Maximal regularity for local minimizers of non-autonomous functionals

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This is a joint work with Peter Hästö.

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Non-autonomous functionals

\[ W^{1,1}(\Omega) \ni v \mapsto \mathcal{F}(v, \Omega) = \int_{\Omega} f(x, Dv) \, dx. \]  \hspace{1cm} (1)

\( u \) is always a minimizer (or a weak solution to a PDE problem).
Non-autonomous functionals

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- If the integrand \( f \) is independent of \( x \), i.e., \( \mathcal{F}(v, \Omega) = \int_{\Omega} f(Dv) \, dx \), the functional \( \mathcal{F} \) is said to be autonomous.

- If the integrand \( f \) depends also on \( x \), the functional \( \mathcal{F} \) is said to be non-autonomous.
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- If the integrand \( f \) depends also on \( x \), the functional \( \mathcal{F} \) is said to be non-autonomous.
- In particular, if \( z \mapsto f(x, z) \) depends only on \( |z| \), i.e.,

\[ \mathcal{F}(v, \Omega) = \int_{\Omega} \varphi(x, |Dv|) \, dx \quad (f(x, z) \equiv \varphi(x, |z|)), \]

then we say that \( \mathcal{F} \) has so-called Uhlenbeck’s structure.
Functional with $p$-growth

\[
\begin{aligned}
&z \mapsto f(x, z) \text{ is } C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}), \\
&1 < p < \infty, \\
&\nu|z|^p \leq f(x, z) \leq L|z|^p, \\
&\nu|z|^{p-2}|\lambda|^2 \leq \partial^2 f(x, z)\lambda \cdot \lambda \leq L|z|^{p-2}|\lambda|^2.
\end{aligned}
\]

\[
W^{1,p}(\Omega) \ni v \mapsto \int_{\Omega} f(x, Dv) \, dx.
\]

E-L eq: \( \text{div} \left( \partial f(x, Du) \right) = 0 \).

- Model case

\[
f(x, z) \equiv \varphi(x, |z|) = a(x)|z|^p, \quad 0 < \nu \leq a(\cdot) \leq L.
\]

E-L eq: \( \text{div} \left( a(x)|Du|^{p-2}Du \right) = 0 \).
Functional with $p$-growth

\[ \omega(r) := \sup_{z \in \mathbb{R}^n \setminus \{0\}} \sup_{x,y \in B_r, B_r \subset \Omega} \frac{|f(x, z) - f(y, z)|}{|z|^p} \leq 2L. \]

\[ \left( \text{model case: } \omega(r) := \sup_{x,y \in B_r, B_r \subset \Omega} |a(x) - a(y)| \right) \]
Functional with $p$-growth

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\omega(r) := \sup_{z \in \mathbb{R}^n \setminus \{0\}} \sup_{x,y \in B_r, B_r \subset \Omega} \frac{|f(x, z) - f(y, z)|}{|z|^p} \leq 2L.
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(model case: $\omega(r) := \sup_{x,y \in B_r, B_r \subset \Omega} |a(x) - a(y)|$)

**DeGiorgi theory**

- (No additional condition) $\implies u \in C^\alpha$ for some $\alpha \in (0, 1)$.

**Continuous for $x$**

- $\lim_{r \to 0^+} \omega(r) = 0 \implies u \in C^\alpha$ for any $\alpha \in (0, 1)$.
- $\omega(r) \lesssim r^\beta \implies u \in C^{1,\alpha}$ for some $\alpha \in (0, 1)$. (Maximal regularity)

- (model case) It is well known that if $a(\cdot)$ is VMO then $u \in W^{1,q}$ for any $q > p$ hence $u \in C^\alpha$ for any $\alpha \in (0, 1)$.

Moreover, if $a(\cdot)$ is VMO only for $x_1, \ldots, x_{n-1}$ ($x = (x_1, \ldots, x_n)$), then $u \in W^{1,q}$ for any $q > p$. (Byun-Kim(16), Kim(18))
Non-autonomous functionals with \((p, q)\)-growth

Marcellini introduced a general class of functionals.

\[(p, q)\)-growth condition (Marcellini, 1989)

\[
\begin{align*}
    z \mapsto f(x, z) & \text{ is } C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}), \\
    1 < p \leq q, \\
    \nu |z|^p \leq f(x, z) \leq L(1 + |z|^q), \\
    \nu |z|^{p-2}|\lambda|^2 \leq \partial^2 f(x, z) \lambda \cdot \lambda \leq L(|z|^{p-2} + |z|^{q-2})|\lambda|^2.
\end{align*}
\]
In this talk, I will introduce regularity results for the following problems.
Non-autonomous functionals with $(p, q)$-growth

In this talk, I will introduce regularity results for the following problems.

- $(p, q)$-growth functionals with Uhlenbeck’s structure

$$f(x, z) \equiv \varphi(x, |z|)$$

$$W^{1,\varphi}(\Omega) \ni v \mapsto \int_{\Omega} \varphi(x, |Dv|) \, dx.$$ (2)

In this case, the previous $(p, q)$-growth condition can be simplified as

\[
\begin{cases}
t \mapsto \varphi(x, t) \text{ is in } C^1([0, \infty)) \cap C^2((0, \infty)), \\
1 < p \leq q, \\
p - 1 \leq \frac{t\varphi''(x, t)}{\varphi'(x, t)} \leq q - 1 \quad (\iff t\varphi''(x, t) \approx \varphi'(x, t)).
\end{cases}
\] (3)

Bounded or Hölder continuous minimizers
Non-autonomous functionals with \((p, q)\)-growth

In this talk, I will introduce regularity results for the following problems.

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\end{cases}
\] (3)

- **Bounded or Hölder continuous minimizers**
Non-autonomous functionals with \((p, q)\)-growth

- functionals and PDEs with generalized Orlicz growth

\[
W^{1, \varphi}(\Omega) \ni v \mapsto \int_{\Omega} f(x, Dv) \, dx.
\]

\[
\begin{align*}
z &\mapsto f(x, z) \in \mathbb{R} \text{ is } C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}), \\
\nu \varphi(x, |z|) &\leq f(x, z) \leq L \varphi(x, |z|), \\
\nu \varphi'(x, |z|) |\lambda|^2 &\leq \partial^2 f(x, z) \lambda \cdot \lambda \leq L \varphi'(x, |z|) |\lambda|^2.
\end{align*}
\]

\[
\text{div } A(x, Du) = 0.
\]

\[
\begin{align*}
z &\mapsto A(x, z) \in \mathbb{R}^n \text{ is } C^1(\mathbb{R}^n \setminus \{0\}), \\
|A(x, |z|)| &+ |z| |\partial A(x, z)| \leq L \varphi'(x, |z|), \\
\nu \varphi'(x, |z|) |\lambda|^2 &\leq \partial A(x, z) \lambda \cdot \lambda.
\end{align*}
\]
We say $f : U \subset \mathbb{R} \to \mathbb{R}$ is almost increasing (or almost decreasing) if $f(t) \leq Lf(s)$ (or $f(s) \leq Lf(t)$) for any $t < s$ for some $L \geq 1$.

