Gradient estimates for fully nonlinear models with non-homogeneous degeneracy

João Vitor da Silva

Universidade Estadual de Campinas  (jdasilva@unicamp.br)

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Outline

1. Introducing the problem
2. Our main result and its natural obstacles
3. Further improved results
Before presenting our main problems and results, let us revisit some well-known elliptic PDEs models:

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In this point will be natural to consider the following **non-homogeneous model case**:

$$\mathcal{L}[u] = [||Du|^p + a(x)|Du|^q]\text{Tr}(A(x)D^2u) \quad \text{for} \quad 0 < p \leq q < \infty \quad \text{and} \quad 0 \leq a \in C^0(\Omega),$$

i.e. the counterpart of certain variational problems from Calculus of Variations with double phase structure.
In the last decades have appeared a huge amount of literature on double phase problems \(^a\)

\[
(w,f) \mapsto \min \int_{\Omega} \left( \frac{1}{p} |\nabla w|^p + \frac{a(x)}{q} |\nabla w|^q - fw \right) \, dx \quad \Rightarrow \quad \mathcal{L}[u] = -\text{div} \left( \left( |\nabla u|^{p-2} + a(x)|\nabla u|^{q-2} \right) \nabla u \right) = f(x)
\]

---

\(a\)

L. Beck, **Elliptic regularity theory.** A first course. *Lecture Notes of the Unione Matematica Italiana*, 19. Springer, Cham; Unione Matematica Italiana, Bologna, 2016. xii+201 pp. ISBN: 978-3-319-27484-3; 978-3-319-27485-0.

I. Chlebicka *et al*, **Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces.** Monograph.

C. De Filippis’ contributions - https://sites.google.com/view/cristianadefilippis/home


G. Mingione’s contributions - https://sites.google.com/site/giuseppemingionemath/

Introducing the problem
Our main result and its natural obstacles
Further improved results

Presenting the problem

In this Lecture we are interested in studying quantitative features for fully nonlinear elliptic models of double degenerate type as follows:

\[ G[u] := \mathcal{H}(x, Du)F(x, D^2 u) = f(x, u) \quad \text{in} \quad \Omega \subset \mathbb{R}^n \quad \text{(bounded domain)}, \quad (1.1) \]

where we will suppose the following Structural Conditions (SC):

✓ \; f \in C_0^0(\Omega \times \mathbb{R}) \cap L_\infty(\Omega \times \mathbb{R})

✓ \; F: \Omega \times \text{Sym}(n) \rightarrow \mathbb{R} is a \quad (\lambda, \Lambda) -\text{elliptic operator with}\n
\omega - \text{continuous coefficients:} \lambda \|X - Y\| \leq F(x, X) - F(x, Y) \leq \Lambda \|X - Y\| \quad \text{and} \quad \Theta F(x, y) := \sup_X X \neq 0 |F(x, X) - F(y, X)| \|X\| \leq C F \omega(|x - y|).

✓ \; L_1[|\xi|^p + a(x)|\xi|^q] \leq \mathcal{H}(x, \xi) \leq L_2[|\xi|^p + a(x)|\xi|^q];

✓ \; 0 < p \leq q < \infty, 0 < L_1 \leq L_2 < \infty \quad \text{and} \quad 0 \leq a \in C_0^0(\Omega).
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\[ \lambda \|X - Y\| \leq F(x, X) - F(x, Y) \leq \Lambda \|X - Y\| \quad \text{and} \quad \Theta_F(x, y) := \sup_{X \in \text{Sym}(n) \atop X \neq 0} \frac{|F(x, X) - F(y, X)|}{\|X\|} \leq C \omega(|x - y|). \]

\[ \checkmark \quad L_1 \left[ ||\xi||^p + a(x) ||\xi||^q \right] \leq \mathcal{H}(x, \xi) \leq L_2 \left[ ||\xi||^p + a(x) ||\xi||^q \right]; \]
\[ \checkmark \quad 0 < p \leq q < \infty, \quad 0 < L_1 \leq L_2 < \infty \quad \text{and} \quad 0 \leq a \in C^0(\overline{\Omega}). \]
Some Motivations

