Boundary regularity of minimizers of double phase functionals

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Monday’s Nonstandard Seminar 2020/21
30 November 2020
(Warsaw, On-line seminar)

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Introduction
Nonstandard growth

\[ \Omega \subset \mathbb{R}^m, \partial \Omega: \text{sufficiently smooth}, \quad u : \Omega \to \mathbb{R}^n \]

\[ \mathcal{F}(u; \Omega) := \int_{\Omega} f(x, u, Du) dx, \]

- Growth condition:

\[ \lambda |z|^p \leq f(x, u, z) \leq \Lambda (1 + |z|)^q \quad (1 \leq p \leq q) \]

- \( p = q \): Standard growth
- \( p < q \): Non-standard growth (introduced by P. Marcellini ’89)

* Before this talk, in “Non-standard Seminar”, in many talks, there have been many good introduction for non-standard growth problems. See, for example, presentation file by C. De Filippis.

* In this talk, we treat only vectorial cases \((n \geq 2)\). For scaler-valued case \((n = 1)\), see for example presentation files for this seminar by P. Hästö or by P. Harjuleht.
Functionals with variable exponents

V.V. Zhikov '95, '97

\[ \int |Du|^{p(x)} \, dx \]

- Higher integrability of minimizers for continuous \( p(\cdot) \).
- Levrentiev phenomenon for discontinuous \( p(x) \):
  
  \( m = 2, \Omega = B_1 := \{ x = (x^1, x^2) \in \mathbb{R}^2 ; |x| < 1 \}, \)
  
  \( 1 < \alpha < 2 < \beta, \)

  \[ p(x) = \begin{cases} 
  \alpha & \text{for } x^1x^2 > 0, \\
  \beta & \text{for } x^1x^2 \leq 0. 
\end{cases} \]

  (This functional is also considered as “double phase”-type.)
Functionals of double phase have been introduced in M.Colombo-G.Mingione ('15,'15), P.Baroni-M.Colombo-G.Mingione ('16).

\[ H(u) := \int H(x, Du) dx, \quad H(x, \xi) := |\xi|^p + a(x)|\xi|^q, \]

where \( q > p > 1 \) and \( a(\cdot) \geq 0 \) and \( a \in C^{0,\alpha} \). Let \( u \) be a local minimizer of \( \mathcal{H} \) (or more general type of functionals of double phase), then we have the following regularity results:

- \( \frac{q}{p} < 1 + \frac{\alpha}{m} \implies u \in C^{1,\gamma}_{\text{loc}}(\Omega). \)
  (M.Colombo-G.Mingione ARMA. 215 (2015))

- \( u \in L^\infty(\Omega), \ \frac{q}{p} < \alpha \ (q/p \leq \alpha \text{ for } n = 1) \implies u \in C^{1,\gamma}_{\text{loc}}(\Omega). \)
  (M.Colombo-G.Mingione ARMA. 218 (2015))

- Manifold constrained, \( \frac{q}{p} < \alpha \implies \text{partial } C^{1,\gamma}-\text{regularity} \)
  (C. De Filippis-G.Mingione JGA (2020))
Double phase with variable exponents

\[ F(u; K) := \int_K \left( |Du|^{p(x)} + a(x)|Du|^{q(x)} \right) \, dx \]

This problem was suggested by Mingione to us (M.A Ragusa and T.) in 2016.

**Interior regularity:** (Ragusa-T. 2020)

Assume the following conditions on \( p(\cdot), q(\cdot) \) and \( a(\cdot) \):

- \( p(\cdot), q(\cdot) \in C^{0,\sigma} (\Omega) \) (\( \sigma \in (0, 1) \)),
  \[ q(x) \geq p(x) \geq p_0 > 1 \quad (\forall x \in \Omega), \]
- \( a(\cdot) \in C^{0,\alpha} (\Omega), a(x) \geq 0, \alpha \in (0, 1] \).
- \( \sup_{x \in \Omega} \frac{q(x)}{p(x)} < 1 + \frac{\beta}{m}, \quad \forall x \in \Omega, \quad \beta = \min\{\alpha, \sigma\}, \)

Let \( u \) be a local minimizer of \( F \).

\[ \implies u \in C^{1,\gamma}_{\text{loc}} (\Omega) \quad \text{for some} \quad \gamma \in (0, 1). \]
Boundary regularity of minimizers
Some known facts and results on the boundary regularity

Roughly speaking, for the case in which we can obtain full-interior regularity, we can get boundary regularity by reflection or other classical methods etc.. (The main result of this talk is in this case.)

