A variational approach to doubly nonlinear equations with nonstandard growth

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Overview

1. Variational formulation of the problem
2. Existence results
3. Strategy of the proof

The main results are due to a joint work with
- Verena Bögelein (Salzburg)
- Frank Duzaar (Erlangen-Nürnberg)
- Paolo Marcellini (Firenze)
I. Variational formulation of the problem
The model problem

Cauchy-Dirichlet problem:

Find \( u : \Omega_T \to [0, \infty) \) with

\[
\begin{cases}
\partial_t u^m - \text{div}(|Du|^{p-2}Du) = 0 \quad \text{in } \Omega_T, \\
u = g \quad \text{on } \partial_{\text{par}} \Omega_T,
\end{cases}
\]

where here,

- \( \Omega \subset \mathbb{R}^n \) is a bounded domain, \( T > 0, \Omega_T := \Omega \times (0, T) \).
- \( m > 0, \ p > 1 \).
- \( g : \partial_{\text{par}} \Omega_T \to [0, \infty) \) are prescribed boundary values.

This generalizes both the porous medium equation (\( p = 2 \)) and the parabolic \( p \)-Laplace equation (\( m = 1 \)).
Variational Formulation I

We test (1) with

$$\varphi = w - u$$

for a comparison map $w: \Omega_T \to [0, \infty)$ with $w = u$ on the lateral boundary $\partial\Omega \times (0, T)$. This leads to

$$
\int_{\Omega_T} \partial_t u^m (w - u) \, dx \, dt + \int_{\Omega_T} |Du|^{p-2} Du \cdot (Dw - Du) \, dx \, dt = 0
$$

By the convexity of $\mathbb{R}^n \ni \xi \mapsto \frac{1}{p} |\xi|^p$ we have

$$
\frac{1}{p} |Du|^p + |Du|^{p-2} Du \cdot (Dw - Du) \leq \frac{1}{p} |Dw|^p.
$$
Variational Formulation II

\[ \begin{align*}
\iint_{\Omega_T} \partial_t u^m (w - u) \, dx \, dt + \iint_{\Omega_T} |Du|^{p-2} Du \cdot (Dw - Du) \, dx \, dt &= 0 \\
&=: I \\
\iint_{\Omega_T} |Du|^{p-2} Du \cdot (Dw - Du) \, dx \, dt &= 0 \\
&=: II
\end{align*} \]

By convexity,

\[ II \leq \frac{1}{p} \iint_{\Omega_T} |Dw|^p \, dx \, dt - \frac{1}{p} \iint_{\Omega_T} |Du|^p \, dx \, dt \]

Integration by parts and elementary calculations imply

\[ I = \iint_{\Omega_T} \partial_t w (w^m - u^m) \, dx \]

\[ + \int_{\Omega} b[u(0), w(0)] \, dx - \int_{\Omega} b[u(T), w(T)] \, dx \]

with

\[ b[u, w] := \frac{1}{m+1} w^{m+1} - \left[ \frac{1}{m+1} u^{m+1} + u^m (w - u) \right] \geq 0 \]
This leads to the variational inequality

\[
\iint_{\Omega_T} \frac{1}{p}|Du|^p \, dx \, dt \leq \iint_{\Omega_T} \left[ \frac{1}{p}|Dw|^p + \partial_t w(w^m - u^m) \right] \, dx \, dt
\]

\[
+ \int_{\Omega} b[u(0), w(0)] \, dx - \int_{\Omega} b[u(T), w(T)] \, dx
\]

\[
=: \mathcal{B}[u(0), w(0)] - \mathcal{B}[u(T), w(T)]
\]

for any \( w : \Omega_T \to [0, \infty) \) with \( \partial_t w \in L^{\frac{m+1}{m}}(\Omega_T) \) and \( w = u \) on \( \partial\Omega \times (0, T) \).

In the case \( m = 1 \) the boundary term simplifies to

\[
b[u, w] = \frac{1}{2}|u - w|^2
\]

so that the usual \( L^2(\Omega) \)-boundary terms appear in the variational inequality.
We replace the term $u^m$ by a nonlinearity $b: [0, \infty) \to [0, \infty)$ which is continuous and piecewise $C^1$ with $b(0) = 0$, and satisfies

$$0 < \ell \leq \frac{ub'(u)}{b(u)} \leq m$$

whenever $u > 0$, $b(u) > 0$ and $b'(u)$ exists.