Let $\varphi = \varphi(x, t) : \Omega \times [0, \infty) \to [0, \infty)$ and $\gamma > 0$.

\((\text{aInc})_\gamma : t \mapsto \frac{\varphi(x, t)}{t^\gamma}\) is almost increasing uniformly $x$ with $L \geq 1$.

\((\text{aDec})_\gamma : t \mapsto \frac{\varphi(x, t)}{t^\gamma}\) is almost decreasing uniformly $x$ with $L \geq 1$.

When $L = 1$, \((\text{Inc})_\gamma = (\text{aInc})_\gamma\) and \((\text{Dec})_\gamma = (\text{aDec})_\gamma\).

\[ p - 1 \leq \frac{t \varphi''(x, t)}{\varphi'(x, t)} \leq q - 1 \quad \overset{\varphi \in C^2}{\Leftrightarrow} \quad \varphi' \text{ satisfies } (\text{Inc})_{p-1} \text{ and } (\text{Dec})_{q-1}. \]

\[ \Rightarrow \quad \varphi \text{ satisfies } (\text{Inc})_{p} \text{ and } (\text{Dec})_{q}. \]
Generalized Orlicz function

- We say \( f : U \subset \mathbb{R} \to \mathbb{R} \) is almost increasing (or almost decreasing) if \( f(t) \leq L f(s) \) (or \( f(s) \leq L f(t) \)) for any \( t < s \) for some \( L \geq 1 \).

Let \( \varphi = \varphi(x, t) : \Omega \times [0, \infty) \to [0, \infty) \) and \( \gamma > 0 \).

\((\text{alnc})_\gamma : t \mapsto \frac{\varphi(x, t)}{t^\gamma}\) is almost increasing uniformly \( x \) with \( L \geq 1 \).

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\[ p - 1 \leq \frac{t \varphi''(x, t)}{\varphi'(x, t)} \leq q - 1 \quad \Leftrightarrow \quad \varphi' \text{ satisfies } (\text{Inc})_{p-1} \text{ and } (\text{Dec})_{q-1}. \]

\[ \implies \varphi \text{ satisfies } (\text{Inc})_p \text{ and } (\text{Dec})_q. \]

\((\text{A0}) \ \varphi(\cdot, 1) \approx 1 \) (i.e., \( \exists L \geq 1 \) s.t. \( L^{-1} \leq \varphi(x, 1) \leq L \ \forall x \in \Omega \)).

From now on, \( \varphi \in \Phi_w(\Omega) \) and satisfies \((\text{A0}), (\text{alnc})_p \) and \((\text{aDec})_q\).
Perturbed Orlicz (general growth)

\[ \varphi(x, t) = a(x)\varphi_0(t), \]

where

\[
\begin{cases}
0 < \nu \leq a(\cdot) \leq L & (\iff \varphi \text{ is (A0) and } \varphi \approx \varphi_0), \\
t\varphi''_0(t) \approx \varphi'_0(t) & (\implies \varphi \text{ is (Inc)}_p \text{ and (Dec)}_q, 1 < p \leq q).
\end{cases}
\]

\[ W^{1,\varphi_0}(\Omega) \ni v \mapsto \int_{\Omega} a(x)\varphi_0(|Dv|) \, dx. \]
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\[ \begin{cases} 0 < \nu \leq a(\cdot) \leq L & (\iff \varphi \text{ is (A0) and } \varphi \approx \varphi_0), \\ t\varphi''_0(t) \approx \varphi'_0(t) & (\implies \varphi \text{ is (Inc)}_p \text{ and (Dec)}_q, 1 < p \leq q). \end{cases} \]

\[ W^{1,\varphi_0}(\Omega) \ni v \mapsto \int_{\Omega} a(x)\varphi_0(|Dv|) \, dx. \]

Define

\[ \omega(r) := \sup_{x, y \in B_r, B_r \subset \Omega} |a(x) - a(y)| \leq 2L. \]

### Regularity (Lieberman(91), \ldots)

- (No additional condition) \implies u \in C^\alpha \text{ for some } \alpha \in (0, 1).
- \[ \lim_{r \to 0^+} \omega(r) = 0 \ (a(\cdot) \in C^0) \implies u \in C^\alpha \text{ for any } \alpha \in (0, 1). \]
- \[ \omega(r) \lesssim r^\beta \ (a(\cdot) \in C^\beta) \implies u \in C^{1,\alpha} \text{ for some } \alpha \in (0, 1). \]
Non-standard growth

Standard growth cases:

\[ \varphi(x, t) = a(x)t^p \quad \text{(or} \quad a(x)\varphi_0(t)) \].

The power, or growth, of \( t \) is independent of \( x \).
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Zhikov’s examples

(Anna Balci’s talk on next week!)*

- **Variable exponent**

\[ \varphi(x, t) = t^{p(x)}, \]

\[ 1 < p \leq \inf p(\cdot) \leq \sup p(\cdot) \leq q \quad \text{and} \quad p(\cdot) \in C^0. \]

- **Double phase**

\[ \varphi(x, t) = t^p + b(x) t^q, \]

\[ 0 \leq b(\cdot) \leq L \quad \text{and} \quad b(\cdot) \in C^{0,\beta}, \quad \beta \in (0, 1]. \]

In last two decades, there have been a lot of researches on regularity theory for these problems.
Harjulehto and Hästö found the following crucial conditions on $\varphi$:

1. There exists $L \geq 1$ such that for any $B_r \subset \Omega$ with $|B_r| < 1$, $\varphi(U(t)) \leq L \varphi - B_r(U(t))$ for any $t > 0$ with $\varphi - B_r(U(t)) \in [1, |B_r| - 1]$. (A1)

(A1-) There exists $L \geq 1$ such that for any $B_r \subset \Omega$ with $|B_r| < 1$, $\varphi(U(t)) \leq L \varphi - B_r(U(t))$ for any $t > 0$ with $t \in [1, |B_r| - 1]$. (A1-)

Here, $\varphi(U(t)) := \sup_{x \in U} \varphi(x, t)$ and $\varphi - U(t) := \inf_{x \in U} \varphi(x, t)$. (A1) means $\varphi(U(t))$ and $\varphi - B_r(U(t))$ are comparable uniformly for $t > 0$ with $\varphi - B_r(U(t)) \in [1, |B_r| - 1]$. 

