Non Linear Asymptotic Mean Value Properties for Monge-Ampère Equations

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Monday's Nonstandard Seminar

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Mean Value Property for Harmonic functions

Theorem

$$\Delta u(x) = 0$$

if and only if

(1)
$$u(x) = \fint_{\partial B(x,r)} u(y)\, d\sigma(y)$$

or

$$(\lim_{r \to 0}) \frac{1}{r^2} \left[\fint_{\partial B(x,r)} u(y)\, d\sigma(y) - u(x) \right] = 0.$$

(2)
$$u(x) = \fint_{B(x,r)} u(y)\, d\sigma(y)$$

or

$$(\lim_{r \to 0}) \frac{1}{r^2} \left[\fint_{B(x,r)} u(y)\, d\sigma(y) - u(x) \right] = 0.$$

Theorem (Blasche, 1916)

An upper-semicontinuous function u is subharmonic, $\Delta u \geq 0$, if and only if

$$\limsup_{\epsilon\to 0} \frac{1}{\epsilon^2}\left[⨍_{\partial B(x,\epsilon)} u(y)\, d\sigma(y) - u(x) \right] \geq 0$$

Theorem (Privaloff, 1916)

An upper-semicontinuous function u is subharmonic, $\Delta u \geq 0$, if and only if

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Theorem

A function u is harmonic, $\Delta u = 0$, if and only if

$$u(x) = \frac{1}{n} \sum_{j=1}^{n} \left\{ \frac{1}{2} u(x + \varepsilon e_j) + \frac{1}{2} u(x - \varepsilon e_j) \right\} + o(\varepsilon^2) \quad as\ \varepsilon \to 0,$$

where $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^n$, or

$$u(x) = \left[\fint_{B(x,\epsilon)} u(y)\, d(y) \right] + o(\varepsilon^2) \quad as\ \varepsilon \to 0.$$

Modern Linear results

If we replace the Laplace equation $\Delta u = 0$ by a linear elliptic equation with constant coefficients $Lu = \sum_{i,j} a_{ij} u_{x_i x_j} = 0$ then mean value formulas hold for appropriate ellipsoids instead of balls.

Nonlinear operators. (Manfredi-Parvianen-R., 2010)

Viscosity solutions to the 1−homogeneous p-Laplacian

$$\Delta_p^N u = \frac{1}{p-2}|\nabla u|^{2-p}\,\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = \frac{1}{p-2}\Delta u + \Delta_\infty^N u = 0,$$

for $1 < p \leq \infty$ are characterized by a mean value formula

$$u(x) - \left(\frac{p-2}{p+n}\right)\left(\frac{\max_{\overline{B}_\varepsilon(x)} u + \min_{\overline{B}_\varepsilon(x)} u}{2}\right) + \left(\frac{2+n}{p+n}\right) \fint_{B_\varepsilon(x)} u(y)\,dy$$
$$= o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0.$$

We will discuss mean value properties for solutions to the

Monge-Ampère equation

$$\det D^2 u = f,$$

with $f \geq 0$ in a convex domain Ω.

As usual, we look for convex solutions u, thus the $D^2 u \geq 0$ and hence f is non-negative. In terms of eigenvalues of $D^2 u$ we have

$$\min_{\lambda \text{ eigenvalue of } D^2 u} \{\lambda\} \geq 0.$$

Let $\phi(\epsilon)$, $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be such that

$$\lim_{\epsilon \to 0} \phi(\epsilon) = +\infty$$

and

$$\lim_{\epsilon \to 0} \epsilon\, \phi(\epsilon) = 0.$$

$$(\phi(\epsilon) = \epsilon^{-1/2} \text{ works}).$$

Theorem (Convex C^2 Case)

Let u be convex and C^2 in Ω. Fix $x \in \Omega$. We have

$$u(x) - \inf_{\substack{\det A = 1 \\ A \le \phi(\epsilon) I}} \left\{ \fint_{B_\epsilon(0)} u(x+Ay)\, dy \right\} + \frac{n}{2(n+2)} \left(\det D^2 u(x)\right)^{1/n} \epsilon^2 = o(\epsilon^2),$$

as $\epsilon \to 0$.

• Notice that for every A with $\det A = 1$, it holds

$$\left\{ \fint_{B_\epsilon(0)} u(x + Ay)\, dy \right\} = \left\{ \fint_{E_\epsilon(A,x)} u(z)\, dz \right\}$$

where $E_\epsilon(A, x) = \{x + Ay : \; y \in B_\epsilon(0)\}$.

