Density of smooth functions in Musielak-Orlicz spaces

Iwona Chlebicka

MIMUW @ University of Warsaw

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Modular density of smooth functions

In Musielak-Orlicz-Sobolev spaces the norm is defined by the means of functional

$$\xi \mapsto \int_Z M(z, \xi)\, dz$$

with typical choice of $Z = \Omega$ or $Z = \Omega \times [0, T]$.

Smooth functions are not dense in norm (in general) in the spaces, but in modular topology and only the growth of M with respect to ξ is well balanced with small perturbations of z-variable.

Y. Ahmida, I. C., P. Gwiazda, A. Youssfi,
Gossez's approximation theorems in Musielak-Orlicz-Sobolev spaces,
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there will appear also references to several papers joint with Gwiazda & Zatorska-Goldstein with applications of similar density results in existence to renormalized solutions to elliptic and parabolic problems with merely integrable data

Examples

Musielak-Orlicz spaces unify theory for spaces with norm given by

- $\xi \mapsto \int_\Omega |\xi|^p \, dx$ (Lebesgue/Sobolev spaces)
- $\xi \mapsto \int_\Omega |\xi|^{p(x)} \, dx$ (variable exponent Lebesgue/Sobolev spaces)
- $\xi \mapsto \int_\Omega |\xi|^p + a(x)|\xi|^q \, dx$, $\xi \mapsto \int_\Omega |\xi|^p + a(x)|\xi|^p \log(1 + |\xi|) \, dx$ (double-phase spaces)
- $\xi \mapsto \int_\Omega B(|\xi|) \, dx$ (Orlicz/Orlicz-Sobolev spaces; also $L \log L$, $\exp L$)
- $\xi \mapsto \int_\Omega \Phi(\xi) \, dx$ (anisotropic Orlicz-Sobolev spaces)
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reflexivity!

C., A pocket guide to nonlinear differential equations in Musielak-Orlicz spaces, Nonl. Analysis 2018.

(Generalized) N-function $M : \Omega \times [0, \infty) \to \mathbb{R}$

1. M is a Carathéodory function, such that $M(x, 0) = 0$, $s \neq 0 \Rightarrow \inf_{x \in \Omega} M(x, s) > 0$ and $\sup_{x \in \Omega} M(x, s) < \infty$,
2. $s \mapsto M(x, s)$ is a convex function,
3. $\lim_{s \to 0} \frac{M(x,s)}{s} = 0$ for a.e. $x \in \Omega$,
4. $\lim_{s \to \infty} \frac{M(x,s)}{s} = \infty$ for a.e. $x \in \Omega$.

Examples of $M(x, s)$

$|s|^p$, $\omega(x)|s|^p$, $|s|^{p(x)}$, $|s|^q + a(x)|s|^p$, more phases...
Orlicz: $B(|s|)$, $\omega(x)B(|s|)$, $B_0(|s|) + \sum_i \omega_i(x)B_i(|s|)$...

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Anisotropy...

Doubling conditions

We say that an N-function $M : \Omega \times [0, \infty) \to \mathbb{R}$ satisfies Δ_2-**condition** if there exists a constant $c > 0$ and $h \in L^1(\Omega)$, $h \geqslant 0$, such that

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- in Δ_2: $M(x, s) = |\xi|^p$; $a(x)|\xi|^p + |\xi|^q$; $|\xi|^{p(x)}$; $|\xi|^p \log^\alpha(1 + |\xi|)$;
- **NOT** in Δ_2:
  (fast) $M(x, \xi) = a(x)\,(\exp(|\xi|) - 1 + |\xi|)$;
  (irregularly increasing) even trapped between t^p and $t^{p+\varepsilon}$,
  $1 < p < \infty$, cf. C.–Giannetti–Zatorska-Goldstein, JMAA 2019.

Musielak-Orlicz spaces

Suppose $\Omega \subset \mathbb{R}^N$.

- $L_M(\Omega)$ is defined as the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega M(x, |u(x)|/\lambda)dx < +\infty$ for some $\lambda > 0$.
- $E_M(\Omega)$ is defined as the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega M(x, |u(x)|/\lambda)dx < +\infty$ for all $\lambda > 0$.

