Removable sets with generalized Orlicz growth

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18.1.2021

Based on joint work with Iwona Chlebicka
Removable sets are essentially null sets for some function classes. For Lebesgue integrable functions $L^1(\Omega)$ over a domain $\Omega$, zero measured sets $E$ are removable since $L^1(\Omega \setminus E) = L^1(\Omega)$.

In Sobolev spaces $W^{1,p}(\Omega)$ this is not enough since removing a zero measured set changes the test functions:

$$\int_\Omega \frac{\partial f(x)}{\partial x_j} \phi(x) \, dx = -\int_\Omega f(x) \frac{\partial \phi(x)}{\partial x_j} \, dx$$

for all $\phi \in C_0^\infty(\Omega)$. 
A notion of $p$-capacity yields the characterization of removable sets in $W^{1,p}_0(\Omega)$. For compact set $E \subset \Omega$ we define

$$\text{cap}_p(E, \Omega) = \inf_{v \in S_E} \int_{\Omega} |\nabla v|^p \, dx,$$

where $S_E := \{ f \in C^\infty_0(\Omega) : f \geq 1 \text{ in } E \}$. A set $E$ satisfies $W^{1,p}_0(\Omega \setminus E) = W^{1,p}_0(\Omega)$ if and only if $\text{cap}_p(E, \Omega) = 0$.

In this talk we are interested in removable sets of continuous $\varphi$-harmonic functions, where $\varphi$ is a generalized Orlicz function.
\( \varphi : \Omega \times [0, \infty) \to [0, \infty) \) is a convex \( \Phi(\Omega) \)-function if

- For every measurable function \( f : \Omega \to \mathbb{R} \) the function \( x \mapsto \varphi(x, f(x)) \) is measurable and for every \( x \in \Omega \) the function \( t \mapsto \varphi(x, t) \) is non-decreasing.
- \( \varphi(x, 0) = \lim_{t \to 0^+} \varphi(x, t) = 0 \) and \( \lim_{t \to \infty} \varphi(x, t) = \infty \) for almost every \( x \in \Omega \).
- \( t \mapsto \varphi(x, t) \) is convex and left-continuous.

\( u \in W^{1, \Phi}(\Omega) \) if \( u \) and its weak gradient have finite norms

\[
\| f \|_{L^{\Phi}(\Omega)} = \inf \left\{ \lambda : \int_{\Omega} \varphi \left( x, \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\} < \infty.
\]
Generalized Orlicz functions have familiar special cases:

- $\phi(x, t) = t^p$ is the classical $p$-growth
- $\phi(x, t) = \phi_0(t)$ is the Orlicz growth (for example $\phi_0(t) = \log(e + t)t^p$)
- $\phi(x, t) = a(x)t^{p(x)}$ is the variable exponent growth
- $\phi(x, t) = t^p + a(x)t^q$ is the double phase growth
- $\log(e + t)t^{p(x)}$, $t^{p(x)} + a(x)t^{q(x)}$, ...
Let us write \(\varphi_B^+(s) := \sup_{x \in B \cap \Omega} \varphi(s)\) and 
\(\varphi_B^-(s) := \inf_{x \in B \cap \Omega} \varphi(s)\). We need \(\varphi\) to satisfy the following regularity assumptions

(A0) There exists \(\beta > 0\) such that \(\varphi^+ (\beta) \leq 1 \leq \varphi^- (1/\beta)\)

(A1) There exists \(\beta > 0\) such that \(\varphi_B^+(\beta s) \leq \varphi_B^-(s)\) for every \(s \in [1, (\varphi_B^-)^{-1}(1/|B|)]\)

(alnc)\(_p\) There exists \(L_p \geq 1\) such that \(\frac{\varphi(x,t)}{t^p} \leq L_p \frac{\varphi(x,s)}{s^p}\) for all \(t < s\)

(aDec)\(_q\) There exists \(L_q \geq 1\) such that \(\frac{\varphi(x,t)}{t^q} \leq L_q \frac{\varphi(x,s)}{s^q}\) for all \(t > s\)

