Potential estimates for solutions of nonstandard growth measure data problems

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Goals

$$-\mathrm{div}\mathcal{A}(x, Du) = \mu \quad \text{in} \quad \Omega \subset \mathbb{R}^N$$

with nonnegative bounded measure μ and Carathéodory's function $\mathcal{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N \implies$ nonlinear operator (including Δ and Δ_p).

Solutions can be unbounded, but we can control them precisely by a certain potential and infer local properties such as Hölder continuity.

Chlebicka, Giannetti, AZG, Wolff potentials and local behaviour of solutions to measure data elliptic problems with Orlicz growth, arXiv:2006.02172

Problems:

- definition of solution
- Orlicz growth (no homogeneity $\mathcal{A}(x, k\xi) = |k|^{p-2}k\mathcal{A}(x, \xi)$)
- measurable dependence $x \mapsto \mathcal{A}(x, \xi)$

The operator of general growth

Growth & ellipticity condition - Orlicz framework

$$c_1^{\mathcal{A}} G(|\xi|) \le \mathcal{A}(x,\xi)\cdot\xi \qquad \text{and} \qquad |\mathcal{A}(x,\xi)| \le c_2^{\mathcal{A}} g(|\xi|),$$

$g = G'$ and $G \in \Delta_2 \cap \nabla_2$ ($\Rightarrow$ G is sandwiched between power functions)
e.g. Zygmund-type function $G_{p,\alpha}(s) = s^p \log^\alpha(1+s)$

Examples

$$-\operatorname{div}(a(x)Du) = \mu \quad \text{with} \quad 0 \ll a \in L^\infty(\Omega)$$

$$-\operatorname{div}\left(a(x)|Du|^{p-2}Du\right) = \mu \quad \text{with} \quad 0 \ll a \in L^\infty(\Omega)$$

$$-\operatorname{div}\left(a(x)\frac{G(|Du|)}{|Du|^2}Du\right) = \mu \quad \text{with} \quad 0 \ll a \in L^\infty(\Omega)$$

Potential estimate in the linear case 1/2

Global case

If u solves $-\Delta u = \mu$ in $\mathbb{R}^N$, with μ - a locally integrable function, and $u \to 0$ at ∞, then

$$u(x) = \int_{\mathbb{R}^N} E(x,y)\, d\mu(y)$$

where E is the fundamental solution, i.e.,

$$E(x) = c_n \begin{cases} |x-y|^{2-n} & \text{if } n > 2, \\ -\log|x-y| & \text{if } n = 2, \end{cases}$$

so, for $n > 2$, it can be estimated as follows:

$$|u(x)| \lesssim \int_{\mathbb{R}^N} \frac{d|\mu|(y)}{|x-y|^{n-2}} =: \mathrm{I}_2(|\mu|)(x) \quad \Leftarrow \text{Riesz potential}$$

Potential estimate in the linear case 2/2

Local behaviour of solutions to $-\Delta u = \mu$

Localized/trucated Riesz potential of a nonnegative measure

$$\mathrm{I}_2^{\mu}(x,R) := \int_0^R \frac{|\mu|(B_\varrho(x))}{\varrho^{n-2}}\frac{d\varrho}{\varrho} \lesssim_n \int_{B_R(x)} \frac{d|\mu|(y)}{|x-y|^{n-2}}$$
$$\leq \int_{\mathbb{R}^N} \frac{d|\mu|(y)}{|x-y|^{n-2}} = \mathrm{I}_2(|\mu|)(x) \quad \Leftarrow \text{Riesz potential}$$

Then locally

$$|u(x)| \leq C\left(\mathrm{I}_2^{\mu}(x,R) + \text{`}\mathit{sth\ not\ that\ much\ important}'\right).$$

Potential estimate in the power growth case

$-\Delta_p u = -\mathrm{div}(|Du|^{p-2}Du) = \mu$ **for** $1 < p < \infty$

For the nonlinear operator we have

$$|u(x)| \le C\left(\mathcal{W}_p^{\mu}(x,R) + \text{`}sth(u,R)\ not\ that\ much\ important'\right),$$

with

$$\mathcal{W}_p^{\mu}(x,R) = \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-1}}\right)^{\frac{1}{p-1}} d\varrho$$

called Wolff potential (similar ones were considered by Havin & Maz'ya).
For $p = 2$ we are back with Riesz potential.

Kilpeläinen & Malý ['92,'94] proven that for $\mu \ge 0$ we actually have

$$\mathcal{W}_p^{\mu}(x,R) \lesssim u(x) \lesssim \mathcal{W}_p^{\mu}(x,2R) + \text{`}sth(u,R)'$$

Trudinger & Wang [2002], Korte & Kuusi [2010], Kuusi & Mingione [2018]

Measure data problems

Let μ be a nonnegative Radon measure. Consider problems

- $-\Delta u = \mu$
- $-\Delta_p u = -\mathrm{div}(|Du|^{p-2}Du) = \mu$
- $-\mathrm{div}\mathcal{A}(x, Du) = \mu$,

where $\mathcal{A}(x,\xi)\cdot\xi \simeq G(|\xi|)$, and G is an Orlicz function.

