A modular Poincaré inequality

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Motivation

Dealing with capacities often implies dealing with fuctionals more general than norms

$\Downarrow$

we constructed a modular capacity theory
and
introduced two modular capacities

A comparison between the zero capacity sets with respect to the two different notions was obtained through a modular Poincaré inequality

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The setting

$\Omega \subset \mathbb{R}^n$ open set,

$$\mathcal{M}(\Omega) = \{f : \Omega \to \mathbb{R}, \text{ measurable w.r.t Lebesgue measure}\}$$

Given a mapping $\rho_X(\cdot) : \mathcal{M}(\Omega) \to [0, \infty]$, the set

$$X(\Omega) = \{u \in \mathcal{M}(\Omega) : \rho_X(u) < \infty\},$$

is a modular function space over Ω if the pair $(X(\Omega), \rho_X)$ satisfies the following properties:

- *i* $\rho_X(u) = \rho_X(|u|)$ and $\rho_X(u) = 0$ if and only if $u \equiv 0$
- *ii* $|u| \leq |v|$ a.e. $\Rightarrow \rho_X(u) \leq \rho_X(v)$
- *iii* $\rho_X(u + v) \leq \rho_X(u) + \rho_X(v) \qquad \forall u, v \ : \ uv \equiv 0$
- *iv* if $E \subset \Omega$ is measurable set and $|E| < \infty$, then $\rho_X(\chi_E) < \infty$
- *v* $|u_j| \uparrow |u|$ a.e. $\Rightarrow \ \rho_X(u_j) \uparrow \rho_X(u)$
- *vi* $\forall k > 1 \exists c_k > 1 \ : \rho_X(ku) \leq c_k \rho_X(u)$

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AS A CONSEQUENCE of the previous properties we have the following type-convexity of ρ_X:

$$\rho_X(\alpha u + \beta v) \leq \rho_X(u) + \rho_X(v) \qquad \forall \alpha, \beta \geq 0\,, \alpha + \beta = 1.$$

Examples of modular

EXAMPLE 1

Consider

$$\rho_X(u) = \|u\|_X$$

where $X(\Omega)$ is a Banach function space in the sense given by Bennett & Sharpley.

In particular, it holds for

$$\rho_X(u) = \|u\|_A$$

where $X(\Omega) = L^A(\Omega)$ is an Orlicz space

NOTE that it is not required that A satisfies the Δ_2 condition, but rather that ρ_X has to satisfy property ***vi***.

$$\left(\|u\|_A = \inf \left\{ \lambda > 0 : \int_\Omega A\left(\frac{|u|}{\lambda}\right) \leq 1 \right\} \right)$$

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EXAMPLE 2

Consider

$$\rho_X(u) = \int_\Omega A(|u|)\,dx$$

where $A : [0, \infty[\to [0, \infty[$ is a Young function (i.e. an increasing, continuous, convex, and such that $A(0) = 0$, $A(t) > 0$ for $t > 0$) satisfying the Δ_2 condition.

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EXAMPLE 3

Consider

$$\rho_X(u) = \int_\Omega P(|u|)\,dx$$

where $P : [0, \infty[\to [0, \infty[$ is increasing, continuous, strictly CONCAVE, unbounded and such that $P(0) = 0$.

(HERE ρ_X is not convex!)

Some definitions

We consider the generalized Sobolev space defined as

$$W^1X(\Omega) = \{u \text{ weakly differentiable } : u \in X(\Omega) \text{ and } |\nabla u| \in X(\Omega)\}$$

$u_j \to u$ with respect to the ρ_X-convergence in $W^1X(\Omega)$

$$\Longleftrightarrow$$

$$\rho_X(u_j) \to \rho_X(u)\ , \quad \rho_X(|\nabla u_j|) \to \rho_X(|\nabla u|)\,.$$

REMARK In the case ρ_X is the norm in a Banach function space X, the convergence in norm of the function and of its gradient is not equivalent to the ρ_X-convergence, but the former implies the latter.