On the other hand, even for the case for which we can expect only partial regularity, if so-called the \textit{blow-up method} does work, we can show the regularity near the boundary.

\begin{itemize}
  \item Harmonic maps between Riem. Mfds.: J.Jost-M.Meier (1983), R.Schoen-K.Uhlenbeck(1983),
  \item $p$-growth: F.Duzaar-J.F. Grotowski-M.Kronz (2004),
  \item $p(x)$-growth: M.A.Ragusa-T. (2016), T.-K.Usuba(2017),
  \item $p(x)$-growth, Mfd. constrained: I.Chlebicka-C.De Filippis-L.Koch (2020)
\end{itemize}
Writing $|\xi|^2_g := \left( \delta_{ij} g^{\alpha \beta}(x) \xi^i_{\alpha} \xi^j_{\beta} \right)$, $G(x, \xi) := |\xi|^p_g(x) + a(x)|\xi|^q_g(x)$ for $\xi \in \mathbb{R}^{mn}$, we consider the following functional:

$$G(u; K) := \int_K G(x, Du) dx, \quad (K \subset \mathbb{R}^m, \text{compact})$$

We suppose the following conditions:

$$a(x) \geq 0 \quad \forall x \in \Omega, \quad a(\cdot) \in C^{0, \alpha}(\overline{\Omega}) \quad \text{for some } \alpha \in (0, 1].$$

$$p(x), q(x) \in C^{0, \sigma}(\overline{\Omega}) \quad \text{for some } \sigma \in (0, 1).$$

Putting

$$\beta := \min\{\alpha, \sigma\}$$

We assume that $p(x)$ and $q(x)$ satisfy

$$q(x) \geq p(x) > 1 \quad \text{on } \overline{\Omega}, \quad \sup_{x \in \Omega} (q(x) - p(x)) < \beta$$
Conditions on $g$

\[ g^{\alpha \beta}(x) = g^{\beta \alpha}(x) \quad \forall x \in \Omega, \quad \alpha, \beta = 1, \ldots, n, \quad (6) \]
\[ g^{\alpha \beta}(\cdot) \in C^0(\overline{\Omega}), \quad \alpha, \beta = 1, \ldots, n. \quad (7) \]

We assume also that for some constants $0 < \lambda_g \leq \Lambda_g$, $g$ satisfies

\[ \lambda_g |z|^2 \leq g^{\alpha \beta}(x) z_{\alpha} z_{\beta}, \quad \max_{1 \leq \alpha, \beta \leq 1} |g^{\alpha \beta}(x)| \leq \Lambda_g, \quad (8) \]

for all $(x, z) \in \Omega \times \mathbb{R}^n$.

Let us write the modulus of continuity of $g$ by $\omega_g$, namely $\omega_g$ is an increasing continuous function with $\omega_g(0) = 0$ which satisfies

\[ \max_{1 \leq \alpha, \beta \leq m} \left| g^{\alpha \beta}(x) - g^{\alpha \beta}(y) \right| \leq \omega_g(|x - y|) \quad (9) \]

for all $x, y \in \Omega$. 

Atsushi TACHIKAWA  Boundary regularity
Main Theorem

Theorem 1 (T. to appear in JMAA)

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with the boundary $\partial \Omega$ of class $C^{1,1}$. Suppose that $a(x)$, $p(x)$ and $q(x)$ satisfy (2), (3), (5), and that $g$ satisfies (6), (7), (8). Let $h$ be a function in the class $W^{1,s}(\Omega; \mathbb{R}^n)$ for some $s > \max \left\{ \sup_{x \in \Omega} q(x), m \cdot \sup_{x \in \Omega} q(x)/p(x) \right\}$.

