Hölder regularity for nonlocal double phase equations

Giampiero Palatucci
(Università di Parma)

Based on a joint work\footnote{http://palatucci.altervista.org} with
Cristiana De Filippis (Università di Torino)

Monday’s Nonstandard Seminar 2020/21

University of Warsaw, Faculty of Mathematics, Informatics and Mechanics

January 25, 2021
H"older regularity for nonlocal double phase equations

\[ \(-\Delta\)_{p}^{s}u + a(x, y)(-\Delta\)_{q}^{t}u = f \]

We deal with a class of possible degenerate and singular integro-differential equations whose leading operator switches between two different types of fractional elliptic phases, according to the zero set of a modulating coefficient \(a=a(\ldots)\).

The model case is driven by

\[
\mathcal{L}u(x) := \int_{\mathbb{R}^{n}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} \, dy + \int_{\mathbb{R}^{n}} a(x, y) \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))}{|x - y|^{n+ tq}} \, dy,
\]

where \( q \geq p \) and \( a(\cdot, \cdot) \geq 0 \).

More in general, we will deal with inhomogeneous equations, for very general classes of measurable kernels.
Non-uniformly elliptic functionals

The nonlocal double phase operator $\mathcal{L}$ can be plainly seen as the nonlocal analog of the classical double phase functional,

$$\mathcal{F}(u) := \int \left( |D u|^p + a(x)|D u|^q \right) \, dx, \quad 1 < p \leq q,$$

introduced by Zhicov in 1986; related to Homogenization, modeling of strongly anisotropic materials, Elasticity, Lavrentiev phenomenon, etc...
Non-uniformly elliptic functionals

The nonlocal double phase operator $\mathcal{L}$ can be plainly seen as the nonlocal analog of the classical double phase functional,

$$\mathcal{F}(u) := \int \left( |Du|^p + |Du|^q \right) \, dx, \quad 1 < p \leq q,$$

introduced by Zhicov in 1986; related to Homogenization, modeling of strongly anisotropic materials, Elasticity, Lavrentiev phenomenon, etc...

From a regularity point of view, even without the presence of the modulating coefficient $a(\cdot)$, such functional presents very interesting features, falling in the class of the non-uniformly elliptic ones having $(p, q)$-growth conditions. Basically, one can prove that

$$\frac{q}{p} < 1 + o(n)$$


First several fundamental contributions on non-uniformly elliptic operators: Hong, Fusco-Sbordone, Leon Simon, Lieberman, Uraltseva-Urdal etova; and more recently Fiscella, Fonseca, Maly, Mingione, Pucci, Radulescu, and many others.
Non-autonomous functionals

\[ \mathcal{F}(u) := \int \left( |Du|^p + a(x)|Du|^q \right) \, dx \]

Because of the modulating coefficient, the functional \( \mathcal{F} \) is the prototype of a bad kind of \text{interplay} between a coefficient in \( x \) and the \((p, q)\)-growth, since it brings a change of ellipticity occurring on the set \( \{ a = 0 \} \):

\begin{itemize}
  \item in the points where \( a > 0 \), \( \mathcal{F} \) reduces to a non-standard \((p, q)\)-growth functional, which exhibits a \( q \)-growth in the gradient (in the relevant case when \( q > p \)).
\end{itemize}
Non-autonomous functionals

\[ \mathcal{F}(u) := \int \left( |Du|^p + a(x)|Du|^q \right) \, dx \]

Because of the modulating coefficient, the functional \( \mathcal{F} \) is the prototype of a bad kind of interplay between a coefficient in \( x \) and the \( (p, q) \)-growth, since it brings a change of ellipticity occurring on the set \( \{ a = 0 \} \):

- **in the points where** \( a > 0 \), \( \mathcal{F} \) reduces to a non-standard \( (p, q) \)-growth functional, which exhibits a \( q \)-growth in the gradient (in the relevant case when \( q > p \)).
Non-autonomous functionals

\[ \mathcal{F}(u) := \int \left( |Du|^p + a(x)|Du|^q \right) \, dx \]

Because of the modulating coefficient, the functional \( \mathcal{F} \) is the prototype of a bad kind of interplay between a coefficient in \( x \) and the \((p, q)\)-growth, since it brings a change of ellipticity occurring on the set \( \{a = 0\} \):

- in the points where \( a > 0 \), \( \mathcal{F} \) reduces to a non-standard \((p, q)\)-growth functional, which exhibits a \( q \)-growth in the gradient (in the relevant case when \( q > p \)).
Because of the modulating coefficient, the functional $\mathcal{F}$ is the prototype of a bad kind of interplay between a coefficient in $x$ and the $(p, q)$-growth, since it brings a change of ellipticity occurring on the set $\{a = 0\}$:

- in the points where $a > 0$, $\mathcal{F}$ reduces to a non-standard $(p, q)$-growth functional, which exhibits a $q$-growth in the gradient (in the relevant case when $q > p$).