A key issue in linear/nonlinear PDEs consists in inferring which is the expected regularity to corresponding solutions.
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By way of motivation, let us visit the uniformly elliptic theory: Let $u$ be a solution to

$$G[u] := \text{Tr} \left( A(x) D^2 u \right) = f(x) \quad \text{in} \quad B_1.$$  \hspace{1cm} (1.2)
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(1.2)

There are two important aspects which we must take into account:

- A priori estimate to Hom. PDE
- Integrability of the source term
- With “frozen” coeff.

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- A priori estimate to Hom. PDE
- Integrability of the source term
- with “frozen” coeff.

As a matter of fact, \( v(x) := \frac{u(\rho x)}{\rho^\kappa} \), for \( \kappa \in (0, 2] \) verifies:

\[
G_\rho[v] = \text{Tr} \left( A_\rho(x) D^2 v \right) = \rho^{2-\kappa} f(\rho x) := f_\rho(x) \quad \Rightarrow \quad \|f_\rho\|_{L^r(B_1)} \leq \rho^{2-\kappa - \frac{n}{r}} \|f\|_{L^r(B_1)}. \]
Sharp regularity estimates

Better integrability/regularity of \( f \) (resp. \( A \)) \Rightarrow Better (local) regularity of \( u \)


*Let \( u \) be a bounded viscosity solution\(^a\) to (1.2) then*

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\(^a\) \( u \in C^0(B_1) \) is a viscosity super-solution (resp. sub-solution) to (1.2) if whenever \( \varphi \in C^2(B_1) \) and \( x_0 \in B_1 \) such that \( u - \varphi \) has a local minimum (resp. local maximum) at \( x_0 \), then

\[
\text{Tr} \left( A(x_0)D^2 \varphi(x_0) \right) \leq f(x_0) \quad \text{resp.} \quad \text{Tr} \left( A(x_0)D^2 \varphi(x_0) \right) \geq f(x_0).
\]

Let \( u \) be a bounded viscosity solution to (1.2) then

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\[ \xi := 2 - \frac{n}{r} \quad \text{and} \quad \hat{\xi} := \min \left\{ \alpha_{\text{Hom}}, 1 - \frac{n}{r} \right\} \]
Introducing the problem
Our main result and its natural obstacles
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**Sharp Lipschitz Logarithmical moduli of continuity**


*Let* $u$ *be a bounded viscosity solution to* (1.2) *then*

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$\tau(s) := s |\log s|$ \text{ and } $\psi(s) := s^2 |\log s|$.

Similar borderline/regularity results were addressed by Daskalopoulos et al.

Recently, by combining geometric methods and analytic techniques the $C^{1,\alpha}_{\text{loc}}$ regularity estimate was addressed by De Filippis for equations as follow:

\[ |Du|^p + a(x)|Du|^q F(D^2 u) = f \in L^\infty(B_1) \cap C^0(B_1), \text{ for some } \alpha(\text{universal}) \in (0,1). \]

---


Nevertheless, De Filippis’ work leaves some open issues regards the general scenario:

\[ G[u] := \mathcal{H}(x, Du) F \left( x, D^2 u \right) = f(x, u) \quad \text{in} \quad \Omega \quad \text{under the Structural Conditions (SC)}. \]

We stress that the regularity theory for uniformly elliptic models is available in the Caffarelli and Trudinger’s works:

\[ ^a \]

We will provide an affirmative answer in following sceneries:

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<td>$C_{\text{loc}}^{1, \frac{1}{p+1}}$</td>
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<tr>
<td>(SC) + F a concave/convex operator</td>
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Another pivotal question:

Are there significant changes between De Filippis’ approach and ours?