How the equation can be interpreted?
What is a correct notion of a solution?

The function G generates an Orlicz space $L^G(\Omega)$ and a Sobolev-type space $W^{1,G}(\Omega)$ which is reflexive and separable if $G \in \Delta_2 \cap \nabla_2$.

Who can be called 'a solution'?

A function $u \in W^{1,G}_{loc}(\Omega)$ is called a weak solution to a problem

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, Du) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\text{if} \quad \int_\Omega \mathcal{A}(x, Du) \cdot D\phi \, dx = \int_\Omega \phi \, d\mu(x) \quad \text{for every } \phi \in C^\infty_c(\Omega).$$

- weak solutions are too restrictive;
- distributional solutions can be too wild;
- already for $-\Delta u = \delta_0$ in $B(0,1)$ we have the fundamental solution $E(x) = c_n|x-y|^{2-n}$, $(n > 2)$, which does not belong to the enegy space $W^{1,2}_0(B(0,1))$, but we like it!

Different notions of very weak solutions

One may study various kids of very weak solutions:

- SOLA (Boccardo&Gallouët),
- renormalized solutions (DiPerna&Lions, Boccardo, Giachetti, Diaz, Murat),
- entropy solutions (Bénilan, Boccardo, Gallouët, Gariepy, Pierre, Vazquez, Murat),
- approximable solutions (Cianchi-Mazya),
- $\mathcal{A}$-superharmonic functions (Kilpeläinen, Maly, Korte, Kuusi, Tuhola-Kujanpää '11) – nonlinear counterpart of the Perron method for Laplacian;

The class of $\mathcal{A}$-superharmonic functions is wide enough to solve the equation. Conversely, each $\mathcal{A}$-superharmonic function solves equation with some nonnegative (not necessarily finite) Radon measure μ on Ω.

$\mathcal{A}$-superharmonic functions

$\mathcal{A}$-harmonicity

A <u>continuous</u> function $u \in W^{1,G}_{loc}(\Omega)$ is an $\mathcal{A}$-*harmonic* function in an open set Ω if it is a (weak) solution to $-\mathrm{div}\mathcal{A}(x, Du) = 0$.

$\mathcal{A}$-super/subharmonicity

We say that a lower semicontinuous function u is $\mathcal{A}$-superharmonic if for any $K \Subset \Omega$ and any $\mathcal{A}$-harmonic $h \in C(\overline{K})$ in K, $u \geq h$ on ∂K implies $u \geq h$ in K (u is $\mathcal{A}$-subharmonic if $(-u)$ is $\mathcal{A}$-superharmonic).

An $\mathcal{A}$-superharmonic function

- is defined everywhere,
- can be unbounded,
- has a generalized gradient Du;
- generates a measure: $-\mathrm{div}\mathcal{A}(x, Du) = \mu_u$;

This object we want to 'control by a potential' and prove its regularity.

Potential theory in the Musielak-Orlicz setting

Chlebicka, AZG, Generalized superharmonic functions with strongly nonlinear operator, arXiv:2005.00118

Properties of $\mathcal{A}$-harmonic and $\mathcal{A}$-superharmonic functions involving an operator having generalized Orlicz growth (reflexive Orlicz spaces, natural variants of variable exponent and double-phase spaces).
In particular: Harnack's Principle, Minimum Principle, boundary Harnack inequality, etc.

For more references: see last presentation of Petteri Harjulehto
`https://www.mimuw.edu.pl/ ichlebicka/nonstandard-seminar.html`

Theorem - potential estimates

Assume that u is a nonnegative function being $\mathcal{A}$-superharmonic and finite a.e. in $B(x_0, R_{\mathcal{W}}) \Subset \Omega$ for some $R_{\mathcal{W}}$. Let (Havin-Mazy'a-)Wolff potential be given by

$$\mathcal{W}_G^{\mu_u}(x_0, R) = \int_0^R g^{-1}\left(\frac{\mu_u(B(x_0, r))}{r^{n-1}}\right) dr$$

with μ_u generated by u and $g = G'$. Then for $R \in (0, R_{\mathcal{W}}/2)$ we have

$$C_L\left(\mathcal{W}_G^{\mu_u}(x_0, R) - R\right) \le u(x_0) \le C_U\left(\inf_{B(x_0,R)} u(x) + \mathcal{W}_G^{\mu_u}(x_0, R) + R\right)$$

with $C_L, C_U > 0$ depending only on parameters $i_G, s_G, c_1^{\mathcal{A}}, c_2^{\mathcal{A}}, n$.