$$W^{1, \varphi}(\Omega) \ni v \mapsto \int_{\Omega} \varphi(x, |Dv|) \, dx.$$
General non-autonomous problems

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Harjulehto and Hästö found the following crucial conditions on \( \varphi \):

**\( \text{(A1)} \)** There exists \( L \geq 1 \) such that for any \( B_r \subseteq \Omega \) with \( |B_r| < 1 \),

\[ \varphi_{B_r}^+(t) \leq L \varphi_{B_r}^-(t) \quad \text{for any } t > 0 \quad \text{with } \varphi_{B_r}^-(t) \in [1,|B_r|^{-1}]. \]

**\( \text{(A1-s)} \)** There exists \( L \geq 1 \) such that for any \( B_r \subseteq \Omega \) with \( |B_r| < 1 \),

\[ \varphi_{B_r}^+(t) \leq L \varphi_{B_r}^-(t) \quad \text{for any } t > 0 \quad \text{with } t^s \in [1,|B_r|^{-1}]. \]

Here, \( \varphi_U^+(t) := \sup_{x \in U} \varphi(x,t) \) and \( \varphi_U^-(t) := \inf_{x \in U} \varphi(x,t) \).

\( \bullet \) **\( \text{(A1)} \)** means \( \varphi_{B_r}^+(t) \) and \( \varphi_{B_r}^-(t) \) are comparable uniformly for \( t > 0 \) with \( \varphi_{B_r}^-(t) \in [1,|B_r|^{-1}] \approx t \in [1,(\varphi_{B_r}^-)^{-1}(|B_r|^{-1})]. \)
Hölder continuity for non-autonomous problems


If \( \varphi \) satisfies (A1), then \( u \in C^\alpha_{\text{loc}}(\Omega) \) for some \( \alpha = \alpha(n, p, q, L) \in (0, 1) \).

If \( \varphi \) satisfies (A1-n) and \( u \in L^\infty(\Omega) \), then \( u \in C^\alpha_{\text{loc}}(\Omega) \) for some \( \alpha = \alpha(n, p, q, L) \in (0, 1) \).

- The paper considers more general functionals and quasi-minimizers and proves Harnack’s inequality.
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What about \( C^\alpha \)-regularity for any \( \alpha \in (0, 1) \) and \( C^{1,\alpha} \)-regularity for some \( \alpha \in (0, 1) \)?

Note: In particular cases, the proofs of these regularities use perturbation arguments that depend on their particular structures.
(VA1): Vanishing (A1)

There exists a non-decreasing, bounded, continuous function $\omega : [0, \infty) \rightarrow [0, 1]$ with $\omega(0) = 0$ such that for any small $B_r \subset \Omega$,

$$\varphi^+_B(t) \leq (1 + \omega(r))\varphi^-_B(t) \quad \forall t > 0 \text{ with } \varphi^-_B(t) \in [\omega(r), |B_r|^{-1}].$$

(A1): $\varphi^+_B(t) \leq L\varphi^-_B(t) \quad \forall t > 0 \text{ with } \varphi^-_B(t) \in [1, |B_r|^{-1}].$

- (VA1) implies (A1).
- (VA1) implies that $x \mapsto \varphi(x, t)$ is continuous for all $t \in (0, \infty)$. 
Theorem (Hästö-Ok, to appear in JEMS)

Let \( \varphi(x, \cdot) \in C^1([0, \infty)) \) for every \( x \in \Omega \) with \( \varphi' \) satisfying \((A0), (Inc)_{p-1}\) and \((Dec)_{q-1}\) for some \( 1 < p \leq q \).

1. If \( \varphi \) satisfies \((VA1)\), then \( u \in C^{\alpha}_{loc}(\Omega) \) for any \( \alpha \in (0, 1) \).
2. If \( \varphi \) satisfies \((VA1)\) and \( \omega(r) \lesssim r^\beta \) for some \( \beta > 0 \), then \( u \in C^{1,\alpha}_{loc}(\Omega) \) for some \( \alpha \in (0, 1) \).
Theorem (Hästö-Ok, to appear in JEMS)

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(1) If \( \varphi \) satisfies \((VA1)\), then \( u \in C^{\alpha}_{\text{loc}}(\Omega) \) for any \( \alpha \in (0, 1) \).

(2) If \( \varphi \) satisfies \((VA1)\) and \( \omega(r) \lesssim r^{\beta} \) for some \( \beta > 0 \), then \( u \in C^{1,\alpha}_{\text{loc}}(\Omega) \) for some \( \alpha \in (0, 1) \).

- \( \varphi \) is assumed to be \( C^1 \) for \( t \) (In former regularity results in the case \( \varphi(x, t) = \varphi(t) \), \( \varphi \) is always assumed to be \( C^2((0, \infty)) \)).

In fact, the assumption implies \( W^{2,\infty}_{\text{loc}}((0, \infty)) \).

- Recall that

\[
p - 1 \leq \frac{t \varphi''(t)}{\varphi'(t)} \leq q - 1 \quad \varphi \in C^2 \quad \varphi'(t) : \ (Inc)_{p-1} \text{ and } (Dec)_{q-1}.
\]

- For instance \( \varphi(t) := \int_0^t \max\{s^{p-1}, s^{q-1}\} \, ds \) satisfies the assumptions in the above theorem, but is not \( C^2 \).
Let $B_r = B_r(x_0)$. Since $\sup_{x,y \in B_r} |\varphi(x,t) - \varphi(y,t)| = \varphi_{B_r}^+(t) - \varphi_{B_r}^-(t)$, (A1)

$$\sup_{x,y \in B_r} \frac{|\varphi(x,t) - \varphi(y,t)|}{\varphi(x_0,t)} \leq \sup_{x,y \in B_r} \frac{|\varphi(x,t) - \varphi(y,t)|}{\varphi_{B_r}^-(t)} \leq L - 1,$$

$\forall t > 0$ with $\varphi_{B_r}^-(t) \in [1, |B_r|^{-1}]$. (VA1)

$$\sup_{x,y \in B_r} \frac{|\varphi(x,t) - \varphi(y,t)|}{\varphi(x_0,t)} \leq \omega(r), \forall t > 0 \text{ with } \varphi_{B_r}^-(t) \in [\omega(r), |B_r|^{-1}]$$. 