We write just (alnc) if there exists \(p > 1\) such that \(\varphi\) satisfies (alnc)\(_p\), similarly for (aDec).
<table>
<thead>
<tr>
<th>Expression</th>
<th>(A0)</th>
<th>(A1)</th>
<th>(aInc)</th>
<th>(aDec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi(x, t)$</td>
<td>True</td>
<td>True</td>
<td>$p &gt; 1$</td>
<td>$p &lt; \infty$</td>
</tr>
<tr>
<td>$t^p$</td>
<td>True</td>
<td>True</td>
<td>$p &gt; 1$</td>
<td>$p &lt; \infty$</td>
</tr>
<tr>
<td>$a(x)t^{p(x)}$</td>
<td>$a(\cdot) \approx 1$</td>
<td>$p \in C^{\log}$</td>
<td>$\text{ess inf } p(x) &gt; 1$</td>
<td>$\text{ess sup } p(x) &lt; \infty$</td>
</tr>
<tr>
<td>$\log(e + t)t^p$</td>
<td>True</td>
<td>True</td>
<td>$p &gt; 1 \ (\nabla_2)$</td>
<td>$p &lt; \infty \ (\Delta_2)$</td>
</tr>
<tr>
<td>$t^p + a(x)t^q$</td>
<td>$a \in L^\infty$</td>
<td>$a \in C^{0, \frac{n}{p}(q-p)}$</td>
<td>$p &gt; 1$</td>
<td>$q &lt; \infty$</td>
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</table>
**Definition**

A function \( u \in W^{1,\varphi}(\Omega) \) is a A-harmonic function in \( \Omega \) if

\[
\int_{\Omega} A(x, \nabla u) \cdot \nabla w \, dx = 0
\]

for all \( w \in C_0^\infty(\Omega) \).

We assume that \( \Omega \subset \mathbb{R}^n, n \geq 2 \) is an open bounded set and \( A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) satisfies

- \( x \mapsto A(x, z) \) is measurable
- \( z \mapsto A(x, z) \) is continuous
- \( |A(x, z)| \leq c_1 \frac{\varphi(x, |z|)}{|z|} \)
- \( c_2 \varphi(x, |z|) \leq A(x, z) \cdot z \)
- \( 0 < (A(x, z_1) - A(x, z_2)) \cdot (z_1 - z_2) \) for almost every \( x \in \Omega \) and distinct \( z_1, z_2 \)

for fixed \( c_1, c_2 > 0 \) and a convex generalized Orlicz function \( \varphi \).
Let \( \phi \) be a convex \( \Phi(\Omega) \)-function and define

\[
h_\phi(B(y, r)) := \int_{B(y, r)} \phi \left( x, \frac{1}{r} \right) dx.
\]

We get a Hausdorff measure of a set \( E \) by a standard construction

\[
\mathcal{H}_\phi(E) = \lim_{\delta \to 0} \mathcal{H}_{\phi, \delta}(E) = \lim_{\delta \to 0} \inf_{C_\delta^E} \sum_j h_\phi(B_j),
\]

where \( C_\delta^E \) is a countable collection of balls \( B_j \subset \Omega \) such that they cover \( E \) and have radii less than \( \delta \).
We can also define a relative $\varphi$-capacity following the standard construction as in Baruah, Harjulehto & Hästö (2018): for a compact $K \subset \Omega$

$$\text{cap}_\varphi(K, \Omega) := \inf_{v \in S_K} \int_{\Omega} \varphi(x, |\nabla v|) \, dx,$$

with similar test functions

$$S_K := \{ v \in W^{1, \varphi}(\Omega) \cap C_0(\Omega) : v \geq 1 \text{ in } K \text{ and } v \geq 0 \}.$$ 

For open sets $U$ we define

$$\text{cap}_\varphi(U, \Omega) = \sup_{K \subset U} \text{cap}_\varphi(K, \Omega)$$

and for any set $E$

$$\text{cap}_\varphi(E, \Omega) = \sup_{E \subset U \subset \Omega} \text{cap}_\varphi(U, \Omega).$$
Theorem

Suppose $E$ is a relatively closed subset of $\Omega$. Then the following are equivalent

- $W^{1,\phi}_0(\Omega) = W^{1,\phi}_0(\Omega \setminus E)$
- $\text{cap}_\phi(E, \Omega) = 0$.

Since $\mathcal{H}_\phi(E) < \infty$ implies that $\text{cap}_\phi(E, \Omega) = 0$ by De Filippis & Mingione (2020), we get the following result (without harmonicity of $u$):

Corollary

Let $E \subset \Omega$ be a relatively closed subset of $\Omega$ such that $\mathcal{H}_\phi(E) < \infty$ and $u \in W^{1,\phi}(\Omega)$ satisfying

$$\int_{\Omega \setminus E} A(x, \nabla u) \cdot \nabla w \, dx = 0$$

for all $w \in C^\infty_0(\Omega \setminus E)$. Then $u$ is a $A$-harmonic on the whole $\Omega$. 
In the case of $\varphi(x, t) = t^p$, a full characterization was obtained by Kilpeläinen and Zhong (2000).