Assumption (2) implies the nonstandard growth condition

$$b(1) \min\{u^\ell, u^m\} \leq b(u) \leq b(1) \max\{u^\ell, u^m\} \quad \text{for all } u > 0.$$
Orlicz Spaces

The primitive $\phi: [0, \infty) \to [0, \infty)$ is defined by

$$\phi(u) := \int_{0}^{u} b(s) \, ds \quad \forall u \geq 0.$$ 

Note that $\phi$ is convex with $\phi(0) = 0$ and that (2) implies that $\phi$ satisfies the $\Delta_2$- and $\nabla_2$-conditions. We consider the Orlicz-space

$$L^\phi(\Omega) = \left\{ w: \Omega \to \mathbb{R}, \text{measurable} : \int_{\Omega} \phi(|w|) \, dx < \infty \right\}$$

Assumption (2) implies the nonstandard growth condition

$$\phi(1) \min\{u^{\ell+1}, u^{m+1}\} \leq \phi(u) \leq \phi(1) \max\{u^{\ell+1}, u^{m+1}\}$$

for all $u \geq 0$. 

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A variational approach to doubly nonlinear equations
Instead of the model integrand $\frac{1}{p}|\xi|^p$, we consider an general integrand $f: \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that the following convexity and coercivity conditions hold:

\[
\begin{align*}
\mathbb{R} \times \mathbb{R}^n \ni (u, \xi) &\rightarrow f(x, u, \xi) \text{ is convex for a.e. } x \in \Omega, \\
f(x, u, \xi) &\geq \nu|\xi|^p \text{ for } (x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n. 
\end{align*}
\]

- We do not require any growth condition from above.
- More generally, we can assume

\[
f(x, u, \xi) \geq \nu|\xi|^p - g(x)(1 + |u|)
\]

for $(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, with $g \in L^1(\Omega) \cap L^{\phi^*}(\Omega)$. 

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A variational approach to doubly nonlinear equations
We consider initial values $u_0 : \Omega \to [0, \infty)$ (which also play the role of time-independent boundary values) that satisfy

$$u_0 \in L^\phi(\Omega) \quad \text{and} \quad \int_\Omega f(x, u_0, Du_0) \, dx < \infty.$$ 

For the given data, we wish to solve the Cauchy-Dirichlet problem

$$\begin{cases}
\partial_t b(u) - \text{div} \, D_\xi f(x, u, Du) = -D_u f(x, u, Du) & \text{in } \Omega_T, \\
u = u_0 & \text{on } \partial_{\text{par}} \Omega_T,
\end{cases}$$

which generalizes the model case (1) from above.
Variational formulation and integral convexity

Similarly as in the model case, the convexity of \( f \) implies

\[
D_\xi f(x, u, Du) \cdot D(w - u) + D_u f(x, u, Du)(w - u) 
\leq f(x, w, Dw) - f(x, u, Du).
\]

Actually, what we need in the argument is the **convexity of the integral**

\[
F(u) := \int_\Omega f(x, u, Du) \, dx
\]

rather than the convexity of the integrand.

**Theorem (Bögelein, Dacorogna, Duzaar, Marcellini, S., 2020)**

*For gradient flows \((b(u) = u)\) for functionals with \(p\)-growth, integral convexity is necessary and sufficient for existence of variational solutions. (Also in the case of systems.)*
Variational formulation: the time term

For the term involving the time derivative, we calculate (formally)

\[
\iint_{\Omega_T} \partial_t b(u)(w - u) \, dx = \iint_{\Omega_T} \partial_t w (b(w) - b(u)) \, dx
\]

\[
+ \int_{\Omega} b[u(0), w(0)] \, dx - \int_{\Omega} b[u(T), w(T)] \, dx
\]

with the boundary term

\[
b[u, w] := \phi(w) - \left[ \phi(u) + b(u)(w - u) \right] \geq 0
\]

As mentioned before, \( b[u, w] = \frac{1}{2}|u - w|^2 \) in the case \( b(u) = u \).
Variational Solutions