Then a minimizer $u \in W^{1,1}(\Omega; \mathbb{R}^n)$ of $G(\cdot; \Omega)$ with the boundary condition $u = h$ on $\partial \Omega$ is in the class $C^{0,\gamma_0}(\overline{\Omega}; \mathbb{R}^n) \cap C^{0,\gamma}_{\text{loc}}(\Omega; \mathbb{R}^n)$ for any $\gamma_0 \in \left( 0, 1 - \frac{m}{s} \sup_{\Omega} \frac{q(x)}{p(x)} \right)$ and $\gamma \in (0, 1)$.

When $g$ is Hölder continuous, then $Du \in C^{0,\zeta}_{\text{loc}}(\Omega; \mathbb{R}^{mn})$ for some $\zeta \in (0, 1)$. 
Outline of the Proof of Hölder continuity
Roughly speaking, the proof is divided to the following steps:

**Step 1** Fix $x_0 \in \partial \Omega$. By flatten $\partial \Omega$ near $x_0$ by a suitable transformation $\Psi$ s.t. $\Psi(x_0) = 0$, $\Psi(\partial \Omega \cap B_{\delta}(x_0)) \subset \{x; x^m = 0\}$ and $g(x_0) \mapsto I_m$ (identity matrix). (As in the paper by F. Duzaar-J. F. Grotowsky-M. Kronz ('04))

**Step 2** For the transformed functional $\tilde{G}$, we consider *semi-frozen functional* of type

$$\mathcal{H}(u) := \int \left( |Du|^p + b(x)|Du|^q \right) dx$$

**Step 3** By a reflexion argument we prove Hölder regularity on the flat part of the boundary for weak solutions $w$ of the Euler-Lagrange eq. for $\mathcal{H}$.

**Step 4** By estimating the difference between minimizers $v$ of $\mathcal{H}$ and $u$ of $\tilde{G}$, we get Morrey type estimate for $Du$. 
We use the following notation:
for \( x_0 = (x_0^1, \cdots, x_0^{m-1}, 0) \in \mathbb{R}^m \) and \( M > 0 \),
\[
B_M^+(x_0) := \{ x \in \mathbb{R}^m ; \ |x - x_0| < M, \ x^m > 0 \}, \quad B_M^+ := B_M^+(0),
\]
\[
\Gamma := \{ x \in \mathbb{R}^m ; \ x^m = 0 \}, \quad \Gamma_M(x_0) := B_M(x_0) \cap \Gamma, \quad \Gamma_M := \Gamma_M(0).
\]

Choosing a suitable diffeomorphism which flatten locally the boundary, and a suitable linear transformation, we can find the coordinate transformation \( \Psi \) with desired properties.
Step 2: Transformed functional and semi-frozen one

By $\Psi$, locally the functional $G$ is transformed to the following type:

$$
\overline{G}(u; K) := \int_K \overline{G}(x, Du) dx, \quad \overline{G}(x, \xi) = |\xi|^p g + a(x)|\xi|^q g.
$$

with $g(0) = I_n$. We consider the above functional on $B_M^+$ with the boundary condition on $\Gamma_M$.

For $x_0 \in \Gamma_M$ and sufficiently small $R > 0$ with $B_R^+(x_0) \subset B_M^+$, we put

$$
p_2 = p_2(x_0, R) := \sup_{B_R^+(x_0)} p(x), \quad q_2 = q_2(x_0, R) := \sup_{B_R^+(x_0)} q(x),
$$

We consider a semi-frozen functional for $\overline{G}$:

$$
\mathcal{H}_R(u) := \int_{B_R^+(0)} H_R(x, Du) dx, \quad H_R(x, \xi) := |\xi|^{p_2} + a(x)\frac{q_2}{q(x)}|\xi|^{q_2}.
$$

Let us put $a(x)^{q_2/q(x)} = b(x)$. 

**A remark on the exponents of $H_R$**

- When we use, for example, Hölder or reverse Hölder inequalities, we must treat $a(x)$ and $|Du|^{q(x)}$ together. So, the second term of the semi-frozen functional should be $\left(a(x)^{1/q(x)}|Du|\right)^{q_2}$.