**Theorem**

Let $E \subset \Omega$ be closed and $s > 0$. Suppose that $u$ is a continuous function in $\Omega$, $A$-harmonic in $\Omega \setminus E$ such that

$$|u(x_0) - u(y)| \leq C|x_0 - y|^{(s+p-n)/(p-1)}$$

for all $y \in \Omega$ and $x_0 \in E$. If $E$ is of $s$-Hausdorff measure zero, then $u$ is $A$-harmonic in $\Omega$.

**Corollary**

Let $0 < \alpha < 1$. A closed set $E$ is removable for $\alpha$-Hölder continuous $p$-harmonic functions if and only if $E$ is of $n - p + \alpha(p - 1)$ Hausdorff measure zero.

See also Carleson (1967), Hirata (2011), Ono (2013).
A variable exponent analogue of the classical case can be found in paper of Latvala, Lukkari & Toivanen (2010).

**Theorem**

Let $p(\cdot)$ be a log-Hölder continuous and $E \subset \Omega$ be closed and let $u \in C(\Omega)$ be a weak solution to $-\text{div}(|\nabla u|^{p(x)-2}\nabla u) = 0$ in $\Omega \setminus E$, and assume that

$$|u(x_0) - u(y)| \leq M|x_0 - y|^\alpha$$

for all $y \in \Omega$ and $x_0 \in E$ for some $0 < \alpha < 1$. If $\mathcal{H}_s(\cdot)(E) = 0$, where

$$s(x) = n - p(x) + \alpha(p(x) - 1),$$

then $u$ is a weak solution in $\Omega$.

See also Fu & Shan (2015).
In Orlicz growth the result the following result was proven by Challal & Lyaghfouri (2011).

**Theorem**

Let \( E \subset \Omega \) be a closed set and \( s > 0 \). Assume that \( u \) is a continuous function in \( \Omega \), \( A_{\varphi_0} \)-harmonic in \( \Omega \setminus E \), and such that for some \( \alpha \in (0, 1) \)

\[
|u(x) - u(y)| \leq L|x - y|^\alpha \quad \forall y \in \Omega, \forall x \in E.
\]

If \( E \) is of \( m \)-Hausdorff measure zero, with \( m = \tau(\alpha) \), then \( u \) is \( A \)-harmonic in \( \Omega \).

Here \( \tau(\alpha) = (\alpha - 1) \frac{a_0}{a_0 + 1} (1 + a_1) + \left( \frac{a_0}{a_0 + 1} + \frac{1}{a_1 + 1} \right) n - 1 \)

and \( a_0 \) and \( a_1 \) correspond to \( p - 1 \) and \( q - 1 \) from \((\text{alnc})_p\) and \((\text{aDec})_q\).
Double phase case was settled by Chlebicka & De Filippis (2020)

**Theorem**

Let \( \frac{q}{p} \leq 1 + \frac{\alpha}{n} \) and \( E \subset \Omega \) be a closed subset and \( u \in C(\Omega) \) be a continuous solution to \(-\text{div} A_{H}(x, Du) = 0 \) in \( \Omega \setminus E \) such that, for all \( x_1 \in E, x_2 \in \Omega \),

\[
|u(x_1) - u(x_2)| \leq C_u |x_1 - x_2|^\beta_0
\]

for a positive, absolute constant \( C_u \) and some \( \beta_0 \in (0, 1] \). If \( \mathcal{H}_{H_{\sigma}}(E) = 0 \), for \( \sigma := 1 - \frac{\beta_0}{q}(p - 1) \) then \( u \) is a solution in \( \Omega \).

Here \( H_{\sigma}(x, z) := |z|^{p\sigma} + a(x)^{\sigma} |z|^{q\sigma}, \frac{1}{p} < \sigma \leq 1 \).
Corresponding result in generalized Orlicz spaces:

**Theorem (Chlebicka & K)**

Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded open set and $A$ satisfies structural conditions with a convex $\Phi$–function $\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty)$ satisfying (A0), (A1), (aInc)$_p$ and (aDec)$_q$ with some $1 < p \leq q \leq n$. Let $E \subset \Omega$ be a closed subset and $u \in C(\Omega) \cap W^{1,\varphi}(\Omega \setminus E)$ be a continuous solution to $-\text{div}(A(x, \nabla u)) = 0$ in $\Omega \setminus E$ such that there exist some $C_u > 0$ and $\theta \in (0, 1]$

$$|u(x_1) - u(x_2)| \leq C_u |x_1 - x_2|^\theta \quad \text{for all } x_1 \in E, \ x_2 \in \Omega.$$