Definition

A non-negative measurable map \( u : \Omega_T \to [0, \infty) \) in the class

\[
\begin{align*}
u &\in C^0([0, T]; L^\phi(\Omega)) \cap L^p(0, T; u_o + W^{1,p}_0(\Omega))
\end{align*}
\]

is called variational solution if and only if the variational inequality

\[
\begin{align*}
\mathcal{B}[u(\tau), w(\tau)] + \iint_{\Omega_T} f(x, u, Du)dxdt &
\leq \mathcal{B}[u_o, w(0)] + \iint_{\Omega_T} \left[ f(x, w, Dw) + \partial_t w (b(w) - b(u)) \right]dxdt
\end{align*}
\]

holds true for any \( \tau \in [0, T] \), and any \( w \in L^p(0, T; u_o + W^{1,p}_0(\Omega)) \) with \( \partial_t w \in L^\phi(\Omega_T) \) and \( w(0) \in L^\phi(\Omega) \).

Here, \( \mathcal{B}[u, w] := \int_{\Omega} \left[ \phi(w) - \phi(u) - b(u)(w - u) \right]dx \). Here,

\[
\mathcal{B}[u, w] := \frac{1}{2} \int_{\Omega} |w - u|^2 dx \quad \text{if} \quad b(u) = u.
\]
II. Existence results
The General Existence Result

Theorem (Bögelein, Duzaar, Marcellini, S., ARMA 2018)

Suppose that the non-linearity $b$, the integrand $f$ and the initial datum $u_0$ are as before. Then there exists a variational solution

$$
u \in C^0([0, T]; L^\phi(\Omega)) \cap L^p(0, T; u_0 + W^{1,p}_0(\Omega))$$

in the sense of the previous definition. The solution satisfies

$$\partial_t \sqrt{\phi(u)} \in L^2(\Omega_T)$$

and attains the initial datum $u_0$ in the $C^0 - L^\phi$-sense.
Weak solutions

Under additional assumptions, the variational solutions constructed in the preceding theorem are **distributional solutions** of

\[
\begin{align*}
\partial_t b(u) - \text{div} D_x f(x, u, Du) &= -D_u f(x, u, Du) \quad \text{in } \Omega_T, \\
u &= u_0 \quad \text{on } \partial_{\text{par}} \Omega_T,
\end{align*}
\]

The examples include general nonlinearities \( b(u) \) with

\[
1 \leq \ell \leq \frac{ub'(u)}{b(u)} \leq m
\]

and functionals with nonstandard growth such as

- \( f(x, \xi) := \alpha(x)|\xi|^p + \beta(x)|\xi|^q, \quad (1 < p < q \leq p+1, \ \alpha(x) + \beta(x) > 0) \);
- \( f(\xi) := |\xi|^p \log(1 + |\xi|); \)
- \( f(\xi) := e^{\sqrt{1+|\xi|^2}}. \)
The preceding existence result has been extended in various directions:

- to unbounded domains, in particular to the Cauchy-problem on $\Omega = \mathbb{R}^n$ (Bögelein, Duzaar, Marcellini, S., ARMA 2018);
- to time-dependent boundary values (Bögelein, Duzaar, Marcellini, S., Rend. Lincei 2018);
- to doubly nonlinear systems with $b(u) = u^m$, $m > 1$, and time-dependent boundary values (Schätzler, J. Elliptic Parabol. Equ 2019)
Known Results I

- Grange & Mignot (1972), Alt & Luckhaus (1983): equations/systems of the type

\[
\partial_t b(u) - \text{div}(A(b(u), Du)) = f(b(u))
\]

with \( u = g \) on \( \partial_{\text{par}} \Omega_T \). The coefficients \( A \) satisfy

\[
\begin{cases}
|A(s, \xi)| \leq C(1 + |\xi|^{p-1}), \\
(A(s, \xi) - A(s, \eta)) \cdot (\xi - \eta) \geq C_0 |\xi - \eta|^p,
\end{cases}
\]

\( b \) is the continuous gradient of a convex \( C^1 \)-function \( \phi \).

The boundary values satisfy

\[
\begin{cases}
g \in L^p(0, T; W^{1,p}(\Omega)) \cap L^\infty(\Omega_T), \\
\partial_t g \in L^1(0, T; L^\infty(\Omega))
\end{cases}
\]

Proof by Galerkin type method.
Known Results II


\[
\partial_t u - \text{div}\left( u^{m-1} |Du|^{p-2} Du \right) = 0.
\]

Existence via approximation by strictly positive solutions and a-priori Hölder-estimates.

