On the globally Lipschitz minimizers to variational problems

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Problem formulation

- let Ω be a regular bounded domain in $\mathbb{R}^d$, $d \geq 2$
- we study existence of a minimizer $u \in u_0 + W_0^{1,1}(\Omega)$ to

$$\int_\Omega F(|\nabla u|)\,\mathrm{d}x \leq \int_\Omega F(|\nabla u_0 + \nabla\varphi|)\,\mathrm{d}x \tag{F}$$

  for all $\varphi \in \mathcal{D}(\Omega)$ and F strictly convex having linear growth
- existence of minimizer to (F) $\Leftrightarrow$ existence of (unique) solution $u \in W^{1,1}(\Omega)$ to

$$\begin{aligned} -\operatorname{div}(a(|\nabla u|)\nabla u) &= 0 \quad \text{in } \Omega, \\ u &= u_0 \quad \text{on } \partial\Omega \end{aligned} \tag{a}$$

  for $s \mapsto a(s)s$ increasing bounded function, where F and a are connected via $\mathbf{F'(s) = a(s)s}$ for all $s \in \mathbb{R}^+$
- goal: characterize integrands F (or coefficient functions a) in terms of properties only such that the minimization (or Dirichlet) problem admits a regular solution which attains the trace for any regular domain Ω and regular boundary values u_0

Prototypic coefficient functions

- special case: for $p > 0$ and $s \in \mathbb{R}^+$, consider $a(s) = a_p(s) := \frac{1}{(1+s^p)^{\frac{1}{p}}}$
- $p = 2$ represents the minimal surface problem, i.e.

$$-\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \quad \text{in } \Omega, \quad u \quad = u_0 \quad \text{on } \partial\Omega.$$

- result by Finn (1965) - if Ω is not (at least) pseudoconvex then there always exist smooth boundary data u_0 for which the problem does not admit a minimizer (solution)
- result by Miranda (1971) - for any Ω locally pseudoconvex and any u_0 continuous there exists a unique classical solution $u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$
- however, the solution in $BV(\Omega)$ exists for any domain and $u_0 \in L^1(\partial\Omega)$ (this is not of our interest because the trace may not be attained)
- we want to characterize the functions a_p (in terms of p) in such way that the geometry of the domain does not play role anymore, only regularity

Continuum mechanics motivation

- deformation of the body $\Omega \subset \mathbb{R}^d$ ($d = 3$) with $\Gamma_D \cap \Gamma_N = \emptyset$, $\overline{\Gamma_D \cup \Gamma_N} = \partial\Omega$

$$-\operatorname{div}\mathbb{T} = \boldsymbol{f} \quad \text{in } \Omega, \quad \boldsymbol{u} \quad = \boldsymbol{u}_0 \quad \text{on } \Gamma_D, \quad \mathbb{T}\boldsymbol{n} = \boldsymbol{g} \quad \text{on } \Gamma_N, \tag{1}$$

  where $\boldsymbol{u}$ is displacement, $\mathbb{T}$ the Cauchy stress tensor, $\boldsymbol{f}$ the external body forces, $\boldsymbol{g}$ the external surface forces
- $\boldsymbol{\varepsilon}$ is the linearized strain tensor, i.e., $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$
- under the assumption $|\nabla u| \ll 1$, in the constitutive relation for the Cauchy stress we can replace the full strain tensor by the linearised strain tensor
- model suggested by Rajagopal (R., Walton 2011; Kulvait, Málek, R. 2013): constitutive relation between the Cauchy stress tensor and the strain

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^*(\mathbb{T}), \quad \text{where} \quad \boldsymbol{\varepsilon}^*(\mathbb{T}) := \frac{\mathbb{T}}{(1 + |\mathbb{T}|^p)^{\frac{1}{p}}}$$

  for $p > 0$ (admits $|\boldsymbol{\varepsilon}(\boldsymbol{u})| \ll 1$ and $|\mathbb{T}| \gg 1$ at the same time)

Special geometry

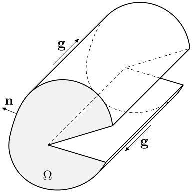

Figure: Anti-plane stress geometry.