(A1) and (VA1)
Let $B_r = B_r(x_0)$. Since

$$\sup_{x,y\in B_r} |\varphi(x,t) - \varphi(y,t)| = \varphi^+_{B_r}(t) - \varphi^-_{B_r}(t),$$

(A1)

$$\sup_{x,y\in B_r} \frac{|\varphi(x,t) - \varphi(y,t)|}{\varphi(x_0,t)} \leq \sup_{x,y\in B_r} \frac{|\varphi(x,t) - \varphi(y,t)|}{\varphi^-_{B_r}(t)} \leq L - 1,$$

\forall t > 0 \text{ with } \varphi^-_{B_r}(t) \in \left[1, |B_r|^{-1}\right].$

(VA1)

$$\sup_{x,y\in B_r} \frac{|\varphi(x,t) - \varphi(y,t)|}{\varphi(x_0,t)} \leq \omega(r), \quad \forall t > 0 \text{ with } \varphi^-_{B_r}(t) \in \left[\omega(r), |B_r|^{-1}\right].$$

- If the above inequalities hold for all $t \in (0, \infty)$, they imply

$$\varphi(x,t) \approx a(x)\varphi(x_0,t) \quad \text{(perturbed Orlicz case)},$$

where $\omega_a(r) \lesssim L - 1$ and $\omega_a(r) \lesssim \omega(r)$, respectively.
(Recall) Functional with $p$-growth

\[
\omega(r) := \sup_{z \in \mathbb{R}^n \setminus \{0\}} \sup_{x,y \in B_r, B_r \subset \Omega} \frac{|f(x, z) - f(y, z)|}{|z|^p}.
\]

**DeGiorgi theory**

- (No additional condition) \( \implies u \in C^\alpha \) for some \( \alpha \in (0, 1) \).

**Continuous for \( x \)**

- \( \lim_{r \to 0^+} \omega(r) = 0 \implies u \in C^\alpha \) for any \( \alpha \in (0, 1) \).
- \( \omega(r) \lesssim r^\beta \implies u \in C^{1,\alpha} \) for some \( \alpha \in (0, 1) \). (Maximal regularity)

If \( f(x, z) \Rightarrow \varphi(x, |z|) \) and \( |z|^p \Rightarrow \varphi(x_0, |z|) \),

\[
\omega(r) = \sup_{t \in (0, \infty)} \sup_{x,y \in B_r, B_r \subset \Omega} \frac{|\varphi(x, t) - \varphi(y, t)|}{\varphi(x_0, t)}.
\]
Non-autonomous functional with Uhlenbeck’s structure

The previous theorem covers most known regularity results with continuity assumption for $x$ in special cases.
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Double phase problem

$$\varphi(x, t) = t^p + b(x)t^q, \quad b \in C^{0, \beta} \quad \text{and} \quad 0 \leq b(\cdot) \leq L.$$ 

- If $\frac{q}{p} < 1 + \frac{\beta}{n}$, $\varphi$ satisfies (VA1) with $\omega(r) \lesssim r^\gamma$, $\gamma = \beta - \frac{n(q-p)}{p} > 0$, hence $u \in C^{1,\alpha}_{\text{loc}}$. (Colombo-Mingione (2015))
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- **Double phase problem**

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- If $\frac{q}{p} < 1 + \frac{\beta}{n}$, $\varphi$ satisfies (VA1) with $\omega(r) \lesssim r^\gamma$, $\gamma = \beta - \frac{n(q-p)}{p} > 0$, hence $u \in C^{1,\alpha}_{loc}$. (Colombo-Mingione (2015))

- If $\frac{q}{p} \geq 1 + \frac{\beta}{n}$, $\varphi$ does not satisfy (VA1).
The previous theorem covers most known regularity results with continuity assumption for $x$ in special cases.

- Double phase problem

$$\varphi(x, t) = t^p + b(x)t^q, \quad b \in C^{0,\beta}, \quad \text{and} \quad 0 \leq b(\cdot) \leq L.$$  

- If $\frac{q}{p} < 1 + \frac{\beta}{n}$, $\varphi$ satisfies (VA1) with $\omega(r) \lesssim r^\gamma$, $\gamma = \beta - \frac{n(q-p)}{p} > 0$, hence $u \in C^{1,\alpha}_{\text{loc}}$. (Colombo-Mingione (2015))

- If $\frac{q}{p} \geq 1 + \frac{\beta}{n}$, $\varphi$ does not satisfy (VA1).

- However, if $\frac{q}{p} = 1 + \frac{\beta}{n}$, $u \in C^{1,\alpha}_{\text{loc}}$. (Baroni-Colombo-Mingione (2018))
$u$ is a minimizer of

$$ u \mapsto \int_{\Omega} \varphi(x, |Dv|) \, dx $$

if and only if it is a weak solution to

$$ \text{div} \left( \frac{\varphi'(x, |Du|)}{|Du|} Du \right) = 0. $$

(4)

If we regard $u$ as a weak solution to (4) and do not return to a variational problem, (as far as we have checked) the approach used in the proof implies the same regularity results except for replacing the inequality in (VA1) by

$$(\varphi')^+_B(t) \leq (1+\omega(r))(\varphi')^-_B(t) \quad \forall t > 0 \quad \text{with} \quad \varphi^-_B(t) \in [\omega(r), |B_r|^{-1}].$$

\textbf{Note} The above inequality is not comparable to the original one.
For any $\epsilon > 0$, there exists a non-decreasing, bounded, continuous function $\omega = \omega_\epsilon : [0, \infty) \to [0, 1]$ with $\omega(0) = 0$ such that for any small ball $B_r \subset \Omega$,

$$\varphi_{B_r}^+(t) \leq (1 + \omega(r))\varphi_{B_r}^-(t) + \omega(r) \quad \text{with} \quad \varphi_{B_r}^-(t) \in [\omega(r), |B_r|^{-1+\epsilon}].$$

- $(\text{VA1}) \implies (\text{wVA1}) \implies (\text{A1}).$
(wVA1): weak (VA1)

For any $\epsilon > 0$, there exists a non-decreasing, bounded, continuous function $\omega = \omega_\epsilon : [0, \infty) \to [0, 1]$ with $\omega(0) = 0$ such that for any small ball $B_r \subset \Omega$,

$$
\varphi^+_B(t) \leq (1 + \omega(r))\varphi^-_B(t) + \omega(r) \quad \text{with} \quad \varphi^-_B(t) \in [\omega(r), |B_r|^{-1+\epsilon}].
$$

- $(VA1) \implies (wVA1) \implies (A1)$.
- $\varphi(x, t) = t^p + b(x)t^q$ ($b \in C^{0,\beta}$, $0 \leq b(\cdot) \leq L$)
  
  If $\frac{q}{p} \leq 1 + \frac{\beta}{n}$, $\varphi$ satisfies $(wVA1)$ with $\omega_\epsilon(r) \lesssim r^{\gamma_\epsilon}$,

  $$
  \gamma_\epsilon = \beta - \frac{n(1-\epsilon)(q-p)}{p} > 0.
  $$

- The following inequality implies the inequality in $(wVA1)$ (with different $\omega$).

  $$
  (\varphi')^+_B(t) \leq (1 + \omega(r))(\varphi')^-_B(t) + \omega(r) \quad \text{with} \quad \varphi^-_B(t) \in [\omega(r), |B_r|^{-1+\epsilon}]
  $$
Theorem (Hästö-Ok, to appear in JEMS)

Let \( \varphi(x, \cdot) \in C^1([0, \infty)) \) for every \( x \in \Omega \) with \( \varphi' \) satisfying (A0), (Inc)\(_{p-1} \) and (Dec)\(_{q-1} \) for some \( 1 < p \leq q \).