Statement of the result

THEOREM (Modular Poincaré inequality)

$(X(\Omega), \rho_X)$, $(Y(\Omega), \rho_Y)$, and $(Z(\Omega), \rho_Z)$ are such that

$$\rho_Z(u) \le c(Y,Z)\, \rho_Y(|\nabla u|) \quad u \in \overline{C_0^\infty(\Omega)}^{W^1X(\Omega)}.$$

If for some strictly increasing function $\varphi : [0,\infty[\to [0,\infty[$ such that $\varphi^{-1} \in \Delta_2$, it is

$$c_1 \rho_Y(f) \le \varphi(\rho_X(f)) \le c_2 \rho_Z(f) \quad f \in Z(\Omega),$$

$$\Downarrow$$

$$\rho_X(u) \le c_{\varphi^{-1}}(c_1\, c_2\, c(Y,Z))\, \rho_X(|\nabla u|) \quad u \in \overline{C_0^\infty(\Omega)}^{W^1X(\Omega)}.$$

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Actually the proof is a straightforward consequence of the assumptions but examples and comments in the sequel will show:

- the differences with the current results in literature
- we unify and extend known results

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Remarks

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Actually, we underline that a weak form of the Sobolev estimate is sufficient and we don't need the full gain of summability established by the Sobolev inequality: even if the gain is not optimal therein, an improvement of the exponent exists and this is sufficient. (as it happens e.g in *Kovácik-Rákosník:Czechoslovak Math. J (1991)* , in the variable exponent case).

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REMARK 2 It is well known that the classical Poincaré inequality does not hold, in general, for unbounded Ω, the typical counterexample being $\Omega = \mathbb{R}^n$. Unbounded Ω are allowed: Ω can be bounded in one direction or the measure of $\Omega \cap (\mathbb{R}^n \setminus B_R)$ tends sufficiently fast to zero as $R \to \infty$.
(see Frehse:Jahresber.Deutsch.Math.Verein.(1982)).
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REMARK 3 Note that for given $(Y(\Omega), \rho_Y)$ and $(Z(\Omega), \rho_Z)$, choosing in a suitable way $(X(\Omega), \rho_X)$, we may get two "endpoint" cases of our Poincaré inequality: assume the Sobolev inequality

$$\rho_Z(u) \leq c(Y, Z)\, \rho_Y(|\nabla u|),$$

and assume the second inequality

- for $(X(\Omega), \rho_X) = (Y(\Omega), \rho_Y)$, $c_1 = 1$, $\varphi(t) = t$, that is $\rho_Y(u) \leq c_2 \rho_Z(u)$, therefore we get

$$\rho_Y(u) \leq c\, \rho_Y(|\nabla u|) \quad u \in \overline{C_0^\infty(\Omega)}^{W^1 Y(\Omega)} \tag{1}$$

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PRECISELY

Poincaré inequalities (1) and (2) involve respectively the domain space and the target space of the Sobolev inequality

$$\rho_Z(u) \leq c(Y, Z)\, \rho_Y(|\nabla u|)$$

while the Poincaré inequality in the theorem

$$\rho_X(u) \leq c_{\varphi^{-1}}(c_1\, c_2\, c(Y, Z))\, \rho_X(|\nabla u|)$$

involves an intermediate space $X(\Omega)$ (see also next Examples 4, 5).

Examples

EXAMPLE 1

$\Omega \subset \mathbb{R}^n$ open, $|\Omega| < \infty$, $1 \leq p < n$.