- in the formulated problem (1), let

$$\boldsymbol{f}(\boldsymbol{x}) \equiv \mathbf{0} \quad \text{and} \quad \boldsymbol{g}(\boldsymbol{x}) = (0, 0, g(x_1, x_2))$$

be given, we look for $\boldsymbol{u}$, $\mathbb{T}$ of the form

$$\boldsymbol{u}(\boldsymbol{x}) = (0, 0, u(x_1, x_2)),$$

$$\mathbb{T}(\boldsymbol{x}) = \begin{pmatrix} 0 & 0 & T_{13}(x_1, x_2) \\ 0 & 0 & T_{23}(x_1, x_2) \\ T_{13}(x_1, x_2) & T_{23}(x_1, x_2) & 0 \end{pmatrix}$$

Equivalent reformulation

- find $U : \Omega \to \mathbb{R}$, $U(\boldsymbol{x}) = U(x_1, x_2)$ such that $T_{13} = \frac{1}{\sqrt{2}} U_{x_2}$ and $T_{23} = -\frac{1}{\sqrt{2}} U_{x_1}$, then $\operatorname{div} \mathbb{T} = \mathbf{0}$ is fulfiled
- on simply connected domain, U must satisfy (using $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{\mathbb{T}}{(1+|\mathbb{T}|^p)^{1/p}}$)

$$
\begin{aligned}
-\operatorname{div}\left(\frac{\nabla U}{(1+|\nabla U|^p)^{\frac{1}{p}}}\right) &= 0 \qquad \text{in } \Omega, \\
U_{x_2}\boldsymbol{n}_1 - U_{x_1}\boldsymbol{n}_2 = \sqrt{2}g \quad &\text{on } \Gamma_N.
\end{aligned}
$$

- Neumann boundary condition includes the tangential derivative of U,

$$
\nabla U \cdot \tau = (U_{x_1}, U_{x_2}) \cdot (-\boldsymbol{n}_2, \boldsymbol{n}_1) = \sqrt{2}g
$$

- assume that Γ_N is parameterized by a curve $\gamma(s)$, then

$$
U(\gamma(\tau)) = U(\gamma(0)) + \sqrt{2}\int_0^\tau g(\gamma(s))\, |\gamma'(s)|\, \mathrm{d}s =: U_0(\boldsymbol{x})
$$

for $\boldsymbol{x} = \gamma(\tau)$ makes it a Dirichlet problem

By means of U

- we look for $U \in W^{1,1}(\Omega)$, a weak solution to

$$-\operatorname{div}\left(\frac{\nabla U}{(1+|\nabla U|^p)^{\frac{1}{p}}}\right) = 0 \quad \text{in } \Omega, \qquad U = U_0 \quad \text{on } \partial\Omega \tag{a_p}$$

- this is precisely the original problem (a) formulated for the coefficient function $a_p(s) := (1+s^p)^{-\frac{1}{p}}$ for $p > 0$
- there are some positive results by Bulíček, Málek, Rajagopal, Walton (2015), the weak solution exists:
  - ✓ for $p \in (0, \infty)$ and Ω Lipschitz uniformly convex,
  - ✓ for $p \in (0, 2)$ and Ω Lipschitz piece-wise uniformly convex

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  - ✓ for $p \in (0,\infty)$ and Ω Lipschitz uniformly convex,
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  - **Q: for $p \in ?$ and Ω regular**

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  - ✓ for $p \in (0, \infty)$ and Ω Lipschitz uniformly convex,
  - ✓ for $p \in (0, 2)$ and Ω Lipschitz piece-wise uniformly convex,
  - ✓ **for $p \in (0, 1]$ and $\Omega \in \mathcal{C}^1$ satisfying exterior ball condition**