It is easy to see that

$$a(x)\frac{q_2}{q(x)} \in C^{0,\beta}, \quad \beta := \min\{\alpha, \sigma\}.$$

- Since we are assuming

$$(*): \quad q(x) < p(x) + \beta,$$

we can choose $M > 0$ sufficiently small so that

$$q_2(0, M) < p_1(0, M) + \beta, \quad p_1 := \inf_{B_M} p(x).$$

This enables us to apply the previous results by Colombo-Mingione to $H_R$. This is the reason why we assume $(*)$. 

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**Proposition 1**

Assume that \( v \in W^{1,1} \cap L^\infty(B^+_R, \mathbb{R}^n) \) with \( \mathcal{H}_R(v) < \infty \) satisfies

\[
\left\{(**\right) \begin{array}{l}
\int_{B^+_R} (p_2 |Dv|^{p_2-2} + b(x)q_2 |Dv|^{q_2-2}) \langle Dv, D\varphi \rangle dx = 0, \\
v = h \quad \text{on } \Gamma_R,
\end{array}
\]

where \( h \in W^{1,s}(B^+_R; \mathbb{R}^n), (s > q_2) \). Then \( \forall \mu \in (0, n], \exists C \) s.t.

\[
\int_{B^+_r} H_R(x, Dv) dx \\
\leq \left[ \left( \frac{r}{R} \right)^{m-\mu} \int_{B^+_R} H_R(x, Dv) dx + r^m(1-\frac{q_2}{s}) \left( \int_{B^+_R} (1 + |Dh|^s) dx \right)^{\frac{q_2}{s}} \right] \]

\( \forall r \in (0, R). \)
Let $w$ be a minimizer of $\mathcal{H}_R$ with $w - (v - h) = 0$ on $\partial B_R$. Then

$w$ satisfies (***) with $h = 0$. Let $\bar{w}$ and $\bar{b}$ be odd and even extensions of $w$ and $b$ respectively to $B_R$. For a test function $\varphi$, let us put $\bar{\varphi}(x^1, ..., x^{m-1}, x_m) = \varphi(x^1, ..., x^{m-1}, -x_m)$. Then we have

$$
\int_{B_R} \left( p_2 |D\bar{w}|^{p^2-2} + \bar{b}(x) q_2 |D\bar{w}|^{q^2-2} \right) \langle D\bar{w}, D\varphi \rangle dx
$$

$$
= \int_{B_R^+} \left( p_2 |Dw|^{p^2-2} + b(x) q_2 |Dw|^{q^2-2} \right) \langle Dw, D(\varphi - \bar{\varphi}) \rangle dx = 0,
$$

since $\varphi - \bar{\varphi} = 0$ on $\Gamma_R$ and $\varphi = \bar{\varphi} = 0$ on $\partial B_R^+ \setminus \Gamma_R$. 

By the convexity of the functional the weak solution $w$ minimizes 
\[ \tilde{H}_R(w, B_R) := \int_{B_R} (|Dw|^p + b(x)|Dw|^q)\,dx, \]
and therefore $w$ satisfies for every $\mu \in (0, n)$.

\[
\int_{B_r^+} H_R(x, Dw)\,dx \lesssim \left( \frac{r}{R} \right)^{m-\mu} \int_{B_R^+} H_R(x, Dw)\,dx.
\]

(by C.De Filippis-G.Mingione '20, M.Colombo-G.Mingione '15)

Now, mentioning that $w - (v - h) \in W^{1,1}_0(B_R^+)$ and that $w, v$ both minimize $\mathcal{H}_R$, we can estimate the difference between
\[
\int H_R(x, Dv)\,dx \quad \text{and} \quad \int H_R(x, Dw)\,dx
\]
to obtain the desired estimate:

\[
\int_{B_r^+} H_R(x, Dv)\,dx \\
\lesssim \left[ \left( \frac{r}{R} \right)^{m-\mu} \int_{B_R^+} H_R(x, Dv)\,dx + r^m(1 - \frac{q_2}{s}) \left( \int_{B_R^+} (1 + |Dh|^s)\,dx \right)^{\frac{q_2}{s}} \right].
\]
Step 4 : Morrey type estimate for $Du$