• The restriction $A \leq \phi(\epsilon) I$ in the infimum makes the formula local. For every $x \in \Omega$, the conditions $A \leq \phi(\varepsilon) I$ and $|y| \leq \varepsilon$ imply that

$$\mathrm{dist}(E_\epsilon(A, x), x) = \mathrm{dist}(x + Ay, x), x) = |Ay| \leq |A||y| \leq \epsilon\, \phi(\epsilon) \to 0$$

(since $\epsilon\, \phi(\epsilon) \to 0$ as $\epsilon \to 0$).
Hence, $E_\epsilon(A, x) \subset \Omega$ for ε small enough.

Replace the condition $A \leq \phi(\epsilon)$ by requiring

$$E_\epsilon(A,x) = \left\{x + Ay : \; y \in B_\epsilon(0)\right\} \subset \Omega.$$

Theorem (Non-Local Convex C^2-case)

Let u be convex and C^2 in Ω. Fix $x \in \Omega$. We have

$$u(x) - \inf_{\substack{\det A = 1 \\ E_\epsilon(A,x) \subset \Omega}} \left\{ \fint_{B_\epsilon(0)} u(x+Ay)\, dy \right\} + \frac{n}{2(n+2)} \left(\det D^2 u(x)\right)^{1/n} \epsilon^2 = o(\epsilon^2)$$

as $\epsilon \to 0$.

Corollary (Characterization of C^2-solutions)

Let u be convex and C^2 in Ω, $f \geq 0$. TFAE:

$$\left(\det D^2 u(x)\right)^{1/n} = f(x)$$

$$u(x) - \inf_{\substack{\det A = 1 \\ A \leq \phi(\epsilon) I}} \left\{ ⨍_{B_\epsilon(0)} u(x + Ay)\, dy \right\} + \frac{\epsilon^2\, n}{2(n+2)} f(x) = o(\epsilon^2)$$

$$u(x) - \inf_{\substack{\det A = 1 \\ E_\epsilon(A,x) \subset \Omega}} \left\{ ⨍_{B_\epsilon(0)} u(x + Ay)\, dy \right\} + \frac{\epsilon^2\, n}{2(n+2)} f(x) = o(\epsilon^2).$$

Theorem (Characterization of viscosity solutions)

Let $f \in C(\Omega)$ be non-negative and $u \in C(\Omega)$ be convex. TFAE:

u is a viscosity subsolution (respectively, supersolution) of

$$\det D^2 u = f \quad \textit{in} \;\; \Omega, \qquad (\det D^2 u \geq f)$$

$$u(x) \leq \inf_{\substack{\det A = 1 \\ A \leq \phi(\epsilon) I}} \left\{ \fint_{B_\epsilon(0)} u(x + Ay)\, dy \right\} - \frac{\epsilon^2\, n}{2(n+2)}\, f(x) + o(\epsilon^2)$$

$$u(x) \leq \inf_{\substack{\det A = 1 \\ E_\epsilon(A,x) \subset \Omega}} \left\{ \fint_{B_\epsilon(0)} u(x + Ay)\, dy \right\} - \frac{\epsilon^2\, n}{2(n+2)}\, f(x) + o(\epsilon^2)$$

(respectively, $\geq$) in the viscosity sense (the mean value expansions are satisfied for convex paraboloids P that touch u at x).

Discrete asymptotic expansion

For $u \in C^2$ convex we have the asymptotic expansion

$$u(x) = \inf_{V \in \mathbb{O}} \inf_{\alpha_i \in I_\epsilon^n} \left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{u(x + \epsilon\sqrt{\alpha_i} v_i) + u(x - \epsilon\sqrt{\alpha_i} v_i)}{2} \right\} - \frac{\epsilon^2}{2} (\det(D^2 u)(x))^{1/n} + o(\epsilon^2)$$

as $\epsilon \to 0$.