- Akagi & Stefanelli (2011): Equations of the type

\[
b(\partial_t u) - \text{div}\left( |Du|^{p-2} Du \right) = 0
\]

Existence via elliptic regularization.
Known Results III

- Akagi & Stefanelli (2014):

\[ \partial_t b(u) - \text{div}(A(Du)) = f \]

\[ \iff \quad v := b(u) - \text{div}(A(Db^{-1}(v))) = f - \partial_t v \]

for \( b \) and \( A \) with polynomial growth.
Existence for the dual problem via elliptic regularization.

III. Strategy of the proof
Fix a step size $h \in (0, 1]$. The goal is to construct approximations $u_i$ of the solution at times $t = ih$, $i \in \mathbb{N}_0$.

Let $u_0 = u_0$.

Suppose that for some $i \in \mathbb{N}$ with $ih \leq T$ the non-negative map $0 \leq u_{i-1} \in L^\phi(\Omega) \cap (u_o + W^{1,p}_0(\Omega))$ has been defined.

In the case $b(u) = u$, we define $u_i$ as the minimizer of

$$F_i[v] := \int_{\Omega} f(x, v, Dv)dx + \frac{1}{2h} \int_{\Omega} |u_{i-1} - v|^2 \, dx.$$

in the class of functions $0 \leq v \in L^2(\Omega) \cap (u_o + W^{1,p}_0(\Omega))$.

In the general case, we define $u_i$ as the minimizer of

$$F_i[v] := \int_{\Omega} f(x, v, Dv)dx + \frac{1}{h} \int_{\Omega} b[u_{i-1}, v] \, dx.$$

in the class of functions $0 \leq v \in L^\phi(\Omega) \cap (u_o + W^{1,p}_0(\Omega))$.

Minimizers exist by the Direct Method of the Calculus of Variations.
For a test function $\psi \in C_0^\infty(\Omega)$, we consider variations $u_i + s\psi$, $s \in (-\varepsilon, \varepsilon)$, of the minimizers $u_i$ and calculate

$$\frac{d}{ds} \bigg|_{s=0} \left( \frac{1}{h} \int_\Omega b[u_{i-1}, u_i + s\psi] \, dx \right)$$

$$= \frac{1}{h} \int_\Omega \frac{\partial}{\partial s} \bigg|_{s=0} \left[ \phi(u_i + s\psi) - \phi(u_{i-1}) - b(u_{i-1})(u_i + s\psi - u_{i-1}) \right] \, dx$$

$$= \frac{1}{h} \int_\Omega \left[ \phi'(u_i)\psi - b(u_{i-1})\psi \right] \, dx$$

$$= \int_\Omega \frac{b(u_i) - b(u_{i-1})}{h} \psi \, dx$$
Energy Estimates I

Observe that $u_{i-1}$ is an admissible competitor for $u_i$, and therefore $F_i[u_i] \leq F_i[u_{i-1}]$. This can be iterated and leads to

$$
\int_{\Omega} f(x, u_k, Du_k) dx + \frac{1}{h} \sum_{i=1}^{k} \int_{\Omega} b[u_{i-1}, u_i] dx \leq \int_{\Omega} f(x, u_o, Du_o) dx
$$

$$
\geq \nu |Du_k|^p
$$

whenever $k \in \mathbb{N}$ with $kh \leq T$. 

\[ =: M < \infty \]

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A variational approach to doubly nonlinear equations
Monotonicity

For the boundary term $b[u, w]$, we have the following bounds:

**Lemma**

*For any $u, w \geq 0$, we have*

$$b[u, w] \leq (b(u) - b(w))(u - w)$$

$$\leq C \left| \sqrt{\phi(u)} - \sqrt{\phi(w)} \right|^2$$

$$\leq C^2 b[u, w],$$

*with a constant $C = C(\ell, m) \geq 1$.*

The proof relies on the assumption

$$\ell \leq \frac{ub'(u)}{b(u)} \leq m.$$
Energy Estimates II

From the previous energy estimate we immediately obtain:

\[
\frac{1}{h} \sum_{i=1}^{k} \int_{\Omega} \left| \sqrt{\phi(u_i)} - \sqrt{\phi(u_{i-1})} \right|^2 \, dx 
\]

\[
\leq \frac{C}{h} \sum_{i=1}^{k} \int_{\Omega} b[u_{i-1}, u_i] \, dx \leq C(\ell, m) M \quad (3)
\]