- in the points where $a = 0$, $\mathcal{F}$ exhibits a $p$-growth in the gradient.
Non-autonomous functionals

\[ \mathcal{F}(u) := \int \left( |Du|^p + a(x, |Du|^q) \right) \, dx \]

Because of the modulating coefficient, the functional \( \mathcal{F} \) is the prototype of a bad kind of interplay between a coefficient in \( x \) and the (\( p, q \))-growth, since it brings a change of ellipticity occurring on the set \( \{ a = 0 \} \):

- in the points where \( a > 0 \), \( \mathcal{F} \) reduces to a non-standard \( (p, q) \)-growth functional, which exhibits a \( q \)-growth in the gradient (in the relevant case when \( q > p \)).

- in the points where \( a = 0 \), \( \mathcal{F} \) exhibits a \( p \)-growth in the gradient.

Indeed, it was introduced by Zhikov in order to describe strongly anisotropic materials whose hardening properties drastically change with the point: the regulation of the mixture between two different materials, with \( p \) and \( q \) hardening, is modulated by the coefficient \( a(\cdot) \).
Non-autonomous functionals

\[ F(u) := \int \left( |D u|^p + a(x)|D u|^q \right) \, dx \]

Even basic regularity issues for these double phase problems have remained unsolved for the last decades.*
Non-autonomous functionals

\[ \mathcal{F}(u) := \int \left( |Du|^p + a(x)|Du|^q \right) \, dx \]

Even basic regularity issues for these double phase problems have remained unsolved for the last decades.*

The first result in this spirit was recently due to Colombo and Mingione [ARMA 2015, and ARMA 2015]: If the modulating coefficient \( a(\cdot) \) is Hölder continuous, the weak solutions to the double phase equations are Hölder continuous as well, by assuming that \( 1 \leq q/p \leq 1 + \alpha/n \), where \( \alpha \in (0, 1] \) is the Hölder exponent of \( a(\cdot) \).

*A first (counter-)example by Fonseca-Maly-Mingione [ARMA 2004].
Recent developments in the double phase theory (just to name a few…)


- Chlebicka-De Filippis [AMPA 2019]: Removability of the singularities; Obstacle problems.


- De Filippis-Oh [JDE 2019]: Multiphase (different rates of ellipticity with Hölder continuous coefficients).

- De Filippis-Mingione [JDG 2019, JDG 2020]: Manifold constrained problems; Vectorial case and critical systems.

- Hästö-Ok [Preprints 2019]: Maximal regularity for local minimizers; Calderón-Zygmund estimates in Orlicz setting.


We consider the following inhomogeneous nonlocal double phase equation,

\[ \mathcal{L}u = f, \]

where \( f \) is bounded and the integro-differential operator \( \mathcal{L} \) is given by

\[
\begin{align*}
\mathcal{L}u(x) & := P.V. \int_{\mathbb{R}^n} |u(x) - u(x + y)|^{p-2}(u(x) - u(x + y)) K_{sp}(x, y) \, dy \\
& + P.V. \int_{\mathbb{R}^n} a(x, y)|u(x) - u(x + y)|^{q-2}(u(x) - u(x + y)) K_{tq}(x, y) \, dy.
\end{align*}
\]

For \( s, t \in (0, 1) \) and \( p, q > 1 \), the measurable kernels \( K_{sp} \) and \( K_{tq} \) essentially behave like \((s, p)\) and \((t, q)\)-kernels, respectively. More precisely, there exists a positive constant \( \Lambda \) such that

\[
\left\{ \begin{array}{ll}
\Lambda^{-1} |y|^{-n-sp} \leq K_{sp}(x, y) \leq \Lambda |y|^{-n-sp}, \\
K_{sp}(x, y) = K_{sp}(x, -y),
\end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll}
\Lambda^{-1} |y|^{-n-tq} \leq K_{tq}(x, y) \leq \Lambda |y|^{-n-tq}, \\
K_{tq}(x, y) = K_{tq}(x, -y).
\end{array} \right.
\]
Our main result is the following

**Theorem [De Filippis-Palatucci, J. Differential Equations 2019]**

Let $p, q > 1$ be such that

$$p > \frac{1}{1-s} \text{ if } p < 2, \quad q > \frac{1}{1-t},$$

and

$$1 \leq q/p \leq \min\left\{\frac{s}{t}, 1+s\right\},$$

and let $f$ be in $L^\infty(B_2)$.