$(Y, \rho_Y) = (L^p(\Omega), \| \cdot \|_p)$, $\quad (Z, \rho_Z) = (L^{p^*}(\Omega), \| \cdot \|_{p^*})$, $p^* = np/(n-p)$

$$(X, \rho_X) = \left(L^A(\Omega), \int_\Omega A(\cdot)\, dx \right)$$

where A is a Young function such that

$$pA(t) \leq tA'(t) \leq p^* A(t), t \geq 0.$$

$\Rightarrow$ the Sobolev inequality $\rho_Z(u) \leq c(Y, Z)\, \rho_Y(|\nabla u|)$ obviously holds and the second inequality $c_1 \rho_Y(f) \leq \varphi(\rho_X(f)) \leq c_2 \rho_Z(f)$ is satisfied choosing $\varphi(t) = A^{-1}(t)$ (see Fiorenza: NonlinearAnal. (1991))

Hence

$$\int_\Omega A(u)dx \leq c_A \int_\Omega A(|\nabla u|)dx$$

for every u in the closure of $C_0^\infty(\Omega)$ with respect to the ρ_X-convergence in $W^1L^A(\Omega)$.

REMARK Obviously the same inequality holds also for u in the closure of $C_0^\infty(\Omega)$ with respect to the norm convergence in $W^{1,A}(\Omega)$, since the convergence in norm implies the ρ_X-convergence.

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EXAMPLE 2 $\Omega \subset \mathbb{R}^n$, $n \geq 2$ open,

$$(Y, \rho_Y) = (L^n(\Omega), \| \cdot \|_n), \qquad (Z, \rho_Z) = \left(L^A(\Omega), \| \cdot \|_A \right)$$

$$(X, \rho_X) = \left(L^A(\Omega), \| \cdot \|_A \right) \quad \text{with} \quad A(t) = exp(t^{n/(n-1)}) - 1$$

$\Rightarrow$ the Sobolev inequality $\rho_Z(u) \leq c(Y, Z)\, \rho_Y(|\nabla u|)$ holds (see Trudinger:J.Math.Mech., (1967)) and the second inequality $c_1 \rho_Y(f) \leq \varphi(\rho_X(f)) \leq c_2 \rho_Z(f)$ is satisfied choosing

$$\varphi(t) = t, \qquad c_2 = 1$$

and observing that

$$L^A(\Omega) \subset L^n(\Omega)$$

since $A(t)$ dominates $B(t) = t^n$ globally
(see Adams-Fournier: Sobolev Spaces, (2003)).

Hence

$$\|u\|_A \leq c\,\|\nabla u\|_A$$

for every u in the closure of $C_0^\infty(\Omega)$ with respect to the ρ_X-convergence in $W^1L^A(\Omega)$.

REMARK As in the previous example, we note that the same inequality holds also for u in the closure of $C_0^\infty(\Omega)$ with respect to the norm convergence in $W^{1,A}(\Omega)$.

Hence

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REMARK As in the previous example, we note that the same inequality holds also for u in the closure of $C_0^\infty(\Omega)$ with respect to the norm convergence in $W^{1,A}(\Omega)$.

REMARK Obviously, if $L^B(\Omega)$ is any Orlicz space such that

$$L^A(\Omega) \subset L^B(\Omega) \subset L^n(\Omega)\,,$$

choosing $(X, \rho_X) = \big(L^B(\Omega), \|\cdot\|_B\big)$, we have also

$$\|u\|_B \leq c\,\|\nabla u\|_A$$

for every u in the closure of $C_0^\infty(\Omega)$ with respect to the ρ_X-convergence in $W^1L^B(\Omega)$. We note that the same inequality holds also for u in the closure of $C_0^\infty(\Omega)$ with respect to the norm convergence in $W^{1,B}(\Omega)$, and of course also for u in the closure of $C_0^\infty(\Omega)$ with respect to the norm convergence in $W^{1,A}(\Omega)$.