and

\[
\int_{\Omega} |Du_k|^p \, dx \leq \frac{M}{\nu}, \quad (4)
\]

whenever \( k \in \mathbb{N} \) with \( kh \leq T \). Furthermore,

\[
\int_{\Omega} \phi(u_k) \, dx \leq 2k \sum_{i=1}^{k} \int_{\Omega} \left| \sqrt{\phi(u_i)} - \sqrt{\phi(u_{i-1})} \right|^2 \, dx + 2 \int_{\Omega} \phi(u_o) \, dx 
\]

\[
\leq 2TM + 2 \int_{\Omega} \phi(u_o) \, dx. \quad (5)
\]
From now on we consider only such $h \in (0, 1]$ such that $h_k = T/k$ with $k \in \mathbb{N}$. From the construction before we obtain minimizers $u_i^{(k)}$ with $i \in \{0, 1, \ldots, k\}$.

We define $u^{(k)}: \Omega \times (-h_k, T] \to [0, \infty)$ by

$$u^{(k)}(\cdot, t) := u_i^{(k)} \text{ for } t \in ((i - 1)h_k, ih_k], i \in \{0, \ldots, k\}.$$

The preceding estimates for $u_i^{(k)}$ imply the uniform energy bound

$$\sup_{t \in [0, T]} \int_{\Omega} \left[ \phi(u^{(k)}(t)) + |Du^{(k)}(t)|^p \right] dx \leq C,$$

where the constant $C$ is independent of $k$. (6)
From (6) we conclude that

\[(u^{(k)})_{k \in \mathbb{N}} \text{ is uniformly bounded in } L^\infty(0, T; W^{1,p}(\Omega)).\]

Therefore, we find a subsequence (still denoted by \(k\)) and

\[u \in L^\infty(0, T; u_0 + W^{1,p}_0(\Omega))\]

such that

\[u^{(k)} \rightharpoonup^* u \text{ weakly}^* \text{ in } L^\infty(0, T; W^{1,p}(\Omega)).\]
Compactness lemma

The energy estimate

\[
\max_{i \in \{0,1,...,k\}} \int_{\Omega} \phi(u_i^{(k)}) \, dx + \sup_{i \in \{0,1,...,k\}} \int_{\Omega} |Du_i^{(k)}|^p \, dx \leq C, \tag{7}
\]

and the continuity estimate

\[
\frac{1}{h} \sum_{i=1}^{k} \int_{\Omega} \left| \sqrt{\phi(u_i^{(k)})} - \sqrt{\phi(u_{i-1}^{(k)})} \right|^2 \, dx \leq C(\ell, m)M, \tag{8}
\]

imply, after passing to a subsequence, that

\[
\begin{align*}
\sqrt{\phi(u^{(k)})} & \rightarrow \sqrt{\phi(u)} \quad \text{strongly in } L^1(\Omega_T), \\
u^{(k)} & \rightarrow u \quad \text{a.e. in } \Omega_T.
\end{align*}
\]
Compactness lemma

\[
\max_{i \in \{0,1,\ldots,k\}} \int_{\Omega} \phi(u^{(k)}_i) \, dx + \sup_{i \in \{0,1,\ldots,k\}} \int_{\Omega} |Du^{(k)}_i|^p \, dx \leq C, \quad (7)
\]

and
\[
\frac{1}{h} \sum_{i=1}^{k} \int_{\Omega} \left| \sqrt{\phi(u^{(k)}_i)} - \sqrt{\phi(u^{(k)}_{i-1})} \right|^2 \, dx \leq C(\ell, m)M, \quad (8)
\]

\[
\Rightarrow \begin{cases} 
\sqrt{\phi(u^{(k)})} \to \sqrt{\phi(u)} \text{ strongly in } L^1(\Omega_T), \\
u^{(k)} \to u \text{ a.e. in } \Omega_T.
\end{cases}
\]

- This lemma can be interpreted as a Jacques Simon type compactness result adapted to doubly nonlinear equations.
- For the proof, we relied on techniques by Alt & Luckhaus.
Construction of the Limit Map IV: Time Derivative

From (6), we recall the uniform estimate

\[ \iint_{\Omega_T} \left| \partial_t^{(-h_k)} \sqrt{\phi(u^{(k)})} \right|^2 \, dx \, dt \leq CM \]