Assume that the modulating coefficient $a$ is a measurable function such that $0 \leq a(x, y) \leq M$ for a.e. $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. If $u$ is a bounded viscosity solution to

$$Lu = f \text{ in } B_2,$$

then $u \in C^{0,\gamma}(B_1)$ for some $\gamma = \gamma(n, p, q, s, t, M, \Lambda, \|u\|_{L^\infty}, \|f\|_{L^\infty}) \in (0, 1)$.
A few observations are in order

- **local vs nonlocal**: in the local case, regularity results for bounded weak solutions are achieved provided that $1 \leq q/p \leq 1 + \alpha$, with $a \in C^{0,\alpha}$.

In the nonlocal case, assuming only $a(\cdot) \in L^\infty$, we have $1 \leq q/p \leq \min \left\{ \frac{s}{t}, 1 + s \right\}$. 
A few observations are in order

- **local vs nonlocal**: in the local case, regularity results for bounded weak solutions are achieved provided that \( 1 \leq q/p \leq 1 + \alpha \), with \( a \in C^{0,\alpha} \). In the nonlocal case, assuming only \( a(\cdot) \in L^\infty \), we have \( 1 \leq q/p \leq \min \left\{ \frac{s}{t}, 1 + s \right\} \).

In addition, if we assume that \( a(\cdot) \in C^{0,\alpha} \), we have \( 1 \leq q/p \leq 1 + c(\alpha, s, t) \), with \( c \geq \alpha \).
A few observations are in order

- **local vs nonlocal**: in the local case, regularity results for bounded weak solutions are achieved provided that $1 \leq q/p \leq 1 + \alpha$, with $a \in C^{0,\alpha}$. In the nonlocal case, assuming only $a(\cdot) \in L^\infty$, we have $1 \leq q/p \leq \min \{ \frac{s}{t}, 1 + s \}$. In addition, if we assume that $a(\cdot) \in C^{0,\alpha}$, we have $1 \leq q/p \leq 1 + c(\alpha, s, t)$, with $c \geq \alpha$.

- **nonlocal nonlinear nonstandard**: The nonlocal equation inherits both the difficulties newly arising from the double phase problems and those naturally arising from the fractional integro-differential operators.
A few observations are in order

• **local vs nonlocal**: in the local case, regularity results for bounded weak solutions are achieved provided that \( 1 \leq q/p \leq 1 + \alpha \), with \( a \in C^{0,\alpha} \).

In the nonlocal case, assuming only \( a(\cdot) \in L^\infty \), we have \( 1 \leq q/p \leq \min \{ \frac{s}{t}, 1 + s \} \).

In addition, if we assume that \( a(\cdot) \in C^{0,\alpha} \), we have \( 1 \leq q/p \leq 1 + c(\alpha, s, t) \), with \( c \geq \alpha \).

• **nonlocal nonlinear nonstandard**: The nonlocal equation inherits both the difficulties newly arising from the double phase problems and those naturally arising from the fractional integro-differential operators.

• To our knowledge, this is the very first regularity result for solutions to nonlocal double phase equations. Even in the very special case when \( s = t \) and \( p = q \), no related results involving a modulating coefficient could be found in the literature. It is worth mentioning the fine Hölder estimates in a relevant paper by Kassmann, Rang and Schwab [Indiana J. 2014], for elliptic integro-differential operators with kernels satisfying lower bounds on conic subsets, thus strongly directionally dependent.
**Definition [viscosity subsolutions]**

Let $\Omega \subset \mathbb{R}^n$ be an open subset and $\mathcal{L}$ be the nonlocal double phase functional. An upper semicontinuous function $u \in L^\infty_{\text{loc}}(\Omega)$ is a subsolution of $\mathcal{L}(\cdot) = C$ in $\Omega$, and we write