1. Compactness (Hölder estimate) via Ishii-Lions method;
2. $C^{1,\alpha}$ regime via affine approximation scheme (Deviation by Planes);
3. Degeneracy character of operator: a priori estimates for small/large translations;
Therefore, we will establish sharp (geometric) $C^{1,\alpha}$ estimates for solution to (1.1) by making use a systematic and alternative approach, as well as address some improvements.

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Our Main Theorem

Theorem ([da S. - Ricarte, Calc. Var. PDEs, 59 (2020), n. 5, Paper No. 161, 33 pp.])

Let \( K \subset B_1 \), \( u \) be a bounded solution of (1.1) in \( B_1 \) and suppose that (SC) are in force. Then \( u \) is \( C^{1,\alpha}_{loc} \), i.e., there exists a (universal) constant \( M > 0 \) such that

\[
[u]_{C^{1,\alpha}(K)}^* \leq M \left[ \|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}^{\frac{1}{p+1}} + 1 \right]
\]

where,

\[
[u]_{C^{1,\alpha}(K)}^* := \sup_{0 < \rho \leq \rho_0} \left( \inf_{x_0 \in K} \frac{\|u - l_{x_0}(u)\|_{L^\infty(B_{\rho}(x_0) \cap K)}}{\rho^{1+\alpha}} \right)
\]

and \( l_{x_0}(u) := u(x_0) + Du(x_0) \cdot (x - x_0) \).

Our regularity estimates generalize, to some extent, earlier ones via a different approach\(^a\)

Introducing the problem

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It is worth highlight that such estimates play an essential role in proving\(^a\):

1. Blow-up and Liouville type results;
2. Weak geometric properties and Hausdorff measure estimates;
3. Sharp regularity in certain free boundary problems (FBPs for short).

\(^a\)


Chapter 1: An approximation Scheme

A key step in accessing the regularity theory available for “frozen” coefficient, homogeneous operators is the following.

**Lemma (Approximation Lemma)**

If $u$ is a solution of (1.1) in $B_1$ with $\|u\|_{L^\infty(B_1)} \leq 1$, then $\forall \varepsilon > 0$ there exists $\delta = \delta(p, q, n, \lambda, \Lambda, \varepsilon) > 0$ such that whenever

$$\max \left\{ \Theta_F(x), \|f\|_{L^\infty(B_1 \times \mathbb{R})} \right\} \leq \delta \varepsilon$$

there exists a $F$-harmonic function $\phi : B_{1/2} \rightarrow \mathbb{R}$, i.e., $F(D^2 \phi) = 0$, such that

$$\max \left\{ \|u - \phi\|_{L^\infty(B_{1/2})}, \|D(u - \phi)\|_{L^\infty(B_{1/2})} \right\} < \varepsilon$$

where $\|\phi\|_{C^{1,a} \text{Hom} (\Omega')} \leq C(n, \lambda, \Lambda) \cdot \|\phi\|_{L^\infty(\Omega)}$.

**Proof “Just waving hands”:**

The proof is based on a *Reductio ad absurdum* and makes use of compactness, stability, *a priori* estimates and uniqueness of Dirichlet problem.
Chapter 1: An approximation Scheme

Proof “Just waving hands”:

The proof is based on a *Reductio ad absurdum* and makes use of compactness, stability, *a priori* estimates and uniqueness of Dirichlet problem.
Remark (Normalization and “flatness regime”)