(non)Existence of solution on an annulus

- consider problem (a_p) on domain $B_R \setminus B_r \subset \mathbb{R}^d$, $0 < r < R < \infty$,

$$\begin{aligned} -\text{div}\frac{\nabla U}{(1+|\nabla U|^p)^{\frac{1}{p}}} &= 0 \quad \text{in } B_R \setminus B_r, \\ U &= 0 \quad \text{on } \partial B_r, \\ U &= K \quad \text{on } \partial B_R \end{aligned} \tag{an_p}$$

- we demand the solution to attain this boundary value for any $K \in \mathbb{R}^+$

Lemma 1

For $p > 1$, the problem (an_p) has a weak solution $U \in W^{1,1}(B_R \setminus B_r)$ if and only if

$$K < \int_r^R \frac{r}{\left(z^{p(d-1)} - r^p\right)^{\frac{1}{p}}}\, dz.$$

If $p \in (0,1]$, then for any $K \in \mathbb{R}^+$ there exists a weak solution to problem (an_p).

Proof of the lemma

- the solution, if exists, is rotation invariant, i.e. $U(\boldsymbol{x}) =: u(|\boldsymbol{x}|)$ and $\varphi(\boldsymbol{x}) =: g(|\boldsymbol{x}|)$
- problem can be simplified significantly,

$$\int_\Omega \frac{\nabla U(\boldsymbol{x})}{(1+|\nabla U(\boldsymbol{x})|^p)^{\frac{1}{p}}} \cdot \nabla\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0 \;\Leftrightarrow\; H_d \int_r^R \frac{z^{d-1}u'(z)}{(1+(u'(z))^p)^{\frac{1}{p}}} g'(z)\,\mathrm{d}z = 0,$$

  where H_d is Hausdorff measure of the unit sphere in $\mathbb{R}^d$, $g \in \mathcal{D}(r,R)$ arbitrary
- this gives condition $\frac{u'(z)}{(1+(u'(z))^p)^{\frac{1}{p}}} = \frac{c}{z^{d-1}}$ with $c \in (0, r^{d-1})$, which implies

$$u(s) = \int_r^s \frac{c}{(z^{p(d-1)} - c^p)^{\frac{1}{p}}}\,\mathrm{d}z \qquad \left(\text{and that } u'(s) = \frac{c}{(s^{p(d-1)} - c^p)^{\frac{1}{p}}}\right)$$

- therefore $K = u(R)$ is bounded if $p > 1$ and may be arbitrarily large (achieved only by a proper choice of c) for $p \in (0, 1]$

Exterior ball condition

A domain Ω satisfies the exterior ball condition if there exists a number $r_0 > 0$ such that for every point $\boldsymbol{x} \in \partial\Omega$ there is a ball $B_{r_0}(\boldsymbol{y})$ with $\overline{B_{r_0}(\boldsymbol{y})} \cap \overline{\Omega} = \{\boldsymbol{x}\}$.
Convexity or $\mathcal{C}^{1,1}$-regularity of the domain are sufficient for the exterior ball condition.

Main theorem

Theorem 2

Let $F \in \mathcal{C}^2(\mathbb{R}^+)$ *be a strictly convex function with* $\lim_{s\to 0} F'(s) = 0$ *which satisfies, for some constants* $C_1, C_2 > 0$*,*

$$
\begin{aligned}
C_1 s - C_2 \le F(s) \le C_2(1+s) \quad & \text{for all } s \in \mathbb{R}^+, \\
\frac{F''(s)}{F''(t)} \le C_2 \quad & \text{for all } s \ge 1 \text{ and } t \in [s/2, 2s].
\end{aligned}
$$

Then the following statements are equivalent:

i) *For arbitrary domain* Ω *of class* $\mathcal{C}^1$ *satisfying an exterior ball condition and arbitrary prescribed boundary value* $u_0 \in \mathcal{C}^{1,1}(\overline{\Omega})$ *there exists a unique function* $u \in \mathcal{C}^{0,1}(\overline{\Omega})$ *solving* (a).