Powerful corollaries

$u \geq 0$ is $\mathcal{A}$-superharmonic and finite a.e. and $\mu_u := -\mathrm{div}\mathcal{A}(x, Du)$ (distrib.)

- The result is sharp as the same potential controls bounds from above and from below.
- u is continuous in $x_0 \iff \mathcal{W}_G^{\mu_u}(x,r)$ is small for $x \in B(x_0, r)$.
- if $-\mathrm{div}\mathcal{A}(x, Du) = \mu_u = \delta_{x_0}$; x is close to x_0, $r = |x - x_0|$, then

$$c^{-1}\left(\int_r^{2r} g^{-1}\left(s^{1-n}\right)\,ds - r\right) \leq u(x) \leq c\left(\int_r^{2r} g^{-1}\left(s^{1-n}\right)\,ds + \inf_{B_{2r}} u + r\right).$$

If additionally G is so fast in infinity that $\int_0 g^{-1}\left(s^{1-n}\right)\,ds < \infty$, then $u \in L^\infty(B_r)$. This bound is optimal.

Powerful corollaries

$u \geq 0$ is $\mathcal{A}$-superharmonic and finite a.e. and $\mu_u := -\mathrm{div}\mathcal{A}(x, Du)$ (distrib.)

- $u \in C^{0,\beta}_{loc}(\Omega) \iff \mu_{u,\theta}(B(x,r)) \leq c r^{n-1} g(r^{\theta-1}) \simeq r^{n-\theta} G(r^{\theta-1})$ (Orlicz-Morrey-type condition; * [C., Karppinen, 2019])
- Orlicz version of the fact that Lorentz regularity of the datum ($\mu \in L(\frac{n}{p}, \frac{1}{p-1})(\Omega)$) implies continuity of the solution
- Orlicz version of the fact that Marcinkiewicz regularity of the datum ($\mu \in L(\frac{n}{p+\theta(p-1)}, \infty)(\Omega)$) implies Hölder continuity of the solution.
- Orlicz version of the Hedberg–Wolff Theorem yielding full characterization of the natural dual space to $W^{1,G}_0(\Omega)$ by the means of the Wolff potential

The Hedberg–Wolff Theorem

Let μ be a nonnegative bounded Radon measure compactly supported in bounded open set $\Omega \subset \mathbb{R}^N$. Let

$$\mathcal{W}_G^{\mu}(x_0, R) = \int_0^R g^{-1}\left(\frac{\mu(B(x_0, r))}{r^{n-1}}\right) dr$$

be its Wollf potential.
Then

$$\mu \in (W_0^{1,G}(\Omega))'$$

if and only if

$$\int_\Omega \mathcal{W}_G^{\mu}(x, R)\, d\mu(x) < \infty \quad \text{for some} \quad R > 0.$$

Potential estmates - about the proof

It is enough to prove Theorem 1 for continuous $\mathcal{A}$-supersolutions.

- we take a nondecreasing sequence $\{\phi_j\}$ of Lipschitz functions converging pointwise to u.
- we consider the obstacle problem with a nonnegative obstacle ϕ_j, and boundary datum u (Chlebicka-Karppinen, Karppinen-Lee);
- we get a nondecreasing sequence $\{u_j\}$ of nonnegative continuous $\mathcal{A}$-supersolutions converging to u pointwise with $Du_j \to Du$ a.e. for generalized gradient 'D'.

$\{u_j\}$ – nonnegative supersolutions $\to u$

- for every j, we have that $\{T_k u_j\}_k$ is a nondecreasing sequence of continuous functions converging to u_j and they generate a sequence of measures $\{\mu_{T_k u_j}\}_k \subset (W_0^{1,G}(\Omega))'$.
- $\{\mu_{T_k u_j}\}_k$ locally converge weakly-$*$ to μ_{u_j}.
- Choosing diagonally subsequence of $\{T_k u_j\}_{k,j}$, we get a nondecreasing sequence $\{u^i\}_i$ of continuous and bounded $\mathcal{A}$-supersolutions converging pointwise to u and such that $Du^i \to Du$ a.e. in Ω.
- the corresponding measures μ_{u^i} locally converge weakly-$*$ to μ_u.

Upper bound

- we modify u to be a weak solution in a countable union of disjoint annuli shrinking to a point x_0 (we construct a Poisson's modification of u over a family of annuli)
- the corresponding measure in each annulus concentrates on the boundary of the particular annulus
- we can control the concentrations, since the measure corresponding to the new solution stays also in the dual of $W^{1,G}(B(x_0, R))$.
- being a solution is a local property, so we are equipped with a priori estimates for weak solutions in each annulus.

Thank you for your attention!