1. If \( \varphi \) satisfies (wVA1), then \( u \in C^{\alpha}_{\text{loc}}(\Omega) \) for any \( \alpha \in (0, 1) \).
2. If \( \varphi \) satisfies (wVA1) with \( \omega_\epsilon(r) \leq r^{\beta_\epsilon} \) for some \( \beta_\epsilon > 0 \), then \( u \in C^{1, \alpha}_{\text{loc}}(\Omega) \) for some \( \alpha \in (0, 1) \).

(As far as we have checked) The above theorem covers all previous regularity results with continuity assumptions (w.r.t. \( x \)) for special cases: standard growth case, \( p(x) \)-growth case, double phase case, 

(wVA1) can be replaced by the combination of (A1) and (wVA1) with fixed small \( \epsilon \) that depends on the structure contants.
Examples (variable exponent type)

\[ \phi(x,t) = t^p(x) \]

\[ \lim_{r \to 0} \omega_p(r) \ln(1/r) = 0 \iff \phi \text{satisfies (VA1)} \]

\( \omega_p(r) \lesssim r^{\tilde{\beta}} \iff \phi \text{satisfies (VA1) with } \omega(r) \lesssim r^{\beta} \iff u \in C_1,\alpha \).

(Observe-Mingione(2001))

\[ \phi(x,t) = t^p(x) + t^q(x) \]

\[ \lim_{r \to 0} \omega_p(r) = 0, \lim_{r \to 0} \omega_q(r) \ln(1/r) = 0 \iff \phi \text{satisfies (VA1)} \]

\( \omega_p(r),\omega_q(r) \lesssim r^{\tilde{\beta}} \iff \phi \text{satisfies (VA1) with } \omega(r) \lesssim r^{\beta} \iff u \in C_1,\alpha . \)
Examples (variable exponent type)

\[ \varphi(x, t) = t^{p(x)} \]

- \[
\lim_{r \to 0} \omega_p(r) \ln(1/r) = 0 \iff \varphi \text{ satisfies } (VA1) \implies u \in C^\alpha \text{ } \forall \alpha.
\] (Acerbi-Mingione(2001))

- \[
\omega_p(r) \lesssim r^{\beta} \iff \varphi \text{ satisfies } (VA1) \text{ with } \omega(r) \lesssim r^{\beta} \implies u \in C^{1,\alpha}.
\] (Cosica-Mingione(1999))
Examples (variable exponent type)

\[ \varphi(x, t) = t^{p(x)} \]

\[ \lim_{r \to 0} \omega_p(r) \ln(1/r) = 0 \iff \varphi \text{ satisfies (VA1)} \implies u \in C^\alpha \quad \forall \alpha. \]

(Acerbi-Mingione(2001))

\[ \omega_p(r) \lesssim r^{\tilde{\beta}} \iff \varphi \text{ satisfies (VA1) with } \omega(r) \lesssim r^\beta \implies u \in C^{1,\alpha}. \]

(Cosica-Mingione(1999))

\[ \varphi(x, t) = t^{p(x)} + t^{q(x)}, \quad p(\cdot) \leq q(\cdot) \]

\[ \lim_{r \to 0} \omega_p(r) = 0, \quad \lim_{r \to 0} \omega_q(r) \ln(1/r) = 0 \implies \varphi \text{ satisfies (VA1)} \implies u \in C^\alpha \quad \forall \alpha. \]

\[ \omega_p(r), \omega_q(r) \lesssim r^{\tilde{\beta}} \implies \varphi \text{ satisfies (VA1) with } \omega(r) \lesssim r^\beta. \]

\[ \implies u \in C^{1,\alpha}. \]
Examples (double phase type)

\( \xi / \psi \) is almost increasing, \( a(\cdot), b(\cdot) \in C^0 \), \( a(\cdot), b(\cdot) \geq 0 \) and \( a(\cdot) + b(\cdot) \approx 1 \).

\[ \varphi(x, t) := a(x)\psi(t) + b(x)\xi(t), \]

Define \( \omega_\epsilon(r) := \omega_a(r) + \omega_b(r)r^{n(1-\epsilon)}\xi(\psi^{-1}(r^{-n(1-\epsilon)})), \epsilon \in [0, 1). \)

- \( \lim_{r \to 0} \omega_\epsilon(r) = 0 \implies \varphi \) satisfies (wVA1). \( \implies u \in C^\alpha \ \forall \alpha \in (0, 1). \)
- Moreover, \( \omega_\epsilon(r) \lesssim r^{\beta_\epsilon} \implies u \in C^{1,\alpha} \) for some \( \alpha \in (0, 1). \)

If \( \psi \) and \( \xi \) are general Orlicz functions,

- it is natural to assume that
  \[ b \in C^{\omega_b} \iff \sup_{x,y} \frac{|b(x) - b(y)|}{\omega_b(|x-y|)} < \infty \iff |b(x) - b(y)| \lesssim \omega_b(|x - y|). \]
- we can distinguish \( C^{\alpha}-\)regularity for any \( \alpha \in (0, 1) \) and \( C^{1,\alpha}-\)regularity.
Proof (main steps)

**Step 1. Higher integrability**

There exists $\sigma_0 = \sigma_0(n, p, q, L) > 0$ and $c_1 = c_1(n, p, q, L) \geq 1$ such that

$$
\left( \int_{B_r} \varphi(x, |Du|)^{1+\sigma_0} \, dx \right)^{\frac{1}{1+\sigma_0}} \leq c_1 \left( \int_{B_{2r}} \varphi(x, |Du|) \, dx + 1 \right)
$$

for any $B_{2r} \subset \Omega$ with $\int_{B_{2r}} \varphi(x, |Du|) \, dx \leq 1$. Hence

$\varphi(\cdot, |Du|) \in L^{1+\sigma_0}_{\text{loc}}(\Omega)$.