ii) *The function* F *satisfies*

$$\int_1^\infty sF''(s)\,ds = \infty.$$

Main theorem - a_p case

- we prove the Theorem 2 only for the prototypic case with F_p, a_p, (a_p)
- F_p satisfies assumption ii) in Theorem 2 for $p \in (0,1]$ (and does not satisfy it for $p > 1$), since

$$F_p''(s) = (a_p(s)s)' = \frac{1}{(1+s^p)^{\frac{1}{p}}}\left(1 - \frac{s^p}{1+s^p}\right) = \frac{1}{(1+s^p)^{\frac{1}{p}+1}},$$

and therefore

$$\int_1^\infty sF_p''(s)\,\mathrm{d}s = \infty \iff \int_1^\infty \frac{s}{(1+s^p)^{\frac{1}{p}+1}}\,\mathrm{d}s = \infty \iff p \in (0,1]$$

Theorem 3

For any domain $\Omega \subset \mathbb{R}^d$ of class $\mathcal{C}^{1,1}$, boundary condition $u_0 \in \mathcal{C}^{1,1}(\partial\Omega)$ and $p \in (0,1]$, there exists $u_p \in \mathcal{C}^{0,1}(\overline{\Omega})$ a solution to problem (a_p).

Scheme for the proof

- approximate, find uniform estimate and proceed with $\varepsilon \to 0_+$
- elliptic problem, if we estimate the gradient on the boundary, we have boundedness everywhere by the use of the maximum principle
- tangential derivative on the boundary is bounded since $u = u_0$ on $\partial\Omega$
- we have to take care of the normal derivative of u
- it is done by finding proper barrier functions u^b, u_b
- first idea - use the solution u_p from the annulus (for $p \in (0, 1]$), however, this works only for locally constant boundary data
- second idea - add the tangential derivative

  $$u^b(\boldsymbol{x}) := u_p(\boldsymbol{x}) + (\nabla u_0(\boldsymbol{x}_0))_\tau \cdot (\boldsymbol{x} - \boldsymbol{x}_0) + u_0(\boldsymbol{x}_0)$$

  however, this does not work
- third idea - try to weaken the convexity - it works!

Approximative problem

- for $\varepsilon > 0$, approximate the problem (a_p) by

$$\begin{aligned} -\varepsilon \Delta u_\varepsilon - \operatorname{div}\big(a_p(|\nabla u_\varepsilon|)\nabla u_\varepsilon\big) = 0 \qquad & \text{in } \Omega, \\ u_\varepsilon = u_0 \qquad & \text{on } \partial\Omega \end{aligned} \tag{εa_p}$$

- a priori estimate

$$\varepsilon \|\nabla u_\varepsilon\|_2^2 + \|\nabla u_\varepsilon\|_1 + \|u_\varepsilon\|_\infty \le C, \tag{2}$$

  and by difference quotient techniques we also have $u_\varepsilon \in W^{2,2}_{loc}(\Omega)$
- main goal is to show that the uniform estimate holds (for any $p \in (0,1]$)

$$\|\nabla u_\varepsilon\|_\infty \le C(\Omega, F, u_0) \qquad \text{(independent of } \varepsilon\text{!)} \tag{3}$$

- indeed, having (3), there exists a subsequence converging weakly-$*$ to a function $u \in u_0 + W_0^{1,\infty}(\Omega)$; also, when passing to the limit $\varepsilon \to 0_+$ the limit function turns out to be the desired solution u
- it is Lipschitz regular, Theorem 3 is therefore proven, provided that we can show that (3) holds