Let $v$ be the minimizer of $\mathcal{H}_R$ with the boundary condition $v = u$ on $\partial B^+_R$. Here, mention that by virtue of the maximum principle due to F.Leonetti-F.Siepe ('05), $u$ is bounded, and therefore $v$ is also bounded. So, for $v$ we can use Prop. 1 to see that

\[
\int_{B^+_R} H_R(x, Dv) \, dx \lesssim \left( \frac{r}{R} \right)^{m-\mu} \int_{B^+_R} H_R(x, Du) \, dx \\
+ R^m(1 - \frac{q_2(R)}{s}) \int_{B^+_R} (1 + |Dh|^s) \, dx.
\]  

(10)
For $t > 1$, put
\[
V_t(\xi) := \begin{cases} 
|\xi|^{(t-2)/2} \xi & (\xi \neq 0), \\
0 & (\xi = 0).
\end{cases}
\]

Then $|\xi|^t \leq 2(|\eta|^t + |V_t(\xi) - V_t(\eta)|^2$ holds. By this inequality, we have
\[
\int_{B_r^+} H_R(x, Du) dx \leq \int_{B_r^+} (2 + H_R(x, Du)) dx
\]
\[
\lesssim r^m |B_1| + \int_{B_r^+} (H_R(x, Dv)) dx
\]
\[
+ \int_{B_r^+} \left[ |V_{p_2}(Du) - V_{p_2}(Dv)|^2 + b_R(x)|V_{q_2}(Du) - V_{q_2}(Dv)|^2 \right] dx
\]
\[
=: |B_1| r^m + I + II.
\]

$I$ can be estimate by (10).
For $II$, using the fact that $v$ is a weak solution of E-L eq. of $\mathcal{H}_R$ and that $u - v = 0$ on $\partial B_R^+$ (so $u - v$ is an admissible test function for the E-L eq. of $H_R$), we see that

$$II \lesssim [\mathcal{H}_R(u) - \mathcal{H}_R(v)].$$

Now, putting

$$\mathcal{H}_{R,g}(w) := \int_{B_R^+} H_{R,g}(x, Dw) \, dx, \quad H_{R,g}(x, \xi) := |\xi|_{g^{p^2(R)}} + b(x)|\xi|_{g^{q^2(R)}},$$

and using the minimality of $u$, we can estimate $II$ as follows.
\[ II = [\mathcal{H}_R(u) - \mathcal{H}_{R,g}(u) + \mathcal{H}_{R,g}(u) - \overline{\mathcal{G}}(u; B_R^+) \\
+ \overline{\mathcal{G}}(u; B_R^+) - \overline{\mathcal{G}}(v; B_R^+) \\
+ \overline{\mathcal{G}}(v; B_R^+) - \mathcal{H}_{R,g}(v) + \mathcal{H}_{R,g}(v) - \mathcal{H}_R(v)] \]

- **| (black) — (blue) |** : The difference is “| | and | |_g”. So, we can estimate them using the continuity of _g_.
- **green part** : Nonpositive by virtue of the minimality of _u_.
- **magenta parts** : The exponents are different. So, we can estimate them using the following estimate: for _ε_ > 0 there exists a constant _c(ε) > 0_ s.t.

\[ |t^\tau - t^\rho| \leq c(\varepsilon)|\tau - \rho|(1 + t^{\tau + \varepsilon}) \]

holds for any _t > 0_ and _1 < \rho < \tau_ (see A.Coscia-G.Mingione (’99)).
Reverse Hölder inequality (or Gehring-type inequality)

For \( y \in B_M^+ \) and \( R \in (0, (M - |y|)/2) \), let us put
\[ \Omega_R(y) := B_R(y) \cap \Omega. \]
Then for a minimizer \( u \) for \( \bar{G} \) with \( u = h \) on \( \Gamma_M \) we have the following estimate: for sufficiently small \( \delta > 0 \),
\[
\left( \int_{\Omega_R(y)} (\bar{G}(x, Du))^{1+\delta} \, dx \right)^{\frac{1}{1+\delta}}
\leq \int_{\Omega_{2R}(y)} \bar{G}(x, Du) + \left( \int_{\Omega_{2R}(y)} (\bar{G}(x, Dh))^{1+\delta} \, dx \right)^{\frac{1}{1+\delta}} + 1.
\]

When \( B_{2R}(y) \subset B_M \), the above estimate holds without the second term in the right-hand side.
Using also the reverse Hölder inequality (with boundary data) etc..., for sufficiently small $\delta > 0$, we have

$$II \lesssim R^{m+\sigma} + \left( \omega_g(R) + R^{\sigma-\delta n} \right) \int_{B_{2R}^+} H_{2R}(x, Du) \, dx$$

$$+ R^\sigma \int_{B_R^+} H_{2R}(x, Dh) \, dx$$

where $\omega_g$ is the modulus of continuity of $g$ and $\delta$ comes from the reverse Hölder inequality. We can take $\delta > 0$ sufficiently small.