Both are equipped with the Luxemburg norm

$$\|u\|_{L_M(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega M\left(x, \frac{|u(x)|}{\lambda}\right)dx \leqslant 1\right\}.$$

*Then $L_M(\Omega)$ is a Banach space and $E_M(\Omega)$ is its closed subset.
** $E_M(\Omega)$ coincides with norm closure of bounded functions, provided that $\forall_{c>0} \int_\Omega M(x, c)dx < \infty$.

Thus here $E_M(\Omega) = \overline{L^\infty(\Omega)}^{\|\cdot\|_{L_M(\Omega)}}$, but $E_M(\Omega) \neq L_M(\Omega)$.

Musielak-Orlicz-Sobolev spaces

For a positive integer m, we define the Musielak-Orlicz-Sobolev spaces $W^mL_M(\Omega)$ and $W^mE_M(\Omega)$ as follows

$$W^mL_M(\Omega) = \big\{u \in L_M(\Omega) : D^\alpha u \in L_M(\Omega), |\alpha| \leqslant m\big\},$$

$$W^mE_M(\Omega) = \big\{u \in E_M(\Omega) : D^\alpha u \in E_M(\Omega), |\alpha| \leqslant m\big\},$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$, $|\alpha| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_N|$ and D^α denote the distributional derivatives.

$W^mL_M(\Omega)$ and $W^mE_M(\Omega)$ are endowed with the Luxemburg norm

$$\|u\|_{W^mL_M(\Omega)} = \inf\Big\{\lambda > 0 : \sum_{|\alpha|\leqslant m} \int_\Omega M\big(x, \tfrac{1}{\lambda}|D^\alpha u|\big)dx \leqslant 1\Big\}.$$

$\big(W^mL_M(\Omega), \|u\|_{W^mL_M(\Omega)}\big)$ is a Banach space.

Fenchel–Young conjugate function M^*

Fenchel–Young conjugate function (complementary function, the Legendre transform) to a function $M : \Omega \times [0, \infty) \to \mathbb{R}$ is defined by

$$M^*(x,s) = \sup_{r \in \mathbb{R}} (r \cdot s - M(x,r)), \qquad s \geqslant 0,\ x \in \Omega.$$

For $M(x,s) = \frac{1}{p}|s|^p$, then $M^*(x,s) = \frac{1}{p'}|s|^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Fenchel–Young's inequality

$$|\eta \cdot \xi| \leqslant M(x, |\eta|) + M^*(x, |\xi|),$$

Hölder's inequality $\|\eta \cdot \xi\|_{L^1} \leqslant 2\|\eta\|_{L_M} \cdot \|\xi\|_{L_{M^*}}$ but in general

$$(L_M)' \neq L_{M^*}$$

» spaces $(L_M)'$ and L_{M^*} are associate, but not dual «

Doubling condition vs. properties of the spaces

Separability

$$M \in \Delta_2^\infty \qquad \Longrightarrow \qquad E_M(\Omega) = L_M(\Omega)$$

Reflexivity

$$M, M^* \in \Delta_2^\infty \quad \Longleftrightarrow \quad (L_M(\Omega;\mathbb{R}^N))' = L_{M^*}(\Omega;\mathbb{R}^N)$$

and L_M is **reflexive and separable**.

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Modular density 1/3

Modular density - Orlicz case (Gossez, Studia Math.'82)

In Orlicz spaces C_c^∞ is dense in $W_0^1 L_B$ modularly.

Modular convergence

- $u_k \xrightarrow[k\to\infty]{M} u$ $\quad ((u_k)_k$ converges modularly to u in $L_M)$ if

$$\exists_{\lambda>0} \qquad \int_\Omega M\left(x, \frac{|u_k - u|}{\lambda}\right)\, dx \xrightarrow[k\to\infty]{} 0\ .$$

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- $u_k \xrightarrow[k\to\infty]{mod} u$ $\quad$ ($(u_k)_k$ converges modularly to u in $W^m L_M$)

$$\text{if} \qquad \forall_{|\alpha|\leqslant m} \qquad D^\alpha u_k \xrightarrow[k\to\infty]{M} D^\alpha u.$$

Modular density 2/3

Modular density - Orlicz case (Gossez, Studia Math.'82)

In Orlicz spaces C_c^∞ is dense in $W_0^1 L_B$ with respect to sequential modular closure, not the norm one.
The closures concide (only) in doubling case (i.e. reflexive).