Assumptions in the Lemma 3 are not restrictive. Indeed, fixed $\delta \epsilon > 0$, there exist $\kappa, \tau > 0$ such that the function

$$v(x) = \frac{u(\tau x + x_0)}{\kappa},$$

fall into in the conditions of Lemma 3, where

$$\kappa := \|u\|_{L^\infty(\Omega)} + 1 + \delta \epsilon^{-1} \|f\|_{L^\infty(\Omega \times \mathbb{R})}^{\frac{1}{p+1}}$$

and

$$\tau = \min \left\{ \frac{1}{2}, \left( \frac{\delta \epsilon}{\|f\|_{L^\infty(\Omega \times \mathbb{R})} + 1} \right)^{\frac{1}{p+2}}, \omega^{-1} \left( \frac{\delta \epsilon}{C_F + 1} \right) \right\}.$$
In the sequel, the purpose will be to make use of an $\mathcal{F}$–harmonic approximation in a $C^1$–fashion (Approximation Lemma) to ensure that viscosity solutions are “geometrically close” to their tangent plane in a suitable manner, i.e.

\[
C^1 - \text{closeness} \quad \implies \quad \sup_{B_{\rho}(x_0)} \frac{|u(x) - u(x_0) - Du(x_0) \cdot (x - x_0)|}{\rho^{1+\alpha}} \leq 1,
\]

thereby getting a geometric estimate.
Chapter 2: A first approximating estimate

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\]

thereby getting a geometric estimate.

Lemma ("Pseudo" first step of induction)

Let \( u \) be a viscosity solution of (1.1) in \( B_1 \) with \( \|u\|_{L^\infty(B_1)} \leq 1 \). There exist \( \delta_\varepsilon > 0 \) and \( \rho \in \left(0, \frac{1}{2}\right) \) such that if

\[
\max \left\{ \Theta_F(x), \|f\|_{L^\infty(B_1 \times \mathbb{R})} \right\} \leq \delta_\varepsilon, \text{ then}
\]

\[
\sup_{B_\rho(x_0)} \left| u(x) - t_{x_0}(u)(x) \right| \leq \rho^{1+\alpha}.
\]
Chapter 2: A first approximating estimate

(Philosophical) Idea of proof:

\[
\|u - l_{x_0}(u)\|_{L^\infty(B_\rho(x_0))} \leq \|\phi - l_{x_0}(\phi)\|_{L^\infty(B_\rho(x_0))} + |(u - \phi)(x_0)| \\
+ \|u - \phi\|_{L^\infty(B_\rho(x_0))} + |D(u - \phi)(x_0)| \\
\leq C \sup_{B_\rho(x_0)} |x - x_0|^{1+\alpha_{\text{Hom}}} + 3\varepsilon \\
\leq C\rho^{1+\alpha_{\text{Hom}}} + 3\varepsilon \quad \text{(What is the expected estimate?)}
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Chapter 2: A first approximating estimate

(Philosophical) Idea of proof:

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\|u - I_{x_0}(u)\|_{L^\infty(B_\rho(x_0))} \leq \|\phi - I_{x_0}(\phi)\|_{L^\infty(B_\rho(x_0))} + |(u - \phi)(x_0)| \\
+ \|u - \phi\|_{L^\infty(B_\rho(x_0))} + |D(u - \phi)(x_0)| \\
\leq C \sup_{B_\rho(x_0)} |x - x_0|^{1 + \alpha_{\text{Hom}}} + 3\varepsilon \\
\leq C\rho^{1 + \alpha_{\text{Hom}}} + 3\varepsilon \quad (\text{What is the expected estimate?})
\]

Coming back to last estimate we can conclude:

\[
\|u - I_{x_0}(u)\|_{L^\infty(B_\rho(x_0))} \leq \rho^{1 + \alpha}
\]

provided \( \rho \in \left(0, \min \left\{ \frac{1}{2}, \left(\frac{1}{2C}\right)^{\frac{1}{\alpha_{\text{Hom}}}} \right\} \right) \) and \( \varepsilon \in \left(0, \frac{1}{6}\rho^{1 + \alpha}\right) \).
Proceeding with the iteration process

Different from $C^{1,\alpha}$ regularity estimates from linear setting, we can no longer proceed with an iterative scheme, i.e.