EXAMPLE 3

A a Young function satisfying

$$\int_0 \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt < \infty, \qquad n \geq 2,$$

Setting

$$H(r) = \left(\int_0^r \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt\right)^{\frac{n-1}{n}} \qquad r \geq 0$$

$$A_n = A \circ H^{-1}$$

where H^{-1} is the left-continuous inverse of H, the optimal Sobolev inequality

$$\|u\|_{L^{A_n}(\mathbb{R}^n)} \leq c\|\nabla u\|_{L^A(\mathbb{R}^n)}$$

holds (see Cianchi (1999)). Setting $B(t) = \max\{A_n(t), A(t)\}$, $t \geq 0$, obviously $B(t)$ dominates $A_n(t)$ globally.

$\Rightarrow$ the Sobolev inequality

$$\rho_Z(u) \leq c(Y,Z)\,\rho_Y(|\nabla u|) \quad u \in \overline{C_0^\infty(\mathbb{R}^n)}^{W^1X(\mathbb{R}^n)},$$

holds with $(Y,\rho_Y) = \left(L^A(\mathbb{R}^n), \|\cdot\|_A\right)$, $(Z,\rho_Z) = \left(L^B(\mathbb{R}^n), \|\cdot\|_B\right)$ (see Adams-Fournier: Sobolev Spaces, (2003)). Moreover, the second inequality $c_1\rho_Y(f) \leq \varphi(\rho_X(f)) \leq c_2\rho_Z(f)$ is satisfied choosing

$$(X,\rho_X) = \left(L^B(\mathbb{R}^n), \|\cdot\|_B\right), \; \varphi(t) = t, \; c_2 = 1$$

It follows

$$\|u\|_B \leq c\,\|\nabla u\|_B$$

for every u in the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the ρ_X-convergence in $W^1L^B(\mathbb{R}^n)$. We note that the same inequality holds also for u in the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm convergence in $W^{1,B}(\mathbb{R}^n)$.

EXAMPLE 4 $\Omega \subset \mathbb{R}^n$ open, $|\Omega| < \infty$,
$p(\cdot) : \Omega \to [1, p_+]$, $1 \le p_+ < n$ $\quad p^*(\cdot) = \frac{np(\cdot)}{n-p(\cdot)}$

$(Y, \rho_Y) = \left(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}\right)$, $(Z, \rho_Z) = \left(L^{p^*(\cdot)}(\Omega), \|\cdot\|_{p^*(\cdot)}\right)$

$$\Downarrow$$

the Sobolev inequality $\rho_Z(u) \le c(Y, Z)\, \rho_Y(|\nabla u|)$ holds provided the maximal operator is bounded on $L^{(p^*(\cdot)/n')'}(\Omega)$, $n' = \frac{n}{n-1}$ (see e.g. Cruz-Uribe-Fiorenza:Variable Lebesgue spaces, (2013)).
Moreover, the second inequality $c_1\rho_Y(f) \le \varphi(\rho_X(f)) \le c_2\rho_Z(f)$ is satisfied choosing

$$(X, \rho_X) = \left(L^{p^*(\cdot)}(\Omega), \|\cdot\|_{p^*(\cdot)}\right), \; \varphi(t) = t, \; c_2 = 1$$

and observing that $L^{p^*(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$.

Hence

$$\|u\|_{p^*(\cdot)} \leq c\,\|\nabla u\|_{p^*(\cdot)}$$

for every u in the closure of $C_0^\infty(\Omega)$ with respect to the the ρ_X-convergence in $W^1L^{p^*(\cdot)}(\Omega)$. We note that the same inequality holds also for u in the closure of $C_0^\infty(\Omega)$ with respect to the norm convergence in $W^{1,p^*(\cdot)}(\Omega)$.

$$\left(\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|u(x)|}{\lambda}\right)^{p(x)} dx \leq 1\right\}\right)$$

REMARK Obviously, if $L^{q(\cdot)}(\Omega)$ is such that

$$L^{p^*(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)\,,$$

choosing $(X, \rho_X) = \big(L^{q(\cdot)}(\Omega), \|\cdot\|_{q(\cdot)}\big)$, we have also

$$\|u\|_{q(\cdot)} \leq c\,\|\nabla u\|_{p^*(\cdot)}$$

for every u in the closure of $C_0^\infty(\Omega)$ with respect to the ρ_X-convergence in $W^1L^{q(\cdot)}(\Omega)$. We note that the same inequality holds also for u in the closure of $C_0^\infty(\Omega)$ with respect to the norm convergence in $W^{1,p^*(\cdot)}(\Omega)$.