Modular density - Musielak-Orlicz case (Ahmida, Chlebicka, Gwiazda, Youssfi, JFA'2018)

In Musielak-Orlicz spaces to get modular density one needs to exclude Lavrentiev's phenomenon by imposing a condition reflecting log-Hölder continuity of the variable exponent, as well as sharp closeness of powers in double-phase spaces.

We need to control modulus of continuity of function $M(x, s)$: balance its asymptotic behaviour for small perturbations of spacial variable x and big values of s.

Lavrentiev's phenomenon 1/2

Consider variational functional

$$u \mapsto \mathcal{F}[u] = \int_\Omega F(x, u, Du)\, dx$$

with F whose growth is governed by inhomogeneous function M

$$\frac{1}{c} M(x, s) \leqslant F(x, r, s) \leqslant cM(x, s).$$

Then always

$$\inf_{u \in u_0 + W_0^1 L_M(\Omega)} \mathcal{F}[u] \leqslant \inf_{u \in u_0 + C_c^\infty(\Omega)} \mathcal{F}[u]$$

We deal with Lavrentiev's phenomenon if the inequality is strict.

Lavrentiev's phenomenon 2/2

Examples of spaces with functions that cannot be approximated

Variable exponent spaces

$$W^{1,p(\cdot)} = \{f \in W^{1,1}_{loc} :\ |Df|^{p(x)} \in L^1\}$$

when the exponent p is not log-Hölder continuous

($p \in \mathcal{P}_{\log}$ if $|p(x) - p(y)| \leqslant -c/\log|x - y|$)

the condition is essentially sharp

Double phase spaces

$$\{f \in W^{1,1}_{loc} :\ |Df|^p + a(x)|Df|^q \in L^1\}$$

with $a : \Omega \to [0, \infty)$, $a \in C^{0,\alpha}$,

when powers do not satisfy $q/p \leqslant 1 + \alpha/n$

range is sharp due to Colombo & Mingione, ARMA 2015

Doubling conditions

We say that an N-function $M : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies Δ_2-**condition** if there exists a constant $c > 0$ and $h \in L^1(\Omega)$, $h \geqslant 0$, such that

$$M(x, 2\xi) \leqslant cM(x, \xi) + h(x).$$

If $M, M^* \in \Delta_2^\infty$, then $(L_M(\Omega; \mathbb{R}^N))^* = L_{M^*}(\Omega; \mathbb{R}^N)$ and L_M is **reflexive and separable**.

Example-reminder

- in Δ_2: $M(x, \xi) = |\xi|^p$; $a(x)|\xi|^p + |\xi|^q$; $|\xi|^{p(x)}$; $|\xi|^p \log^\alpha(1 + |\xi|)$;
- **NOT** in Δ_2: 'fast growth' like exponential or 'irregular growth'.

Balance condition in the general setting

Ahmida, Chlebicka, Gwiazda, Youssfi, JFA 2018

For small perturbations of x and sufficiently large $s \in \mathbb{R}$ we have

$$\frac{M(x,s)}{M(y,s)} \leqslant \Theta(|x-y|,s)$$

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(M) with $\limsup_{\delta \mapsto 0} \Theta(\delta, \delta^{-N}) < \infty$

or

(M)$_p$ if $M(x,s) \geqslant cs^p$, then we need $\limsup_{\delta \mapsto 0} \Theta(\delta, \delta^{-N/p}) < \infty$

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* fully anisotropic case involves $(\inf_{x \in B_\delta} M(x,\xi))^{**}$.

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   $\implies$ Asymptotical log-Hölder condition.
3. [Double phase] $M(x,s) = s^p + a(x)s^q$, where $1 \leqslant p < q$ and nonnegative $a \in C^{0,\alpha}_{loc}$, then $\Theta(\tau, s) = C_a\tau^\alpha|s|^{q-p} + 1$.