$$\sup_{B_{\rho^k}(x_0)} \frac{|u(x) - l_k(x)|}{\rho^{k(1+\alpha)}} \leq 1$$

Dini-Campanato embedding $\implies u$ is $C^{1,\alpha}$ at $x_0$,

because a priori we do not know the equation which is satisfied by

$$B_{1}(0) \ni x \mapsto \frac{(u - l_k)(\rho^k x)}{\rho^{k(1+\alpha)}}, \text{ for } \{l_k\}_{k \in \mathbb{N}} \text{ affine functions, since } \mathcal{H}(x,Dv)F(x,D^2v) \text{ is not invariant by affine maps.}$$
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Chapter 3: The gap in the standard induction process

Proceeding with the iteration process

Different from $C^{1,\alpha}$ regularity estimates from linear setting, we can no longer proceed with an iterative scheme, i.e.

$$\sup_{B_{\rho^k}(x_0)} \frac{|u(x) - I_k(x)|}{\rho^{k(1+\alpha)}} \leq 1$$

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$$B_1(0) \ni x \mapsto \frac{(u - I_k)(\rho^k x)}{\rho^{k(1+\alpha)}}$$

for $\{I_k\}_{k \in \mathbb{N}}$ affine functions, since $\mathcal{H}(x, Dv)F(x, D^2v)$ is not invariant by affine maps.

For this reason, an alternative approach must be undertaken: quantitative information on the oscillation of $u$

$$\sup_{B_{\rho}(x_0)} \frac{\rho^{-1} |u(x) - u(x_0)|}{\rho^\alpha + |Du(x_0)|} \leq 1$$

Iteration $\implies$

$$\sup_{B_{\rho^k}(x_0)} \frac{\rho^{-k} |u(x) - u(x_0)|}{\rho^{k\alpha} + |Du(x_0)|(1-\rho^{(k-1)\beta})} \leq 1,$$

which proves to be the proper estimate for continuing with an iterative process, provided we get a sort of suitable control under the magnitude of the gradient (point-wisely).
Chapter 4: Iteration scheme: A new oscillation mechanism

**Corollary (The (real) First step of induction)**

Suppose that the assumptions of previous Lemma are in force. Then,

\[
\sup_{B_{\rho}(x_0)} |u(x) - u(x_0)| \leq \rho^{1+\alpha} + \rho |Du(x_0)|.
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Chapter 4: Iteration scheme: A new oscillation mechanism

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\sup_{B_{\rho} (x_0)} |u(x) - u(x_0)| \leq \rho^{1+\alpha} + \rho |Du(x_0)|.
\]

In order to obtain a precise control on the influence of magnitude of the gradient of \(u\), we iterate solutions (using the previous Corollary) in corrected \(\rho\)-adic cylinders.

**Lemma (Iterative process)**

Under the assumptions of previous Corollary one has

\[
\sup_{B_{\rho^k} (x_0)} |u(x) - u(x_0)| \leq \rho^{k(1+\alpha)} + |Du(x_0)| \sum_{j=0}^{k-1} \rho^{k+j\alpha}.
\]

**Proof.**

By induction process – Here we make use of assumption \(\alpha \leq \frac{1}{p+1}\). □
Our next result provides the geometric regularity estimate inside critical zone. We define the critical zone as follows:

\[ C^\alpha_\rho(B_1) := \{ x \in B_1; |Du(x)| \leq \rho^\alpha \}. \]
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**Lemma (Estimate inside critical zone)**

Suppose that the assumptions of previous Lemma are in force. Then, there exists \( M(\text{universal}) > 1 \) such that

\[ \sup_{B_{\rho_0}(x_0)} |u(x) - u(x_0)| \leq M\rho_0^{1+\alpha} \left( 1 + |Du(x_0)|\rho_0^{-\alpha} \right), \quad \forall \rho_0 \in (0, \rho). \]
Proof of the main Theorem.