EXAMPLE 5 $\Omega \subset \mathbb{R}^n$ open, $|\Omega| < \infty$,
$p(\cdot) : \Omega \to [1, p_+]$, $1 \leq p_+ < n$.
Suppose that the maximal operator is bounded on $L^{(p^*(\cdot)/n')'}(\Omega)$,
$p^*(\cdot) = np(\cdot)/(n - p(\cdot))$, $n' = n/(n-1)$.
$(Y, \rho_Y) = \left(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}\right)$, $(Z, \rho_Z) = \left(L^{p^*(\cdot)}(\Omega), \|\cdot\|_{p^*(\cdot)}\right)$,

$$\Downarrow$$

the Sobolev inequality

$$\rho_Z(u) \leq c(Y, Z)\, \rho_Y(|\nabla u|)\,,$$

holds (see e.g. Cruz-Uribe-Fiorenza:Variable Lebesgue spaces, (2013)). Moreover, setting $(X, \rho_X) = \left(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}\right)$, $\varphi(t) = t$, $c_1 = 1$, the second inequality $c_1\rho_Y(f) \leq \varphi(\rho_X(f)) \leq c_2\rho_Z(f)$ is satisfied observing that $L^{p^*(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$.

Hence

$$\|u\|_{p(\cdot)} \leq c\,\|\nabla u\|_{p(\cdot)}$$

for every u in the closure of $C_0^\infty(\Omega)$ with respect to the ρ_X-convergence in $W^1L^{p(\cdot)}(\Omega)$. We note that the same inequality holds also for u in the closure of $C_0^\infty(\Omega)$ with respect to the norm convergence in $W^{1,p(\cdot)}(\Omega)$.

EXAMPLE 6 $\Omega \subset \mathbb{R}^n$ bounded, $n \geq 2$,
$1 < q < \infty$, $A(t) = exp(t^{q/(q-1)}) - 1$.
$(Y, \rho_Y) = \left(L^{n,q}(\Omega), \|\cdot\|_{L^{n,q}(\Omega)}\right)$, $(Z, \rho_Z) = \left(L^A(\Omega), \|\cdot\|_A\right)$,

$$\Downarrow$$

the Sobolev inequality

$$\rho_Z(u) \leq c(Y, Z)\, \rho_Y(|\nabla u|)\,,$$

holds (see Brezis-Wainger: Commun. Partial Differ. Equ., (1980); Alvino-Trombetti-Lions: Nonlinear Anal.(1989)).

Moreover, setting $(X, \rho_X) = \left(L^{n,q}(\Omega), \|\cdot\|_{L^{n,q}(\Omega)}\right)$, $\varphi(t) = t$, the second inequality $c_1\rho_Y(f) \leq \varphi(\rho_X(f)) \leq c_2\rho_Z(f)$ is satisfied observing that $L^{n,q}(\Omega) \subset L^A(\Omega)$.

Hence

$$\|u\|_{L^{n,q}(\Omega)} \leq c \, \|\nabla u\|_{L^{n,q}(\Omega)}$$

for every u in the closure of $C_0^\infty(\Omega)$ with respect to the ρ_X-convergence in $W^1 L^{n,q}(\Omega)$.

$$\left(\|f\|_{n,q}^q = n \int_0^{+\infty} |\Omega_t|^{\frac{q}{n}} t^{q-1} \, dt < \infty \quad |\Omega_t| = \{x \in \Omega : |f(x)| > t\} \, , \; t \geq 0 \right)$$

THANK YOU FOR THE ATTENTION!