Subsolution $|\nabla u_\varepsilon|$, reduction to normal direction

Applying $\frac{\partial}{\partial x_k} =: D_k$ to (εa_p), multiplying the result by $D_k u_\varepsilon$ and summing over $k = 1, \ldots, d$, we obtain

$$
\begin{aligned}
0 &= -\varepsilon D_k u_\varepsilon \Delta D_k u_\varepsilon - D_k u_\varepsilon D_i D_k \left(F_p'(|\nabla u_\varepsilon|) \frac{D_i u_\varepsilon}{|\nabla u_\varepsilon|} \right) \\
&= -\frac{\varepsilon}{2} \Delta |\nabla u_\varepsilon|^2 + \varepsilon |\nabla^2 u_\varepsilon|^2 - D_i \left(D_k \left(F_p'(|\nabla u_\varepsilon|) \frac{D_i u_\varepsilon}{|\nabla u_\varepsilon|} \right) D_k u_\varepsilon \right) + D_{ik} u_\varepsilon D_k \left(F_p'(|\nabla u_\varepsilon|) \frac{D_i u_\varepsilon}{|\nabla u_\varepsilon|} \right) \\
&= -\frac{\varepsilon}{2} \Delta |\nabla u_\varepsilon|^2 - D_i \left(A_{ik}(\nabla u_\varepsilon) D_k |\nabla u_\varepsilon| \right) \\
&\quad + \varepsilon |\nabla^2 u_\varepsilon|^2 + F_p''(|\nabla u_\varepsilon|) |\nabla |\nabla u_\varepsilon||^2 + F_p'(|\nabla u_\varepsilon|) \frac{|\nabla^2 u_\varepsilon|^2 - |\nabla |\nabla u_\varepsilon||^2}{|\nabla u_\varepsilon|},
\end{aligned}
$$

where $A = (A_{ik})$ is positively definite measurable matrix. Consequently, $|\nabla u_\varepsilon|$ is a sub-solution to the elliptic problem (εa_p), we only need a uniform estimate on the boundary,

$$
-\frac{\varepsilon}{2} \Delta |\nabla u_\varepsilon|^2 - D_i \left(A_{ik}(\nabla u_\varepsilon) D_k |\nabla u_\varepsilon| \right) \le 0 \implies \|\nabla u_\varepsilon\|_\infty \le \|\nabla u_\varepsilon\|_{L^\infty(\partial\Omega)}.
$$

In fact, only in the normal direction, as

$$
\|\nabla u_\varepsilon\|_\infty \le \|\nabla u_\varepsilon\|_{L^\infty(\partial\Omega)} \le \|\nabla u_0\|_{L^\infty(\partial\Omega)} + \left\| \frac{\partial u_\varepsilon}{\partial \boldsymbol{n}} \right\|_\infty \le C.
$$

Estimates on normal derivatives

- let $\boldsymbol{x}_0 \in \partial\Omega$ be arbitrary, we want to find u^b, u_b the barriers fulfilling for all (small) $\varepsilon > 0$ and for some $\tilde{\Omega} \subset \Omega$ (such that $\boldsymbol{x}_0 \in \partial\tilde{\Omega}$))

$$u_b \leq u_\varepsilon \leq u^b \quad \text{in } \tilde{\Omega}, \qquad u_b(\boldsymbol{x}_0) = u_\varepsilon(\boldsymbol{x}_0) = u^b(\boldsymbol{x}_0)$$

- this allows to estimate the normal derivative

$$\frac{u_\varepsilon(\boldsymbol{x}) - u_\varepsilon(\boldsymbol{x}_0)}{|\boldsymbol{x} - \boldsymbol{x}_0|} = \frac{u_\varepsilon(\boldsymbol{x}) - u^b(\boldsymbol{x})}{|\boldsymbol{x} - \boldsymbol{x}_0|} + \frac{u^b(\boldsymbol{x}) - u^b(\boldsymbol{x}_0)}{|\boldsymbol{x} - \boldsymbol{x}_0|} \leq \|u^b\|_{1,\infty}$$

- we can use u_b in a similar way and obtain in the passage $\boldsymbol{x} \to \boldsymbol{x}_0$ both lower and upper bounds

$$C(\Omega, \|u_b\|_{1,\infty}) \leq \frac{\partial u_\varepsilon(\boldsymbol{x}_0)}{\partial \boldsymbol{n}} \leq C(\Omega, \|u^b\|_{1,\infty}),$$