Combining the estimate for $I$ and $II$, we get

$$\int_{B_r^+} \mathcal{H}_R(x, Du) \, dx$$

$$\lesssim \left[ \left( \frac{r}{R} \right)^{m-\mu} + \omega_g(R) + R^{\sigma-\delta n} \right] \int_{B_{2R}^+} H_{2R}(x, Du) \, dx$$

$$+ R^{m(1-\frac{q_2(2R)}{s})} \left( 1 + \int_{B_{2R}^+} |Dh|^s \, dx \right)$$

(11)
Hölder continuity (interior)

For an interior point \( y \in \Omega \), proceeding without considering boundary data, we can show the following estimate:

\[
\int_{B_r(y)} H_r(x, Du) \, dx \\
\leq \left[ \left( \frac{r}{R} \right)^{m-\mu} + \omega_g(2R) + R^{\sigma-n\delta} \right] \int_{B_{2R}(y)} H_R(x, Du) \, dx \\
+ CR^m.
\]

From the above estimate, mentioning that we can take \( \mu \in (0, 1) \) arbitrarily, we obtain for any \( \lambda \in (0, m) \)

\[
\int_{B_r(y)} H_r(x, Du) \, dx \leq Cr^{m-\lambda}.
\]

by the following (next page) lemma. Now, by Morrey’s theorem, we see that \( u \in C^{0, \gamma}_{\text{loc}}(B^+_M) \). for any \( \gamma \in (0, 1) \).
**Lemma 2**

Let $A, B, \alpha$ be positive constants and $\beta \in (0, \alpha)$. There exists a positive constant $\varepsilon_0 = \varepsilon_0(\alpha, \beta, A)$ with the following property: if a non-negative and nondecreasing function $\Phi$ defined on $[0, R_0]$ for some $R_0 > 0$ satisfies

$$
\Phi(r) \leq A \left[ \left( \frac{r}{R} \right)^{\alpha} + \varepsilon_0 \right] \Phi(R) + BR^\beta,
$$

all $0 < r < R_0$. Then, for some constant $C = C(\alpha, \beta, A)$

$$
\Phi(r) \leq C(\alpha, \beta, \gamma, A) \left[ \left( \frac{r}{R} \right)^{\beta} \Phi(R) + Br^\beta \right]
$$

holds for any $r \in (0, R_0]$. 
Hölder continuity (up to the boundary)

Fix $x_0 \in B^+_M$ and choose $R > 0$ sufficiently small. For every $y \in B_R(x_0) \cap B^+_M$, using (11) and (12), we can show

$$r^{m-m \frac{q_2(x_0,R)}{s}} \int_{B_r(x_0) \cap \Omega} H_r(x, Du) dx \leq C$$

for some constant $C > 0$.

Thus we obtain

$$\int_{B_r(x_0) \cap \Omega} |Du|^{p_1(x_0,R)} dx \leq C r^{m-m \frac{q_2(x_0,R)}{s}}.$$
Now, for any $\gamma_0 \in (0, 1 - (m/s) \sup_{\Omega}(q(x)/p(x)))$, by virtue of the continuity of the exponents, we can choose $R > 0$ sufficiently small so that

$$\gamma_0 \leq 1 - \frac{mq_2(x_0, R)}{sp_1(x_0, R)}.$$

Then we get

$$\int_{B_r(x_0) \cap \Omega} |Du|^{p_1(x_0, R)} dx \leq Cr^{m-p_1(x_0, R)+p_1(x_0, R)\gamma_0}.$$

By Morrey’s theorem, we have $u \in C^{0, \gamma_0}(B_M^+)$. \qed
Thank you for your attention!