Here $\mathbb{O}$ is the set all orthonormal bases $V = \{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ and

$$I_\varepsilon^n = \left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n : \prod_{j=1}^{n} \alpha_j = 1 \quad \text{and} \quad 0 < \alpha_j < \phi^2(\epsilon) \right\}.$$

Discrete Mean Values, II

Theorem (Characterization of viscosity solutions by Discrete Mean Values)

Let u be a convex function in a domain $\Omega \subset \mathbb{R}^n$. Then, u is a solution to the Monge-Ampère equation

$$\det(D^2 u(x)) = f(x)$$

in the viscosity sense if and only if

$$u(x) = \inf_{V \in \mathbb{O}} \inf_{\alpha_i \in I_\epsilon^n} \left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{u(x + \epsilon\sqrt{\alpha_i} v_i) + u(x - \epsilon\sqrt{\alpha_i} v_i)}{2} \right\} - \frac{\epsilon^2}{2} (f(x))^{1/n} + o(\epsilon^2)$$

as $\varepsilon \to 0$, holds in the viscosity sense.

Determinant identity for $M \geq 0$

$$n\big(\det M\big)^{1/n} = \inf_{\det A = 1} \mathrm{trace}(A^t\, M\, A),$$

where the matrix A is symmetric and positive definite.

Linear averages

Let M be a square matrix of dimension n. We have

$$\mathrm{trace}(M) = \frac{n+2}{\varepsilon^2} \fint_{B_\varepsilon(0)} \langle My, y\rangle \, dy.$$

Suppose $D^2u(x) > 0$. Extra work is needed when $\det D^2u(x) = 0$.

Given $x \in \Omega$, consider the paraboloid

$$P(z) = u(x) + \langle \nabla u(x), z - x \rangle + \frac{1}{2}\langle D^2u(x)(z-x), (z-x)\rangle.$$

Since $u \in C^2(\Omega)$, we have

$$u(z) - P(z) = o(|z-x|^2) \qquad \text{as } z \to x,$$

that is,

$$P(z) - \frac{\eta}{2}|z-x|^2 \leq u(z) \leq P(z) + \frac{\eta}{2}|z-x|^2,$$

for every $z \in B_\delta(x)$, with equality only when $z = x$. Let us denote

$$P_\eta^\pm(z) = P(z) \pm \frac{\eta}{2}|z-x|^2.$$

Main argument

$$\fint_{B_\epsilon(0)} \big(P_\eta^\pm(x+Ay) - u(x)\big)\,dy = \frac{1}{2}\fint_{B_\epsilon(0)} \big(\langle A^t D^2 u(x) Ay, y\rangle \pm \eta |Ay|^2\big)\ dy$$
$$= \frac{1}{2}\fint_{B_\epsilon(0)} \left\langle A^t\left(D^2u(x) \pm \eta I\right) Ay, y\right\rangle\ dy$$
$$= \frac{\epsilon^2}{2(n+2)}\,\mathrm{trace}\left(A^t\left(D^2u(x)\pm\eta I\right)A\right),$$

Using

$$n\big(\det M\big)^{1/n} = \inf_{\det A = 1} \mathrm{trace}(A^t\, M\, A),$$

and that $P_\eta^\pm(x) = u(x)$ we obtain

$$\inf_{\substack{\det A=1\\ A\le\phi(\epsilon) I}} \left\{ \fint_{B_\epsilon(0)} \big(P_\eta^\pm(x+Ay) - P_\eta^\pm(x)\big)\,dy \right\} = \frac{n\,\epsilon^2}{2(n+2)} \big(det\,\big(D^2u(x)\pm\eta I\big)\big)^{1/n}$$

Now we use that

$$P_\eta^-(x+Ay) \le u(x+Ay) \le P_\eta^+(x+Ay) \qquad \text{for every } y \in B_\epsilon(0).$$

to obtain

$$\inf_{\substack{\det A=1,\\ A\le\phi(\epsilon)I}} \left\{ \fint_{B_\epsilon(0)} (u(x+Ay)-u(x))\,dy \right\} \le \frac{n\,\epsilon^2}{2(n+2)} \left(\det\left(D^2u(x)+\eta I\right)\right)^{1/n}$$

$$\inf_{\substack{\det A=1,\\ A\le\phi(\epsilon)I}} \left\{ \fint_{B_\epsilon(0)} (u(x+Ay)-u(x))\,dy \right\} \ge \frac{n\,\epsilon^2}{2(n+2)} \left(\det\left(D^2u(x)-\eta I\right)\right)^{1/n}.$$

The result follows from

$$\left(\det\left(D^2u(x)\pm\eta I\right)\right)^{1/n} \to \left(\det D^2u(x)\right)^{1/n} \qquad \text{as } \eta \to 0.$$