- We have the following reverse Hölder and Jensen type inequalities:

$$
\int_{B_r} \varphi(x, |Du|) \, dx \leq c_t \left[ \left( \int_{B_{2r}} \varphi(x, |Du|)^t \, dx \right)^{\frac{1}{t}} + 1 \right], \quad t \in (0, 1].
$$

$$
\int_{B_r} \varphi(x, |Du|) \, dx \leq c \varphi_{B_{2r}}^-( \int_{B_{2r}} |Du| \, dx + 1).
$$
Proof(main steps)

Step 2. Construction of a regular function
Let $B = B_{2r}(x_0)$, $t_1 := (\varphi_B^-)^{-1}(\omega(2r))$ and $t_2 := (\varphi_B^-)^{-1}(|B|^{-1})$.

We construct $\tilde{\varphi}$ s.t.

1. $\tilde{\varphi} \in C^1([0, \infty)) \cap C^2((0, \infty))$ and $t\tilde{\varphi}''(t) \approx \tilde{\varphi}'(t)$.
2. $0 \leq \tilde{\varphi}(t) - \varphi(x_0, t) \leq c(r\varphi_B^-(t) + \omega(2r)), \ \forall t \in [t_1, t_2]$.
3. $\theta_0(x, t) := \varphi(x, \tilde{\varphi}^{-1}(t))$ satisfies $(A0)$, $(\text{alnc})_1$, $(\text{aDec})_{q/p}$ and $(A1)$.

$$
\psi_B(t) := \begin{cases} 
a_1 \left(\frac{t}{t_1}\right)^{p-1} & \text{if } 0 \leq t < t_1, \\
\varphi'(x_0, t) & \text{if } t_1 \leq t \leq t_2, \\
a_2 \left(\frac{t}{t_2}\right)^{p-1} & \text{if } t_2 < t < \infty,
\end{cases}
$$

where $a_1 = \varphi'(x_0, t_1)$ and $a_2 = \varphi'(x_0, t_2)$, and

$$
\tilde{\varphi}(t) := \int_0^\infty \varphi_B(t\sigma)\eta_r(\sigma - 1) \, d\sigma = \int_1^{1+r} \varphi_B(t\sigma)\eta_r(\sigma - 1) \, d\sigma,
$$

$\eta \in C_0^\infty(\mathbb{R}^+)$ with $\text{supp}\eta \subset (0, 1)$ and $\|\eta\|_1 = 1$, and $\eta_r(t) := \frac{1}{r} \eta\left(\frac{t}{r}\right)$. 

Step 3. Regularity results for the regularized problem

Let $v \in W^{1,\tilde{\varphi}}(B_r)$ be the minimizer of the functional

$$u + W^{1,\tilde{\varphi}}_0(B_r) \ni v \mapsto \int_{B_r} \tilde{\varphi}(|Dv|) \, dx,$$

equivalently, $v$ is a weak solution to

$$\text{div} \left( \frac{\tilde{\varphi}'(|Dv|)}{|Dv|} Dv \right) = 0 \quad \text{in} \quad B_r, \quad v = u \quad \text{on} \quad \partial B_r.$$
Step 3. Regularity results for the regularized problem

Let \( v \in W^{1,\tilde{\varphi}}(B_r) \) be the minimizer of the functional

\[
    u + W_0^{1,\tilde{\varphi}}(B_r) \ni v \mapsto \int_{B_r} \tilde{\varphi}(|Dv|) \, dx,
\]
equivalently, \( v \) is a weak solution to

\[
    \text{div} \left( \frac{\tilde{\varphi}'(|Dv|)}{|Dv|} Dv \right) = 0 \quad \text{in} \quad B_r, \quad v = u \quad \text{on} \quad \partial B_r.
\]

\( C^{1,\alpha} \)-regularity

\( u \in C^{1,\alpha}_{\text{loc}}(B_r) \) for some \( \alpha \in (0, 1) \). For any \( B_\rho(y) \subset B_r \)

\[
    \sup_{B_\rho/2(y)} |Dv| \leq c \int_{B_\rho(y)} |Dv| \, dx
\]

and, for any \( \tau \in (0, 1) \),

\[
    \int_{B_{\tau \rho}(y)} |Dv - (Dv)_{B_{\tau \rho}(y)}| \, dx \leq c \tau^\alpha \int_{B_\rho(y)} |Dv| \, dx.
\]
Proof (main steps)

Calderón-Zygmund type estimates

Suppose $\theta = \theta(x, t)$ satisfies (A0), (aInc)$_{p_1}$, (aDec)$_{q_1}$ and (A1) for some $1 < p_1 < q_1$. Then

\[ \| \tilde{\varphi}(|Du|) \|_{L^\theta(B_r)} \leq c \| \tilde{\varphi}(|Dv|) \|_{L^\theta(B_r)}. \]

This implies that

\[ \int_{B_r} \theta(x, \tilde{\varphi}(|Dv|)) \, dx \leq c \left( \int_{B_r} \theta(x, \tilde{\varphi}(|Du|)) \, dx + 1 \right), \]

provided that $\int_{B_r} \theta(x, \tilde{\varphi}(|Du|)) \, dx \leq 1$.

Set $\theta(x, t) = \theta_0(x, t)^{1+\sigma} = \varphi(x, \tilde{\varphi}^{-1}(t))^{1+\sigma}$.

\[ \int_{B_r} \varphi(x, |Dv|)^{1+\sigma} \, dx = \int_{B_r} \theta(x, \tilde{\varphi}(|Dv|)) \, dx \]

\[ \leq c \left( \int_{B_r} \theta(x, \tilde{\varphi}(|Du|)) \, dx + 1 \right) = c \left( \int_{B_r} \varphi(x, |Du|)^{1+\sigma} \, dx + 1 \right). \]

In particular, $v \in W^{1,\varphi}(B_r)$ and $\varphi(\cdot, |Dv|) \in L^{1+\sigma}(B_r)$. 

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Proof (main steps)

If
\[ \|f\|_{L^p_w(B_r)} \leq c([w]_p) \|g\|_{L^p_w(B_r)} \]
for some \(1 < p < \infty\) and for all weight \(w \in A_p\), then
\[ \|f\|_{L^\theta(B_r)} \leq c\|g\|_{L^\theta(B_r)} \]
for any \(\theta = \theta(x, t)\) satisfying (A0), (alnc)\(_{p_1}\), (aDec)\(_{q_1}\) and (A1).

