} \in W^{1,2}_{\text{loc}}$$

Still true with a right-hand side $f$, under the sharp assumption $f \in W^{s,p'}_{\text{loc}}$, as in B.-Santambrogio [Comm. Cont. Math. (2016)]

$C^1$ regularity
In dimension $N = 2$, local minimizers are such that (Bousquet - B.)

$$\nabla u \in C^0$$

The proof works with a right-hand side $f$, as well....but the paper was already quite complicated with $f = 0$
Still on $C^1$ regularity

$Lindqvist - Ricciotti$ [Nonlinear Anal. (2018)] improved the result to

$$\nabla u \in C^{0,\omega}$$

for some logarithmic modulus of continuity $\omega$

- this is for the homogeneous equation only
- for a right-hand side $f$, one could try to transfer the **excess-decay estimate**

$$\int_{B_r} |\nabla u - \nabla u_{B_r}|^p \, dx \lesssim \omega(r)$$

from solutions of the homogeneous equation...

- ...but the modulus $\omega$ is too weak for this strategy to work (in other words, Campanato’s Theorem fails for $C^{0,\omega}$, see Spanne [Ann. SNS (1965)])
1. The functionals

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Next challenge

What about the orthotropic & non-standard growth?

\[
\sum_{i=1}^{N} \frac{1}{p_i} \int |u_{x_i}|^{p_i} \, dx, \quad 1 < p_1 \leq p_2 \leq \ldots p_N < +\infty
\]

Well-known fact
We **cannot** expect regularity for local minimizers, when

\[
1 \ll \frac{p_N}{p_1}
\]

(Giaquinta-Marcellini’s counterexamples)
In this case, local minimizers may be **unbounded**

Question
What if we impose *a priori* that a local minimizer is bounded?
Orthotropic & non-standard growth

Theorem (Bousquet - B.)

Let $2 \leq p_1 \leq \cdots \leq p_N$. If $u$ is a bounded local minimizer of

$$\sum_{i=1}^{N} \frac{1}{p_i} \int |u_{x_i}|^{p_i} \, dx$$

then $\nabla u \in L^\infty_{loc}$

Remarks

- no upper bound on $p_N/p_1$ is assumed. Under such a generality, the result is claimed in Lieberman [Adv. Diff. Eq. (2005)]

- case $N = 2$ previously proved in B. - Leone - Pisante - Verde, by using a different argument i.e. a two-dimensional trick introduced in Bousquet - B. - Julin

- for $p_1 \geq 4$ and $p_N < 2 p_1$, proved by Uralt’seva & Urdaletova (by Bernstein’s method)
A glimpse of the proof

The proof is composed of two main steps:

**A.** a Moser’s iteration similar to the one for \( p_1 = \cdots = p_N \), to get

\[
\| \nabla u \|_{L^\infty(B_{1/2})} \lesssim \left( \int_{B_1} |\nabla u|^\gamma \right)^{\frac{1+\Theta}{\gamma}}
\]

\( \gamma \) could be very big (here, we **do not** need \( u \in L^\infty_{\text{loc}} \))

**B.** a self-improving scheme for the gradient à la Bildhauer-Fuchs-Zhong [Ann. SNS (2007)]

\[
\int_{B_{\sigma R}} |u_{x_k}|^{p_k+2+\alpha} \, dx \leq C + C \sum_{i \neq k} \int_{B_R} |u_{x_i}|^{\frac{p_i-2}{p_k}} (p_k+2+\alpha) \, dx
\]

The constant \( C \) **depends on** \( \| u \|_{L^\infty_{\text{loc}}} \)

Final gain of **B.**: \( \nabla u \in L^q_{\text{loc}} \) for every \( q \)
Some comments

$L^\infty$ assumption

- Sharp assumptions in order to get $u \in L^\infty_{\text{loc}}$ are in
  
  *Fusco - Sbordone* [Manuscripta Math. (1990)]

- for example, in dimension $N = 2$ local minimizers are always locally bounded

- for more general functionals with nonstandard growth, many authors contributed to local boundedness. Among others, we mention

  *Cupini - Marcellini - Mascolo* [Nonlinear Anal. (2019)]
  
  *Hirsch - Schäffner* [Comm. Contemp. Math. (2020)]
Right-hand side

- Our result does not cover the non-homogeneous case

\[- \sum_{i=1}^{N} \left( |u_{xi}|^{p_i-2} u_{xi} \right)_{x_i} = f\]

- the proof is very likely to be adapted (with some sweat & tears) to include the right-hand side $f$, without sharp assumptions

- The expected sharp assumption on $f$ to get Lipschitz regularity is...? In view of Beck - Mingione it is reasonable to expect $f \in L^{N,1}$

Higher differentiability à la Uhlenbeck

$L^\infty_{loc}$ local minimizers are such that (Bousquet - B.)

\[|u_{xi}|^{\frac{p_i-2}{2}} u_{xi} \in W^{1,2}_{loc}\]
$C^1$ regularity

- in dimension $N = 2$ Lindqvist - Ricciotti [Nonlinear Anal. (2018)] proved also
  \[ \nabla u \in C^{0,\omega} \]
  for some logarithmic modulus of continuity $\omega$, even for
  $2 \leq p_1 \leq p_2$

- again, this is for the homogeneous equation only
Many thanks for your kind attention

“I knew it would take some time to get to that point. And I worked hard to get there”

C. SCHULDINER