We describe a one-player game (or control problem)
RULES OF THE GAME :

• Fix a convex domain $\Omega \subset \mathbb{R}^n$

• Fix a final payoff $g\colon \mathbb{R}^n \setminus \Omega \mapsto \mathbb{R}$ and a running cost $f \in C(\Omega)$, $f \geq 0$,

• Fix $\varepsilon > 0$ small

• Place a token at an initial position $x_0 \in \Omega$

• The player chooses an orthonormal basis $\{v_1, ..., v_n\}$ and n real nonnegative numbers $(\alpha_1, ..., \alpha_n) \in I_\varepsilon^n$ where

$$I_\varepsilon^n = \left\{ (\alpha_i)_{i=1,\dots,n} \in \mathbb{R}^n \colon \prod_{i=1}^{n} \alpha_i = 1 \quad \text{and} \quad 0 < \alpha_i < \frac{1}{\varepsilon} \right\}.$$

• The token is moved to

$$x_1 = x_0 \pm \varepsilon \sqrt{\alpha_i} v_i$$

with equal probabilities $\frac{1}{2n}$

• Player 1 pays $\frac{1}{2}\varepsilon^2 (f(x_0))^{1/n}$

- Repeat the process starting at x_1 to get x_2 and so on
- Get a sequence of positions $\{x_0, x_1, \dots, x_j \dots\}$
- The game stops when the token leaves Ω. Let τ be first time that $x_\tau \notin \Omega$. The player gets paid $g(x_\tau)$
- At the end the player obtains

$$g(x_\tau) - \frac{1}{2}\varepsilon^2 \sum_{j=0}^{\tau-1} (f(x_j))^{1/n}$$

- A STRATEGY S for the player is a choice of orthonormal basis and numbers $(\alpha_i) \in I_\varepsilon^n$.
- Given a strategy S we can look for the expected outcome

$$\mathbb{E}_S^{x_0}\left[g(x_\tau) - \frac{1}{2}\varepsilon^2 \sum_{j=0}^{\tau-1} [f(x_j)]^{1/n} \right]$$

Suppose that the player wants to minimize the payment.

The value of the game at $x_0 \in \Omega$

$$u_\varepsilon(x_0) = \inf_S \mathbb{E}_S^{x_0} \left[g(x_\tau) - \frac{1}{2}\varepsilon^2 \sum_{j=0}^{\tau-1} [f(x_j)]^{1/n} \right]$$

The value function satisfies the DPP

$$\begin{aligned} u_\varepsilon(x) &= \inf_{V\in\mathbb{O}} \inf_{\alpha_i \in I_\varepsilon^n} \left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{u_\varepsilon(x+\varepsilon\sqrt{\alpha_i}v_i) + u_\varepsilon(x-\varepsilon\sqrt{\alpha_i}v_i)}{2} \right\} \\ &\quad - \frac{\varepsilon^2}{2}(f(x))^{1/n} \qquad \text{for } x \in \Omega \\ u_\varepsilon(x) &= g(x) \qquad \text{for } x \notin \Omega \end{aligned}$$

Existence and Uniqueness for the DPP hold.

Solving the DPP for each $\varepsilon > 0$ we get a family of functions $\{u_\varepsilon\}$.

Theorem

When the domain Ω is strictly convex we have

$$u_\varepsilon \to u \qquad \text{as } \varepsilon \to 0$$

uniformly in $\overline{\Omega}$, where u is the unique viscosity solution to the problem

$$\begin{cases} \det D^2 u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

General picture

MVP (continuous, discrete) $\iff$ DPP (usually discrete) $\iff$ PDE (usually continuous)

(Meta) Theorem

For an appropriate real function u we have a meta-equivalence among

1. *u satisfies a Mean Value Property (in an appropriate asymptotic sense)*
2. *u can be approximated by solutions to a Dynamic Programming Principle associated to a game or control problem*
3. *u solves a (possibly nonlinear) PDE*

Flexibility of this approach

Euclidean spaces, Riemannian manifolds, Sub-Riemannian manifolds (Heisenberg group), graphs (trees), metric-measure spaces, parabolic versions.

But limited to $\mathbb{R}$-valued functions and 2nd order PDEs (we use viscosity theory).

References

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Thanks !!! Gracias !!!