- \(f = \tilde{\phi}(|Dv|)\) and \(g = \tilde{\phi}(|Du|)\).
Step 4. Approximating estimate

- Use that
  \[
  \int_{B_r} \varphi(x, |Du|) \, dx \leq \int_{B_r} \varphi(x, |Dv|) \, dx, \quad \int_{B_r} \tilde{\varphi}(|Du|) \, dx \leq \int_{B_r} \tilde{\varphi}(|Dv|) \, dx
  \]

- Separate \( B_{r/2} \) into the three regions:
  \[
  \{|Du| \leq t_1\}, \quad \{t_1 < |Du| \leq t_2\}, \quad \{|Du| > t_2\},
  \]
  where \( t_1 := (\varphi_{B_{2r}}^{-1})(\omega(2r)) \) and \( t_2 := (\varphi_{B_{2r}}^{-1})(|B_{2r}|^{-1}) \), and estimate integrals over the above regions independently.

- By applying reverse type estimates, we obtain \( L^1 \) comparison estimate.
  \[
  \int_{B_{r/2}} |Du - Dv| \, dx \lesssim \tilde{\omega}(r) \int_{B_{2r}} |Du| \, dx,
  \]
  where \( \tilde{\omega}(r) = (\omega(r) + r) \text{ (power)} \).
Step 5. Iteration
By standard iteration arguments with \( B_{r_j} \) with \( r_j = 2 \cdot 4^{-j}r \), we obtain

### Morrey type estimate
For each \( \alpha \in (0, 1) \)

\[
\int_{B_r} |Du| \, dx \lesssim r^{n-1+\alpha}
\]

for any small ball \( B_r \),

which imply \( u \in C^{\alpha}_{\text{loc}} \).

### Campanato type estimate
Suppose \( \omega_\epsilon(r) \lesssim r^{\beta_\epsilon} \). For some \( \alpha \in (0, 1) \)

\[
\int_{B_r} |Du - (Du)_{B_r}| \, dx \lesssim r^{n+\alpha}
\]

for any small ball \( B_r \)

which implies \( u \in C^{1,\alpha}_{\text{loc}} \).
Hölder continuous or bounded minimizers

Let $s > 0$. There exists a non-decreasing, bounded, continuous function $\omega : [0, \infty) \to [0, 1]$ with $\omega(0) = 0$ such that for any small ball $B_r \supseteq \Omega$, $\phi + B_r(t) \leq (1 + \omega(r)) \phi - B_r(t) + \omega(r)$ for all $t \in [\omega(r), |B_r| - 1]$. If $\phi$ satisfies (wVA1-s), then it does (VA1-\tilde{s}) for any $\tilde{s} > s$. 

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**(VA1-\(s\)) : vanishing (A1-\(s\))**

Let \( s > 0 \). There exists a non-decreasing, bounded, continuous function \( \omega : [0, \infty) \to [0, 1] \) with \( \omega(0) = 0 \) such that for any small ball \( B_r \subset \Omega \),

\[
\varphi^+_B(t) \leq (1 + \omega(r)) \varphi^-_{B_r}(t) + \omega(r) \quad \text{for all} \quad t^s \in [\omega(r), |B_r|^{-1}].
\]

**(wVA1-\(s\)) : weak (VA1-\(s\))**

Let \( s > 0 \). For any small \( \epsilon > 0 \), there exists a non-decreasing, bounded, continuous function \( \omega = \omega_\epsilon : [0, \infty) \to [0, 1] \) with \( \omega(0) = 0 \) such that for any small ball \( B_r \subset \Omega \),

\[
\varphi^+_B(t) \leq (1 + \omega(r)) \varphi^-_{B_r}(t) + \omega(r) \quad \text{for all} \quad t^s \in [\omega(r), |B_r|^{-1+\epsilon}].
\]

- If \( \varphi \) satisfies (wVA1-\(s\)), then it does (VA1-\(\tilde{s}\)) for any \( \tilde{s} > s \).
Hölder continuous or bounded minimizers

Theorem (Hästö-Ok, in preparation)

1. If $\varphi$ satisfies (VA1-$\frac{n}{1-\gamma'}$) and $u \in C^\gamma(\Omega)$ for some $0 < \gamma' < \gamma < 1$, then $u \in C^{\alpha}_{\text{loc}}(\Omega)$ for any $\alpha \in (0, 1)$.

2. If $\varphi$ satisfies (VA1-$\frac{n}{1-\gamma'}$) with $\omega(r) \lesssim r^\delta$ and $u \in C^\gamma(\Omega)$ for some $0 < \gamma' < \gamma < 1$ and $\delta > 0$, then $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ for some $\alpha \in (0, 1)$.

- $\varphi(x, t) = t^p + b(x)t^q$ with $b \in C^{0,\beta}$.

  Baroni-Colombo-Mingione (2018) prove that if

  $$q < p + \frac{\beta}{1 - \gamma} \quad \text{with} \quad \gamma \in (0, 1)$$

  and $u \in C^\gamma(\Omega)$, then $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$. 

  Note that (5) implies that $\varphi$ satisfies (VA1-$\frac{n}{1-\gamma'}$) with $\omega(r) \lesssim r^\delta$, where $\gamma' \in (0, \gamma)$ is chosen to satisfy

  $$\delta := \beta - (q - p)(1 - \gamma')^2 > 0.$$
Hölder continuous or bounded minimizers

**Theorem (Hästö-Ok, in preparation)**

1. If $\varphi$ satisfies \((VA1-\frac{n}{1-\gamma'})\) and $u \in C^\gamma(\Omega)$ for some $0 < \gamma' < \gamma < 1$, then $u \in C^{\alpha}_{\text{loc}}(\Omega)$ for any $\alpha \in (0, 1)$.

2. If $\varphi$ satisfies \((VA1-\frac{n}{1-\gamma'})\) with $\omega(r) \lesssim r^\delta$ and $u \in C^\gamma(\Omega)$ for some $0 < \gamma' < \gamma < 1$ and $\delta > 0$, then $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ for some $\alpha \in (0, 1)$.

- $\varphi(x, t) = t^p + b(x)t^q$ with $b \in C^{0,\beta}_{\text{loc}}$.
  Baroni-Colombo-Mingione (2018) prove that if

  \[ q < p + \frac{\beta}{1-\gamma} \quad \text{with} \quad \gamma \in (0, 1) \tag{5} \]

  and $u \in C^\gamma(\Omega)$, then $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$.

  Note that (5) implies that $\varphi$ satisfies \((VA1-\frac{n}{1-\gamma'})\) with $\omega(r) \lesssim r^\delta$, where $\gamma' \in (0, \gamma)$ is chosen to satisfy

  \[ \delta := \beta - (q - p)(1-\gamma') = \frac{\beta - (q - p)(1-\gamma)}{2} > 0. \]
Hölder continuous or bounded minimizers

**Theorem (Hästö-Ok, in preparation)**

1. If $\varphi$ satisfies (wVA1-n) and $u \in L^\infty(\Omega)$, then $u \in C^\alpha_{loc}(\Omega)$ for any $\alpha \in (0, 1)$.