WLOG, we may assume that $K = B_{\frac{1}{2}}$ and $x_0 = 0 \in C^\alpha_{\rho_0}(K)$. Using previous Lemma, we estimate

$$\sup_{B_{\rho_0}} \frac{|u(x) - I_0 u(x)|}{\rho_0^{1+\alpha}} \leq \sup_{B_{\rho_0}} \frac{|u(x) - u(0)|}{\rho_0^{1+\alpha}} + \frac{|Du(0)|\rho_0}{\rho_0^{1+\alpha}}$$

$$\leq M \left( 1 + |Du(0)|\rho_0^{-\alpha} \right) + 1$$

$$\leq 3M.$$
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\leq M \left( 1 + |Du(0)| \rho_0^{-\alpha} \right) + 1 \leq 3M.
$$

On the other hand, if the gradient has a uniform lower bound, i.e. $|Du| \geq L_0 > 0$, then Caffarelli-Trudinger's classical estimates (or Teixeira's work) can be enforced since the operator becomes uniformly elliptic:

$$
P^-_{\lambda, \Lambda}(D^2 u) \leq C_0 \left( L_0^{-1}, p, q, \|a\|_{L^{\infty}(B_1)}, \|f\|_{L^{\infty}(B_1)} \right) \quad \text{and} \quad P^+_{\lambda, \Lambda}(D^2 u) \geq -C_0 \left( L_0^{-1}, p, q, \|a\|_{L^{\infty}(B_1)}, \|f\|_{L^{\infty}(B_1)} \right).
$$
## Closing remarks:

Coming back to the **open issues** (yet):

<table>
<thead>
<tr>
<th>( f \in L^r(B_1) \cap C^0(B_1) ) and ( r = n )</th>
<th>( C^{0,\alpha}_{\text{loc}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (SC) ) and ( n &lt; r &lt; \infty )</td>
<td>Open Problem</td>
</tr>
<tr>
<td>( (SC) )</td>
<td>( 1,\min \left{ \frac{1}{p+1}, \alpha_{\text{Hom}} \right} )</td>
</tr>
<tr>
<td>( (SC) + F ) a concave/convex (or Asympt. convex) operator</td>
<td>( C^{1,\frac{1}{p+1}}_{\text{loc}} )</td>
</tr>
<tr>
<td>( (SC) ) + ( f(x) = f_0(x)u^+_{\mu}(x) )</td>
<td>Better estimates?</td>
</tr>
</tbody>
</table>

Hölder estimates are a consequence of Harnack inequality proved in the following references:

---

\[ a \]


- **J.V. da Silva, G.C. Rampasso, G.C. Ricarte & H. Vivas**, *Free boundary regularity for a class of one-phase problems with non-homogeneous degeneracy*, **Submitted Article**.
Let us consider the dead-core problem for fully nonlinear models with non-homogeneous degeneracy, whose source term presents an absorption term:

\[ u \geq 0 \quad \text{and} \quad \mathcal{H}(x, Du) F(x, D^2 u) = f_0(x) \cdot u^\mu \chi_{\{u > 0\}} \quad \text{in} \quad \Omega, \tag{3.1} \]

where \( 0 < \mu < p + 1 \) is the order of reaction and \( f_0 \) is the Thiele modulus, which is bounded away from zero and infinity.
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(3.1)

where $0 < \mu < p + 1$ is the order of reaction and $f_0$ is the Thiele modulus, which is bounded away from zero and infinity.

What are the expect regularity estimates for dead core models like (3.1)?