"$u$ is such that $\mathcal{L}(u) \leq C$ in $\Omega$ in the viscosity sense,"

if the following statement holds: whenever $x_0 \in \Omega$ and $\varphi \in C^2(B_\varrho(x_0))$ for some $\varrho > 0$ are so that

$$\varphi(x_0) = u(x_0), \quad \varphi(x) \geq u(x) \text{ for all } x \in B_\varrho(x_0) \subseteq \Omega,$$

then we have $\mathcal{L}\varphi_\varrho(x_0) \leq C$, where

$$\varphi_\varrho := \begin{cases} \varphi & \text{in } B_\varrho(x_0) \\ u & \text{in } \mathbb{R}^n \setminus B_\varrho(x_0). \end{cases}$$

A **viscosity supersolution** is defined in an analogous fashion, and a **viscosity solution** is a function which is both a subsolution and a supersolution.
As soon as we can touch a viscosity subsolution with a $C^2$-function, then it behaves as a classical subsolution.

**Proposition [De Filippis-Palatucci, J. Differential Equations 2019]**

Suppose that $Lu \leq C$ in $B_1$ in the viscosity sense. If $\varphi \in C^2(B_\varrho(x_0))$ is such that

$$
\varphi(x_0) = u(x_0), \quad \varphi(x) \geq u(x) \text{ in } B_\varrho(x_0) \subseteq B_1,
$$

for some $0 < \varrho < 1$, then $Lu$ is defined in the pointwise sense at $x_0$ and $Lu(x_0) \leq C$. 
As soon as we can touch a viscosity subsolution with a \(C^2\)-function, then it behaves as a classical subsolution.

**Proposition** [De Filippis-Palatucci, *J. Differential Equations* 2019]

Suppose that \(Lu \leq C\) in \(B_1\) in the viscosity sense. If \(\varphi \in C^2(B_\varrho(x_0))\) is such that

\[
\varphi(x_0) = u(x_0), \quad \varphi(x) \geq u(x) \text{ in } B_\varrho(x_0) \subseteq B_1,
\]

for some \(0 < \varrho < 1\), then \(Lu\) is defined in the pointwise sense at \(x_0\) and \(Lu(x_0) \leq C\).

**Proof.** We plainly extend to the double phase problems a by-now classical approach, as firstly seen in Caffarelli-Silvestre [CPAM 2009] for fully nonlinear integro-differential operators, and successfully applied even for the fractional \(p\)-Laplace equation by Lindgren [NoDEA 2016]. \(\square\)
Proof of the Hölder continuity result

Basically we extend the the approach of Silvestre in *Indiana J.* (2006), where he shows the Hölder continuity of fractional harmonic functions, via a purely analytical proof which goes back to De Giorgi. Not for free, because of the nonstandard \((p,q)\)-growth, and the zero set of \(a(\cdot, \cdot)\).
Proof of the Hölder continuity result

Basically we extend the approach of Silvestre in *Indiana J.* (2006), where he shows the Hölder continuity of fractional harmonic functions, via a purely analytical proof which goes back to De Giorgi. Not for free, because of the nonstandard \((p, q)\)-growth, and the zero set of \(a(\cdot, \cdot)\).

**Sketch of the proof (be aware: lots of cheating)**

For \(\sigma > 0\), let \(\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_{\sigma}\) be a suitable scaling of our functional, and let \(\varphi\) be any radial map which is \(C^2\)-regular, vanishes outside \(B_1\), and it is non-increasing along rays from the origin.

**Step 1 (controlling the energy of smooth maps).**

\[\forall \varepsilon > 0 \ \exists \kappa \in (0, 1/2] \text{ such that } \tilde{\mathcal{L}}\varphi \lesssim \varepsilon \sigma / \kappa^{q-1},\]

which can be check by computations and usual fractional estimates.
Proof of the Hölder continuity result

Basically we extend the approach of Silvestre in Indiana J. (2006), where he shows the Hölder continuity of fractional harmonic functions, via a purely analytical proof which goes back to De Giorgi. Not for free, because of the nonstandard \((p,q)\)-growth, and the zero set of \(a(\cdot, \cdot)\).

Sketch of the proof (be aware: lots of cheating)

For \(\sigma > 0\), let \(\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_\sigma\) be a suitable scaling of our functional, and let \(\varphi\) be any radial map which is \(C^2\)-regular, vanishes outside \(B_1\), and it is non-increasing along rays from the origin.

Step 1 (controlling the energy of smooth maps).

\(\forall \varepsilon > 0 \exists \kappa \in (0, 1/2]\) such that \(\tilde{\mathcal{L}}\varphi \lesssim \varepsilon \sigma / \kappa^{q-1}\),

which can be check by computations and usual fractional estimates

Step 2 (refining).