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   Then $\limsup_{\varepsilon\to 0^+} \Theta(\varepsilon, c\varepsilon^{-N/p}) < \infty$ forces sharp condition $q/p \leqslant 1 + \alpha/N$.
4. [Weighted Orlicz] $M(x,s) = \sum_{i=1}^k k_i(x)M_i(s) + M_0(x,s)$, $k_i : \overline{\Omega} \to (0,+\infty)$, there exists a nondecreasing function Θ_i satisfying $k_i(x) \leqslant \Theta_i(|x-y|)k_i(y)$ with $\limsup_{\varepsilon\to 0^+} \Theta_i(\varepsilon) < \infty$, whereas $M_0(x,s)$ satisfies is Θ-regular with Θ_0.
   Then, we can take $\varphi(\tau,s) = \sum_{i=1}^k \Theta_i(\tau) + \Theta_0(\tau,s)$.

Key ideas of the proof

Construction of approximation

- split domain into starshaped regions
- shrink support of an approximated function
- mollify (re-scaled convolution)
- glue domains

We need: $$\int_\Omega M(x, \tfrac{1}{\lambda} u_\varepsilon)\, dx \leqslant C \int_\Omega M(x, \tfrac{1}{\lambda} u)\, dx.$$

For this, using (M) when x, y are close, we control

$$M(x, \tfrac{1}{\lambda}|u_\varepsilon(x)|) = \frac{M(x, \frac{1}{\lambda}|u_\varepsilon(x)|)}{M(y, \frac{1}{\lambda}|u_\varepsilon(x)|)} \frac{M(y, \frac{1}{\lambda}|u_\varepsilon(x)|)}{(M_{x,\varepsilon})^{**}(\frac{1}{\lambda}|u_\varepsilon(x)|)} (M_{x,\varepsilon})^{**}(\tfrac{1}{\lambda}|u_\varepsilon(x)|).$$

References for approximation

Modular approximation in Orlicz spaces

- isotropic: Gossez, Studia Math.'82
- anisotropic: Alberico, C., Cianchi, Zatorska-Goldstein, arXiv

Modular approximation in Musielak-Orlicz spaces

- isotropic: Ahmida, C., Gwiazda, Youssfi, JFA 2018
- anisotropic: C. (Skrzypczak), Gwiazda, Zatorska-Goldstein, 2xJDE 2018 » covering sharply $p(x)$-case, but not double-phase (but existence does!)
- anisotropic: C., Gwiazda, Świerczewska-Gwiazda, Wróblewska-Kamińska, book » covering sharply $p(x)$-case and double-phase
- anisotropic parabolic with $M(t, x, \xi)$: C-G-ZG (AIHP 2019 & JDE 2019)

Applications to existence and regularity

Alberico, C., Cianchi, Zatorska-Goldstein, *Fully anisotropic elliptic problems with minimally integrable data*, arXiv 2019.

$$-\mathrm{div}\, a(x, \nabla u) = \mu$$

with fully anisotropic operator and measure data.
We prove the existence and regularity in anisotropic Marcinkiewicz scale of *approximable solutions* in particular to

$$\begin{cases} -\sum_{i=1}^{n} \left(b_i(x)|u_{x_i}|^{p_i-2}u_{x_i}\right)_{x_i} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where u_{x_i} denotes partial derivative with respect to the variable x_i, the functions $b_i \in L^\infty(\Omega)$ are such that $b_i(x) \geqslant 1$, and $p_i > 1$.
(but also for problems with slowlier growth like $L\log L$ etc.)

Applications to existence

see https://www.mimuw.edu.pl/~ichlebicka/

- Renormalized solutions to elliptic problem $-\mathrm{div}\, a(x, \nabla u) = f \in L^1$
  Gwiazda, Skrzypczak, Zatorska-Goldstein, JDE 2018
- Renormalized solutions to parabolic problem with $M = M(x, \nabla u)$
  C., Gwiazda, Zatorska-Goldstein, JDE 2018
- Weak solutions to parabolic problem with $M = M(t, x, \nabla u)$
  C., Gwiazda, Zatorska-Goldstein, Ann.I.H.Poincaré 2019
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  C., Gwiazda, Zatorska-Goldstein, JDE 2019

book: *Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces*
C., Gwiazda, Świerczewska-Gwiazda, Wróblewska-Kamińska (soon)

Parabolic existence and uniqueness 1/2

Chlebicka, Gwiazda, Zatorska-Goldstein

$$\partial_t u - \operatorname{div} A(t,x,\nabla u) = f(x,t) \in L^\infty / L^1$$

($\mathcal{A}$1) A is a Carathéodory function

($\mathcal{A}$2) "Growth of $A(t,x,\xi)$ is governed by $M(t,x,\xi)$"