2. If $\varphi$ satisfies (wVA1-n) with $\omega_\varepsilon(r) \lesssim r^{\beta_\varepsilon}$ for some $\beta_\varepsilon > 0$ and $u \in L^\infty(\Omega)$, then $u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0, 1)$.

- This is a corollary of the preceding theorem.

$$(wVA1-n) \quad \Rightarrow \quad \begin{cases} (A1-n) \quad \Rightarrow \quad u \in C^\gamma \text{ for some } \gamma \in (0, 1) \\ (VA1-\gamma') \text{ for any small } \gamma' > 0 \end{cases}$$
H"older continuous or bounded minimizers

\[ \varphi(x, t) = t^p + b(x)t^q \] with \( b \in C^{0, \beta} \).

Colombo-Mingione (2015) prove that if

\[ q \leq p + \beta \] (6)

and \( u \in L^\infty(\Omega) \), then \( u \in C^{1, \alpha}_{\text{loc}}(\Omega) \).

Remark

In the bounded or H"older continuous minimizer case, we cannot take advantage of the extrapolation.
Hölder continuous or bounded minimizers

\[ \varphi(x, t) = t^p + b(x)t^q \] with \( b \in C^{0, \beta} \).

Colombo-Mingione (2015) prove that if

\[ q \leq p + \beta \quad (6) \]

and \( u \in L^\infty(\Omega) \), then \( u \in C^{1, \alpha}_{\text{loc}}(\Omega) \).

Note that (6) implies that \( \varphi \) satisfies \((\text{wVA}1-h)\) with \( \omega(\epsilon) \lesssim r\delta(\epsilon) \), where

\[ \delta(\epsilon) = \beta - (q - p)(1 - \epsilon) > 0. \]
Hölder continuous or bounded minimizers

- \( \varphi(x, t) = t^p + b(x)t^q \) with \( b \in C^{0, \beta} \).

Colombo-Mingione (2015) prove that if

\[ q \leq p + \beta \tag{6} \]

and \( u \in L^\infty(\Omega) \), then \( u \in C^{1, \alpha}_{\text{loc}}(\Omega) \).

Note that (6) implies that \( \varphi \) satisfies (wVA1-n) with \( \omega_{\epsilon}(r) \lesssim r\delta_{\epsilon} \), where

\[ \delta_{\epsilon} = \beta - (q - p)(1 - \epsilon) > 0. \]

**Remark** In the bounded or Hölder continuous minimizer case, we cannot take advantage of the extrapolation.
$W^{1,\varphi}(\Omega) \ni \nu \mapsto \int_{\Omega} f(x, D\nu) \, dx.$

\[
\left\{
\begin{array}{l}
\begin{aligned}
z & \mapsto f(x, z) \text{ is } C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}), \\
\varphi & \in C^1([0, \infty)) \text{ and } \varphi' \text{ satisfies } (A0), (\text{Inc})_{p-1} \text{ and } (\text{Dec})_{q-1}.
\end{aligned}
\end{array}
\right.
\]

\[
\begin{aligned}
\nu \varphi(x, |z|) & \leq f(x, z) \leq L \varphi(x, |z|), \\
\nu \frac{\varphi'(x, |z|)}{|z|} |\lambda|^2 & \leq \partial^2 f(x, z) \lambda \cdot \lambda \leq L \frac{\varphi'(x, |z|)}{|z|} |\lambda|^2.
\end{aligned}
\]

(AF): version of (wVA1) in the functional setting

For any $\epsilon > 0$, there exists a non-decreasing, bounded, continuous function $\omega = \omega_{\epsilon} : [0, \infty) \to [0, 1]$ with $\omega(0) = 0$ such that for any small ball $B_r \subset \Omega$,

\[
f^+_{B_r}(z) \leq (1 + \omega(r)) f^-_{B_r}(z) + \omega(r)
\]

for all $z \in \mathbb{R}^n$ with $\varphi^-_{B_r}(|z|) \in [\omega(r), |B_r|^{-1+\epsilon}]$. 
Theorem (Hästö-Ok, in preparation)

1. If $f$ satisfies (AF), then $u \in C^{\alpha}_{loc}(\Omega)$ for any $\alpha \in (0, 1)$.

2. If $f$ satisfies (AF) with $\omega_{\epsilon}(r) \lesssim r^{\beta_{\epsilon}}$ for some $\beta_{\epsilon} > 0$, then $u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0, 1)$. 
\[ \text{div } A(x, Du) = 0. \]

\[ \begin{cases} 
  z \mapsto A(x, z) \in \mathbb{R}^n \text{ is } C^1(\mathbb{R}^n \setminus \{0\}), \\
  \varphi \in C^1([0, \infty)) \text{ and } \varphi' \text{ satisfies } (A0), \ (\text{Inc})_{p-1} \text{ and } (\text{Dec})_{q-1}. \\
  |A(x, |z|)| + |z| |\partial A(x, z)| \leq L\varphi'(x, |z|), \\
  \nu \frac{\varphi'(x, |z|)}{|z|} |\lambda|^2 \leq \partial A(x, z) \lambda \cdot \lambda.
\end{cases} \]

**{(AP)}: version of (wVA1) in PDE setting**

For any \( \epsilon > 0 \), there exists a non-decreasing, bounded, continuous function \( \omega = \omega_\epsilon : [0, \infty) \to [0, 1] \) with \( \omega(0) = 0 \) such that for any small ball \( B_r \subseteq \Omega \),

\[ |A(x, z) - A(y, z)| \leq \omega(r)((\varphi')_{B_r}(|z|) + 1) \]

for all \( x, y \in B_r \) and for all \( z \in \mathbb{R}^n \) with \( \varphi_{B_r}^{-1}(|z|) \in [\omega(r), |B_r|^{-1+\epsilon}] \).
Theorem (Hästö-Ok, in preparation)

(1) If $A$ satisfies (AP), then $u \in C^\alpha_{loc}(\Omega)$ for any $\alpha \in (0, 1)$.

(2) If $A$ satisfies (AP) with $\omega_\epsilon(r) \lesssim r^{\beta_\epsilon}$ for some $\beta_\epsilon > 0$, then $u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0, 1)$.

Remark In the general functional or PDE case we have to construct $\tilde{f}(z)$ or $\tilde{A}(z)$ with $\tilde{\varphi}$-growth, where $\tilde{\varphi} = \tilde{\varphi}(t)$ is the regular function.
THANK YOU.