If \(u\) is such that \(|B_1 \cap \{u(x) < 0\}| > 0\) and \(\left\{\begin{array}{l}
\tilde{\mathcal{L}}u \leq \sigma \text{ in } B_1 \\
u \leq 1 \text{ in } B_1,
\end{array}\right.\)

then \(u \leq 1 - \kappa\) in \(B_{1/2}\),

which can be proven by working on the function \(u + \kappa \varphi\) thanks to Step 1.
Proof of the Hölder continuity result

Basically we extend the approach of Silvestre in *Indiana J.* (2006), where he shows the Hölder continuity of fractional harmonic functions, via a purely analytical proof which goes back to De Giorgi. Not for free, because of the nonstandard \((p, q)\)-growth, and the zero set of \(a(\cdot, \cdot)\).

**Sketch of the proof** (*be aware: lots of cheating*)

For \(\sigma > 0\), let \(\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_\sigma\) be a suitable scaling of our functional, and let \(\varphi\) be any radial map which is \(C^2\)-regular, vanishes outside \(B_1\), and it is non-increasing along rays from the origin.

**Step 1** (controlling the energy of smooth maps).

\(\forall \varepsilon > 0 \, \exists \kappa \in (0, 1/2]\) such that \(\tilde{\mathcal{L}}\varphi \lesssim \varepsilon \sigma / \kappa^{q-1}\),

which can be check by computations and usual fractional estimates

**Step 2** (refining).

If \(u\) is such that \(|B_1 \cap \{u(x) < 0\}| > 0\) and \(\left\{ \begin{array}{l} \tilde{\mathcal{L}}u \leq \sigma \text{ in } B_1 \\ u \leq 1 \text{ in } B_1, \end{array} \right.\)

then \(u \leq 1 - \kappa\) in \(B_{1/2}\),

which can be proven by working on the function \(u + \kappa \varphi\) thanks to Step 1.

**Step 3** (iterating).

Let \(\tilde{u} := \left( \frac{1}{\|u\|_{L^\infty} + \|f\|_{L^\infty(B_2)} / \sigma^{1/(p-1)}} \right) u\). We have \(\text{osc } \tilde{u} < 1\) and \(\tilde{\mathcal{L}}\tilde{u} = \tilde{f}\), for a suitable \(\tilde{f}\).

By suitably choosing \(\varepsilon\) and \(\sigma\) in the previous steps, we can start an iteration, to get

\[
\text{osc }_{B_\varepsilon(x_0)} u \leq c(\text{data}) \left( \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^{p-1}(B_2)} \right)^{\varphi'},
\]

for some \(\gamma = \gamma(\text{data}) \in (0, 1]\), which implies, by covering, \(u \in C^{0, \gamma}(B_1)\), as desired. \(\square\)
Further clarification (scaling effects on nonlocal double phase equations)

Let \( u \in L^\infty(\mathbb{R}^n) \) be a viscosity solution to \( \mathcal{L}u = f \).
We rescale and blow \( u \) around \( x_0 \in B_1 \) as follows. For \( \lambda, \mu > 0 \) and \( x \in B_1 \), we define the map

\[ u_{\mu, x_0}^{(\lambda)}(x) := \lambda u(\mu x + x_0). \]

Such a function satisfies

\[ \hat{\mathcal{L}}u_{\mu, x_0}^{(\lambda)}(x) := \hat{f}(x) \text{ in } B_1, \]

where

\[
\hat{\mathcal{L}}v(x) := \int_{\mathbb{R}^n} |v(x) - v(x + y)|^{p-2}(v(x) - v(x + y)) \hat{K}_{sp}(x, y) \, dy \\
+ \int_{\mathbb{R}^n} \hat{a}(x, y)|v(x) - v(x + y)|^{q-2}(v(x) - v(x + y)) \hat{K}_{tq}(x, y) \, dy
\]

and

\[ \hat{f}(x) := \lambda^{p-1} \mu^{sp} f(\mu x + x_0). \]

The modulating coefficient and the kernels appearing above are defined as

\[ \hat{a}(x, y) := \lambda^{p-q} \mu^{sp-tq} a(\mu x + x_0, \mu y) \]

and

\[
\begin{cases} 
\hat{K}_{sp}(x, y) := \mu^{n+sp} K_{sp}(\mu x + x_0, \mu y) \\
\hat{K}_{tq}(x, y) := \mu^{n+tg} K_{tq}(\mu x + x_0, \mu y)
\end{cases}
\]

respectively.
Related open problems

- Whether or not, and under which assumptions on the structural quantities, the viscosity solutions to nonlocal double phase equations are indeed **fractional harmonic functions** and/or weak solutions, and vice versa (see, e. g. [Korvempaa-Kuusi-Lindgren, JMPA 2019] for the fractional $p$-Laplace equation).
Related open problems