If $\mathcal{H}_{\mathcal{J}_\theta,\varphi}(E) = 0$, with $\mathcal{J}_{\theta,\varphi}(B(y, r)) = r^{-\theta} \int_{B(y,r)} \varphi(x, r^{\theta-1}) \, dx$, then $u$ is $A$-harmonic in $\Omega$. 
If $\varphi(x, t) = t^p$, then

$$J_{\theta, \varphi}(B(y, r)) = r^{-\theta} \int_{B(y, r)} \varphi(x, r^{\theta-1}) \, dx \leq Cr^{-\theta} r^n r^{(\theta-1)p}$$

$$= Cr^{n-p+\theta(p-1)}$$

If $\varphi(x, t) = t^p + a(x)t^q$, then

$$J_{\theta, \varphi}(B(y, r)) = r^{-\theta} \int_{B(y, r)} r^{p(\theta-1)} + a(x)r^{q(\theta-1)} \, dx$$

$$\leq C \int_{B(y, r)} r^{-p\left(1-\frac{\theta}{q}(p-1)\right)} + a(x)^{1-\frac{\theta}{q}(p-1)} r^{-q\left(1-\frac{\theta}{q}(p-1)\right)}$$
Main steps of the proof:

- Existence and uniqueness of solutions $v$ to obstacle problems
- Hölder continuity of $v$ for Hölder continuous $\psi$
- The following estimate

$$-\text{div} A(x, \nabla v)(B(x_0, r)) :\mu(B(x_0, r)) \leq Cr^{-\theta} \int_{B(x_0, r)} \phi(x, r^{\theta-1}) \, dx,$$

where $v$ is a solution to a an obstacle problem, where $u$ is the obstacle.

- Show what $v$ is actually $A$-harmonic in $\Omega$ and equals to $u$ almost everywhere.
Normally the argument is to use Hölder’s inequality

\[
\mu(B_{2r}(x_0)) \leq \int_{B_{4r}(x_0)} \eta^q \, d\mu = q \int_{B_{4r}(x_0)} \eta^{q-1} A(x, \nabla v) \cdot \nabla \eta \, dx
\]

\[
\leq C \left\| \frac{\phi(\cdot, |\nabla v|)}{|\nabla v|} \right\|_{L^{\phi^*}} \|\nabla \eta\|_{L^\varphi}
\]

However, the Luxemburg norms are difficult to estimate and the best we got was

\[
\mu(B_{2r}(x_0)) \leq C \frac{1}{r \varphi^{-1}(x_0, \frac{1}{|B(x_0, r)|})} \left( \int_{B(x_0, r)} \varphi(x, r^{\theta-1}) \, dx \right)^{1/p'}
\]

This is more or less equivalent to known Orlicz result and recovers classical and variable exponent results.
The way forward was to use Young’s inequality instead of Hölder’s inequality (here $\eta$ is a standard cut-off function with $|\nabla \eta| \leq \frac{4}{r}$)

\[
\mu(B(x_0, 2r)) \leq \int_{B(x_0, 4r)} \eta^q \, d\mu = q \int_{B(x_0, 4r)} \eta^{q-1} A(x, \nabla v) \cdot \nabla \eta \, dx
\]

\[
\leq cqr^{-\theta} \int_{B(x_0, 4r)} \frac{\phi(x, |\nabla v|)}{|\nabla v|} r^\theta |\nabla \eta| \, dx
\]

\[
\leq cqr^{-\theta} \int_{B(x_0, 4r)} \varphi^*(\frac{\phi(x, |\nabla v|)}{|\nabla v|}) + \phi(x, r^\theta |\nabla \eta|) \, dx
\]

\[
\leq cqr^{-\theta} \int_{B(x_0, 4r)} \varphi(x, |\nabla v|) + \phi(x, r^\theta |\nabla \eta|) \, dx
\]

\[
\leq cqr^{-\theta} \int_{B(x_0, 8r)} \varphi\left(\frac{\text{osc} \, v(x)}{r}\right) + \phi(x, r^\theta |\nabla \eta|) \, dx
\]

\[
\leq cqr^{-\theta} \int_{B(x_0, 4r)} \varphi(x, r^{\theta-1}) \, dx
\]


Thank you for your attention!