Extract a further subsequence such that

\[ \partial_t^{(-h_k)} \sqrt{\phi(u^{(k)})} \rightharpoonup w \quad \text{weakly in} \quad L^2(\Omega_T). \]

Since \( \sqrt{\phi(u^{(k)})} \to \sqrt{\phi(u)} \) strongly in \( L^1(\Omega_T) \), we have

\[ w = \partial_t \sqrt{\phi(u)}. \]

We deduce \( \partial_t \sqrt{\phi(u)} \in L^2(\Omega_T) \), with the estimate

\[ \iint_{\Omega_T} \left| \partial_t \sqrt{\phi(u)} \right|^2 \, dx \, dt \leq C \int_{\Omega} f(x, u_0, Du_0) \, dx \equiv M. \]
Variational Inequality for the Limit Map I

Let
\[ F^k[v] := \iint_{\Omega_T} \left( f(x, v, Dv) + \frac{1}{h_k} b[u^{(k)}(t - h_k), v(t)] \right) dx dt. \]

Then, \( u^{(k)} \) minimizes \( F^k \), i.e.
\[ F^k[u^{(k)}] \leq F^k[v] \]
for any \( 0 \leq v \in L^\phi(\Omega_T) \cap L^p(0, T; u_0 + W^{1,p}_0(\Omega)). \) We test the minimality with the admissible comparison map
\[ w_s := u^{(k)} + s(v - u^{(k)}), \quad s \in (0, 1), \]
and use the \textbf{convexity} of \( \int_\Omega f(x, v, Dv) \). Letting \( s \downarrow 0 \), we get
\[
\iint_{\Omega_T} f(x, u^{(k)}, Du^{(k)}) dx dt \\
\leq \iint_{\Omega_T} \left( f(x, v, Dv) + \Delta_{-h_k} b(u^{(k)})(v - u^{(k)}) \right) dx dt
\]
By lower semicontinuity

$$\iint_{\Omega_T} f(x, u, Du) \, dx \, dt \leq \liminf_{k \to \infty} \iint_{\Omega_T} f(x, u^{(k)}, Du^{(k)}) \, dx \, dt.$$ 

Formally, in the limit $k \to \infty$ we have

$$\iint_{\Omega_T} \left( \Delta - h_k b(u^{(k)}) \right) \left( v - u^{(k)} \right) \, dx \, dt \to \iint_{\Omega_T} \partial_t b(u)(v - u) \, dx \, dt$$

$$= \iint_{\Omega_T} \partial_t v(b(v) - b(u)) \, dx \, dt + \mathcal{B}[u_o, v(0)] - \mathcal{B}[u(T), v(T)].$$

This formal argument can be made rigorous by a discrete integration by parts formula.
The Initial Datum I

Testing the variational equation with $u_0$ on $\Omega_\tau = \Omega \times (0, \tau)$ for a (small) time $\tau > 0$, we obtain

$$\mathcal{B}[u(\tau), u_0] + \iint_{\Omega_\tau} f(x, u, Du) \, dx \, dt \leq \tau \int_{\Omega} f(x, u_0, Du_0) \, dx \geq 0 \leq \tau \int_{\Omega} f(x, u_0, Du_0) \, dx =: M < \infty$$

Letting $\tau \downarrow 0$, we deduce

$$\lim_{\tau \downarrow 0} \mathcal{B}[u(\tau), u_0] = 0.$$
The Initial Datum II

Now with the Cauchy-Schwarz inequality and the monotonicity lemma

\[
\int_{\Omega} |\phi(u(\tau)) - \phi(u_o)| \, dx \\
\leq C \left[ \int_{\Omega} \left( \sqrt{\phi(u(\tau))} - \sqrt{\phi(u_o)} \right)^2 \, dx \right]^{1/2} \left[ \int_{\Omega} \left[ \phi(u(\tau)) + \phi(u_o) \right] \, dx \right]^{1/2} \\
\leq C \sqrt{\mathcal{B}[u(\tau), u_o]} \left[ \int_{\Omega} \left[ \phi(u(\tau)) + \phi(u_o) \right] \, dx \right]^{1/2} .
\]

This implies that \( u \) attains the initial data in the sense

\[
u(\tau) \to u_o \text{ in } L^{\phi}(\Omega) \text{ as } \tau \downarrow 0.
\]
Merry Christmas!