- since $\boldsymbol{x}_0 \in \partial\Omega$ was arbitrary and u^b, u_b will be constructed independently of ε,

$$\left\| \frac{\partial u_\varepsilon}{\partial \boldsymbol{n}} \right\|_\infty \leq C$$

General scheme for finding the barrier

- upper barrier u^b a super-solution to (εa_p) in $\tilde{\Omega}$ for any $\varepsilon > 0$,

$$-\varepsilon \Delta u^b - \operatorname{div} \frac{\nabla u^b}{(1 + |\nabla u^b|^p)^{\frac{1}{p}}} \geq 0 \qquad \text{in } \tilde{\Omega}$$
$$u^b \geq u_0 \qquad \text{on } \partial\tilde{\Omega}$$

- it holds for every $\varepsilon > 0$, since $-\Delta u^b \geq 0$ and $-\operatorname{div} \frac{\nabla u^b}{(1+|\nabla u^b|^p)^{\frac{1}{p}}} \geq 0$
- having u^b and using that u_ε is a solution to (εa_p),

$$-\varepsilon\Delta(u^b - u_\varepsilon) - \operatorname{div}\left(\frac{\nabla u^b}{(1 + |\nabla u^b|^p)^{\frac{1}{p}}} - \frac{\nabla u_\varepsilon}{(1 + |\nabla u_\varepsilon|^p)^{\frac{1}{p}}}\right) \geq 0 \quad \text{in } \tilde{\Omega}$$

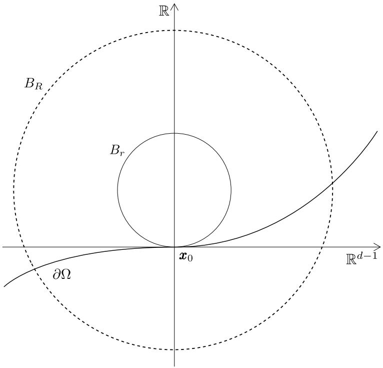
- $0 \leq \int_{\tilde{\Omega}} \left(\frac{\nabla u^b}{(1+|\nabla u^b|^p)^{\frac{1}{p}}} - \frac{\nabla u_\varepsilon}{(1+|\nabla u_\varepsilon|^p)^{\frac{1}{p}}}\right) \cdot \nabla(u^b - u_\varepsilon)_- \leq 0$
- we can also find u_b a lower barrier such that $u_b \leq u_\varepsilon \leq u^b$ in $\tilde{\Omega}$
- the problem admits setting $u_b(\mathbf{x}_0) = u_\varepsilon(\mathbf{x}_0) = u^b(\mathbf{x}_0)$

First idea for constructing the barriers

- first idea: use the solution u_p ($p \in (0,1]$) from the annulus problem (an_p), and define

$$u^b := u_p + u_0(\boldsymbol{x}_0), \qquad u_b := -u_p + u_0(\boldsymbol{x}_0)$$

- good part: these barriers are super-/sub-solutions!
- bad part: it works only for u_0 locally constant..



Other ideas for constructing the barriers

- <u>second idea</u>: use the solution u_p ($p \in (0,1]$) from the annulus problem (an_p), *and include some dependence on the tangential derivative of* u_0

$$u^b := u_p + (\nabla u_0(\boldsymbol{x}_0))_\tau \cdot (\boldsymbol{x} - \boldsymbol{x}_0) + u_0(\boldsymbol{x}_0)$$

- good part: we can control arbitrary, not only constant boundary data!
- bad part: it does not work.. (not a super-solution)

Other ideas for constructing the barriers

- second idea: use the solution u_p ($p \in (0,1]$) from the annulus problem (an_p), *and include some dependence on the tangential derivative of* u_0

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- good part: we can control arbitrary, not only constant boundary data!
- bad part: it does not work.. (not a super-solution)

- third idea: instead of pairing u_p in the definition of barrier with the parameter of the problem (a_p), try to use the limiting admissible solution u_1,

$$u^b := u_1 + (\nabla u_0(\boldsymbol{x}_0))_\tau \cdot (\boldsymbol{x} - \boldsymbol{x}_0) + u_0(\boldsymbol{x}_0) \tag{ub}$$