We shall establish an improved regularity estimate for non-negative solutions of (3.1) along their touching ground boundary $\partial\{u > 0\} \cap \Omega'$:
Final Chapter: An improved regularity estimate

**Theorem (Improved regularity along free boundary)**

Let $u$ be a nonnegative and bounded viscosity solution to (3.1) and consider $z_0 \in \partial\{u > 0\} \cap \Omega'$ a free boundary point with $\Omega' \Subset \Omega$. Then for $r \ll \min \left\{1, \frac{\text{dist}(\Omega', \partial \Omega)}{2}\right\}$ there holds

$$\sup_{B_r(z_0)} u(x) \leq C \cdot r^{\frac{p+2}{p+1-\mu}},$$

where $C > 0$ depends only on $n, \lambda, \Lambda, p, q, \mu, \|f_0\|_{L^\infty(\Omega)}$ and $\text{dist}(\Omega', \partial \Omega)$.

Notice that $\frac{p+2}{p+1-\mu} > 1 + \frac{1}{p+1}$. Moreover, such regularity estimates are natural extension for ones in $^a$

---


Next result regards the first step of a sharp geometric decay, which is a powerful device in nonlinear (geometric) regularity theory and plays a pivotal role in our approach.

**Lemma (Flatness improvement regime)**

Suppose that the assumptions (SC) are in force. Given $0 < \eta < 1$, there exists a $\delta = \delta(n, \lambda, \Lambda, \eta) > 0$ such that if $\phi$ satisfies $0 \leq \phi \leq 1$, $\phi(0) = 0$ and

$$\mathcal{H}(x, D\phi).F(x, D^2\phi) = f_0(x) \cdot (\phi^+)\mu,$$

in the viscosity sense in $B_1(0)$, with $\|f_0\|_{L^\infty(B_1(0))} \leq \delta$. Then,

$$\sup_{B_{1/2}(0)} \phi \leq 1 - \eta.$$
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$$
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$$

**Proof.**

Consequence of Harnack inequality:

$$
\sup_{B_{1/2}(0)} \phi \leq C \cdot \left( \inf_{B_{1/2}(0)} \phi + (q + 1)^{-\frac{1}{q+1}} \max \left\{ \|f_0\phi^\mu\|_{L^\infty(B_1)}, \|f_0\phi^\mu\|_{L^\frac{1}{q+1}(B_1)} \right\} \right).
$$

Finally, by applying Lemma 9 recursively in dyadic balls $B_{\frac{1}{2^n}}(0)$ with $\eta := 1 - \left(\frac{1}{2}\right)^{\frac{p+2}{p+1-\mu}}$, we are able to establish improved regularity estimates along touching ground points.
Introducing the problem
Our main result and its natural obstacles
Further improved results

Final Chapter: An improved regularity estimate

We can show that the maximum of a solution in a ball of radius $r \ll 1$ grows precisely as $r^{\frac{p+2}{p+1-\mu}}$.

Theorem (Non-degeneracy)

Let $u$ be a nonnegative, bounded viscosity solution to (3.1) in $B_1(0)$ with $f(x) \geq m > 0$ and let $x_0 \in \{u > 0\} \cap B_{\frac{1}{2}}(0)$ be a point in the closure of the non-coincidence set. Then for any $0 < r < \frac{1}{2}$, there holds

$$\sup_{\partial B_r(x_0)} u(x) \geq C \cdot r^{\frac{p+2}{p+1-\mu}},$$

where $C = C(m, \|a\|_{L^\infty(\Omega)}, L_1, n, \lambda, \Lambda, p, q, \mu, \Omega) > 0$. 
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**Proof.**

The proof is a consequence of Birindelli-Demengel’s Comparison Principle applied to following profiles:

$$\Xi(x) := C \cdot |x|^{\frac{p+2}{p+1-\mu}} \quad \text{and} \quad u_r(x) := \frac{u(x_0 + rx)}{r^{\frac{p+2}{p+1-\mu}}} \quad \text{for} \quad x \in B_1(0)$$
Thank you very much for your attention! We are waiting your visit as soon as possible at UNICAMP’s Math. Department! Please, follow me on ResearchGate.

https://www.researchgate.net/profile/Joao-Da-Silva-13