• Whether or not, and under which assumptions on the structural quantities, the viscosity solutions to nonlocal double phase equations are indeed fractional harmonic functions and/or weak solutions, and vice versa (see, e. g. [Korvempaa-Kuusi-Lindgren, JMPA 2019] for the fractional $p$-Laplace equation).

• In the same spirit of Baroni-Colombo-Mingione [Nonlinear Anal. 2015, Calc. Var. PDE 2018], one would expect higher differentiability and regularity results for the bounded solutions to nonlocal double phase equations (see, e. g., Brasco-Lindgren-Schikorra [Adv. Math. 2018] for the fractional $p$-Laplace equation).

First relevant results for bounded weak solutions, for the pure fractional double-phase equations when $q$ and $p$ are greater or equal than 2, by Mengesha-Scott [Preprint arXiv, December 2020].
Related open problems

• Whether or not, and under which assumptions on the structural quantities, the viscosity solutions to nonlocal double phase equations are indeed **fractional harmonic functions** and/or weak solutions, and vice versa (see, e. g. [Korvempaa-Kuusi-Lindgren, *JMPA* 2019] for the fractional $p$-Laplace equation).

• In the same spirit of Baroni-Colombo-Mingione [*Nonlinear Anal.* 2015, *Calc. Var. PDE* 2018], one would expect **higher differentiability** and regularity results for the bounded solutions to nonlocal double phase equations (see, e. g., Brasco-Lindgren-Schikorra [*Adv. Math.* 2018] for the fractional $p$-Laplace equation). First relevant results for bounded weak solutions, for the pure fractional double-phase equations when $q$ and $p$ are greater or equal than 2, by Mengesha-Scott [*Preprint arXiv*, December 2020].

• **Harnack-type inequalities.** Preliminary results for weak supersolutions have been proven in De Filippis-Palatucci [*Preprint 2021*], namely by dealing with the resulting error term as a right hand-side (a nonlocal tail), and proving local Boundedness, a Caccioppoli Inequality with tail, and a weak Harnack, in the same flavour of the works by Brasco, Chen, Kassmann, Kuusi, Iannizzotto, Lindgren, Silvestre, Squassina, et Al. (that is, in the spirit of the De Giorgi-Nash-Moser theory).
Related open problems

• Whether or not, and under which assumptions on the structural quantities, the viscosity solutions to nonlocal double phase equations are indeed fractional harmonic functions and/or weak solutions, and vice versa (see, e. g. [Korvempaa-Kuusi-Lindgren, JMPA 2019] for the fractional $p$-Laplace equation).

• In the same spirit of Baroni-Colombo-Mingione [Nonlinear Anal. 2015, Calc. Var. PDE 2018], one would expect higher differentiability and regularity results for the bounded solutions to nonlocal double phase equations (see, e. g., Brasco-Lindgren-Schikorra [Adv. Math. 2018] for the fractional $p$-Laplace equation). First relevant results for bounded weak solutions, for the pure fractional double-phase equations when $q$ and $p$ are greater or equal than 2, by Mengesha-Scott [Preprint arXiv, December 2020].

• Harnack-type inequalities. Preliminary results for weak supersolutions have been proven in De Filippis-Palatucci [Preprint 2021], namely by dealing with the resulting error term as a right hand-side (a nonlocal tail), and proving local Boundedness, a Caccioppoli Inequality with tail, and a weak Harnack, in the same flavour of the works by Brasco, Chen, Kassmann, Kuusi, Iannizzotto, Lindgren, Silvestre, Squassina, et Al. (that is, in the spirit of the De Giorgi-Nash-Moser theory).

• Both in the local and in the nonlocal double phase theory, nothing is known about the regularity for solutions to parabolic double phase equations.
Hölder regularity
for nonlocal double phase equations
Giampiero Palatucci

University of Warsaw, Faculty of Mathematics, Informatics and Mechanics

January 25, 2021
Hölder regularity for nonlocal double phase equations
Giampiero Palatucci

thank you

University of Warsaw, Faculty of Mathematics, Informatics and Mechanics
January 25, 2021