- good part: hooray, this works!
- bad part: only for problems with $p < 1$

The construction itself

Lemma 4

For any $u_0 \in \mathcal{C}^{0,1}$, there exists $M > 0$ such that for all $r > 0$ and u^b defined by (ub),

$$-\varepsilon \Delta u^b - div \frac{\nabla u^b}{(1 + |\nabla u^b|^p)^{\frac{1}{p}}} \geq 0 \text{ in } \tilde{\Omega} \tag{4}$$

for $p \in (0,1)$ and for all $\boldsymbol{x} \in \mathbb{R}^d \setminus B_r$ fulfilling $|u_1'(|\boldsymbol{x}|)| \geq M$.

- find a value of M that satisfies $|u_1'(|\boldsymbol{x}|)| \geq M \implies (4)$
- the condition $|u_1'(|\boldsymbol{x}|)| \geq M$ is achieved by a proper choice of constant $c \in (0, r^{d-1})$ in
$$u_1(s) = \int_r^s \frac{c}{z^{d-1} - c} \, \mathrm{d}z$$
- finally, we fix r in such a way that the exterior ball condition is satisfied in every $\boldsymbol{x}_0 \in \partial\Omega$ and $u^b \geq u_\varepsilon$ also in the rest of $\tilde{\Omega}$

Remarks on regularity of the domain

- convexity or $\mathcal{C}^{1,1}$ regularity of the domain are sufficient for the exterior ball condition, thus, for example, *the theorem holds for all convex domains of class* $\mathcal{C}^1$ *and for arbitrary domains of class* $\mathcal{C}^{1,1}$
- similar proof would work with $\mathcal{C}^{0,1}$ domains which are piece-wise $\mathcal{C}^{1,1}$ as well; except the corner points of the boundary, which one can not attach the ball to - hence we control the trace up to the corner points, which is however the set of zero $((d-1))$ measure
- the goal to get the existence of solution on arbitrary regular domain ONLY by characterizing the functional F (or, equivalently, coefficient function a) was fulfilled

Non-existence result on general domain

Theorem 5

Let F do not satisfy (ii) from Theorem 2. Then for arbitrary smooth domain Ω satisfying interior ball condition there exists a smooth u_0 such that the problem (a) *does not have solution in $W^{1,1}(\Omega) \cap \mathcal{C}(\overline{\Omega})$.*

We say that a domain Ω satisfies the interior ball condition if there exist $\boldsymbol{x}_0 \in \partial\Omega$, $\boldsymbol{y}_0 \notin \Omega$ and $r, \varepsilon > 0$ such that $\boldsymbol{x}_0 \in \partial B_r(\boldsymbol{y}_0)$ and

$$\boldsymbol{x} \in \partial B_r(\boldsymbol{y}_0) \cap (B_\varepsilon(\boldsymbol{x}_0) \setminus \boldsymbol{x}_0) \implies \boldsymbol{x} \in \Omega.$$

For example, every non-convex domain in 2D fulfils the interior ball condition.

More general growth

Theorem 6

Let $F \in \mathcal{C}^2(\mathbb{R}^+)$ *be a strictly convex function with* $\lim_{s\to 0} F'(s) = 0$ *which satisfies, for some constants* $C_1, C_2, \delta_0 > 0$ *and all* $\delta_0 > \delta > 0$,

$$C_1 s - C_2 \leq F(s) \quad \text{for all } s \in \mathbb{R}^+,$$

$$\liminf_{s\to\infty} \frac{s^{2-\delta} F''(s)}{F'(s)} \geq 1$$

Then for arbitrary domain Ω *of class* $\mathcal{C}^1$ *satisfying an exterior ball condition and arbitrary prescribed boundary value* $u_0 \in \mathcal{C}^{1,1}(\overline{\Omega})$ *there exists a unique function* $u \in \mathcal{C}^{0,1}(\overline{\Omega})$ *solving* (a).