($\mathcal{A}$3) $(A(t,x,\xi) - A(t,x,\eta)) \cdot (\xi - \eta) \geqslant 0$

Parabolic existence and uniqueness 2/2

Chlebicka, Gwiazda, Zatorska-Goldstein

Very weak solutions

Suppose $[0, T]$ is a finite interval, Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, $N > 1$, $f \in L^1(\Omega_T)$, $u_0 \in L^1(\Omega)$. Let an N-function M satisfy assumption (M) and function A satisfy assumptions (A1)-(A3). Then there exists **unique** renormalized solution to the problem

$$\begin{cases} \partial_t u - \mathrm{div} A(t, x, \nabla u) = f(t, x) & \text{in } \Omega_T, \\ u(t, x) = 0 & \text{on } \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega. \end{cases}$$

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Very weak solutions

Suppose $[0, T]$ is a finite interval, Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, $N > 1$, $f \in L^1(\Omega_T)$, $u_0 \in L^1(\Omega)$. Let an N-function M satisfy assumption (M) and function A satisfy assumptions (A1)-(A3). Then there exists **unique** renormalized solution to the problem

$$\begin{cases} \partial_t u - \mathrm{div} A(t,x,\nabla u) = f(t,x) & \text{in } \Omega_T, \\ u(t,x) = 0 & \text{on } \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

Remark

Skip balance condition (M) in reflexive spaces if $M = M(x, \nabla u)$.

Main examples not covered before

Chlebicka, Gwiazda, Zatorska-Goldstein, JDE 2018, AIHP 2019, JDE 2019

I. Evolutionary equations with growth below a power $(s^{1+\nu})$

$$\begin{cases} \partial_t u - \mathrm{div}\Big(a(t,x)\frac{\log^\alpha(\mathrm{e}+|\nabla u|)}{|\nabla u|}\nabla u\Big) = f(t,x) \in L^1(\Omega_T), \\ u(t,x) = 0 \quad \text{on} \quad \partial\Omega, \\ u(0,x) = u_0(x) \in L^1(\Omega) \end{cases}$$

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I. Evolutionary equations with growth below a power ($s^{1+\nu}$)

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II. Evolutionary $p(t,x)$-Laplace equation ($1 << p << \infty$)

$$\begin{cases} \partial_t u - \mathrm{div}\Big(a(t,x)|\nabla u|^{p(t,x)-2}\nabla u\Big) = f(t,x) \in L^1(\Omega_T), \\ u(t,x) = 0 \ \text{ on } \ \partial\Omega, \\ u(0,x) = u_0(x) \in L^1(\Omega) \end{cases}$$

Exponent p can vary around 2 – we only need to have it:
$1 << p(\cdot,\cdot) << \infty$ and log-Hölder continuous in Ω_T

Main examples not covered before

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III. Evolutionary double-phase problem

$$\begin{cases} \partial_t u - \operatorname{div}\Big((|\nabla u|^{p-2} + a(t,x)|\nabla u|^{q-2})\nabla u \Big) = f(t,x) \in L^1(\Omega_T), \\ u(t,x) = 0 \ \text{ on } \ \partial\Omega, \\ u(0,x) = u_0(x) \in L^1(\Omega) \end{cases}$$

We cover it provided $a \in C^{0,\alpha}(\Omega_T)$ and $q/p \leqslant 1 + \alpha/n$
(range is sharp for excluding Lavrentiev's phenomenon)

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IV. Anisotropic problems

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Summary

Modular density of smooth functions

In Musielak-Orlicz-Sobolev spaces the norm is defined by the means of functional

$$\xi \mapsto \int_Z M(z,\xi)\,dz$$

with typical choice of $Z = \Omega$ or $Z = \Omega \times [0, T]$.

Smooth functions are not dense in norm (in general) in the spaces, but in modular topology and only the growth of M with respect to ξ is well balanced with small perturbations of z-variable.

Y. Ahmida, I. Chlebicka, P. Gwiazda, A. Youssfi,
Gossez's approximation theorems in Musielak-Orlicz-Sobolev spaces,
J. Functional Analysis 275 (9) (2018), 2538-2571.

see https://www.mimuw.edu.pl/~ichlebicka/publications

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