Removable sets with generalized Orlicz growth

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Removable sets are essentially null sets for some function classes. For Lebesgue integrable functions $L^1(\Omega)$ over a domain Ω, zero measured sets E are removable since $L^1(\Omega \setminus E) = L^1(\Omega)$.

In Sobolev spaces $W^{1,p}(\Omega)$ this is not enough since removing a zero measured set changes the test functions:

$$\int_\Omega \frac{\partial f(x)}{\partial x_j} \varphi(x)\, dx = -\int_\Omega f(x) \frac{\partial \varphi(x)}{\partial x_j}\, dx$$

for all $\varphi \in C_0^\infty(\Omega)$.

A notion of p-capacity yields the characterization of removable sets in $W_0^{1,p}(\Omega)$. For compact set $E \subset \Omega$ we define

$$\operatorname{cap}_p(E, \Omega) = \inf_{v \in S_E} \int_\Omega |\nabla v|^p \, dx,$$

where $S_E := \{ f \in C_0^\infty(\Omega) : f \geqslant 1 \text{ in } E \}$. A set E satisfies $W_0^{1,p}(\Omega \setminus E) = W_0^{1,p}(\Omega)$ if and only if $cap_p(E, \Omega) = 0$.

In this talk we are interested in removable sets of continuous φ-harmonic functions, where φ is a generalized Orlicz function.

$\varphi : \Omega \times [0, \infty) \to [0, \infty]$ is a convex $\Phi(\Omega)$-function if

- For every measurable function $f : \Omega \to \mathbb{R}$ the function $x \mapsto \varphi(x, f(x))$ is measurable and for every $x \in \Omega$ the function $t \mapsto \varphi(x, t)$ is non-decreasing.
- $\varphi(x, 0) = \lim_{t\to 0^+} \varphi(x, t) = 0$ and $\lim_{t\to\infty} \varphi(x, t) = \infty$ for almost every $x \in \Omega$.
- $t \mapsto \varphi(x, t)$ is convex and left-continuous.

$u \in W^{1,\varphi}(\Omega)$ if u and its weak gradient have finite norms

$$\|f\|_{L^\varphi(\Omega)} = \inf \left\{ \lambda : \int_\Omega \varphi\left(x, \frac{|f(x)|}{\lambda}\right) dx \leqslant 1 \right\} < \infty.$$

Generalized Orlicz functions have familiar special cases:

- $\varphi(x, t) = t^p$ is the classical p-growth
- $\varphi(x, t) = \varphi_0(t)$ is the Orlicz growth (for example $\varphi_0(t) = \log(e + t)t^p$)
- $\varphi(x, t) = a(x)t^{p(x)}$ is the variable exponent growth
- $\varphi(x, t) = t^p + a(x)t^q$ is the double phase growth
- $\log(e + t)t^{p(x)}$, $t^{p(x)} + a(x)t^{q(x)}$, $\dots$

Let us write $\varphi_B^+(s) := \sup_{x\in B\cap\Omega} \varphi(s)$ and $\varphi_B^-(s) := \inf_{x\in B\cap\Omega} \varphi(s)$. We need φ to satisfy the following regularity assumptions

(A0) There exists $\beta > 0$ such that $\varphi^+(\beta) \leqslant 1 \leqslant \varphi^-(1/\beta)$

(A1) There exists $\beta > 0$ such that $\varphi_B^+(\beta s) \leqslant \varphi_B^-(s)$ for every $s \in [1, (\varphi_B^-)^{-1}(1/|B|)]$

$(\mathrm{aInc})_p$ There exists $L_p \geqslant 1$ such that $\frac{\varphi(x,t)}{t^p} \leqslant L_p \frac{\varphi(x,s)}{s^p}$ for all $t < s$

$(\mathrm{aDec})_q$ There exists $L_q \geqslant 1$ such that $\frac{\varphi(x,t)}{t^q} \leqslant L_q \frac{\varphi(x,s)}{s^q}$ for all $t > s$

We write just (aInc) if there exists $p > 1$ such that φ satisfies $(\mathrm{aInc})_p$, similarly for (aDec).

$\varphi(x,t)$	(A0)	(A1)	(aInc)	(aDec)
t^p	True	True	$p > 1$	$p < \infty$
$a(x)t^{p(x)}$	$a(\cdot) \approx 1$	$p \in C^{\log}$	$\operatorname{ess\,inf} p(x) > 1$	$\operatorname{ess\,sup} p(x) < \infty$
$\log(e+t)t^p$	True	True	$p > 1$ (∇_2)	$p < \infty$ (Δ_2)
$t^p + a(x)t^q$	$a \in L^\infty$	$a \in C^{0,\frac{n}{p}(q-p)}$	$p > 1$	$q < \infty$

Definition

A function $u \in W^{1,\varphi}(\Omega)$ is a A-harmonic function in Ω if

$$\int_\Omega A(x, \nabla u) \cdot \nabla w \, dx = 0$$

for all $w \in C_0^\infty(\Omega)$.

We assume that $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$ is an open bounded set and $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies

- $x \mapsto A(x, z)$ is measurable
- $z \mapsto A(x, z)$ is continuous
- $|A(x, z)| \leqslant c_1 \frac{\varphi(x,|z|)}{|z|}$
- $c_2 \varphi(x, |z|) \leqslant A(x, z) \cdot z$
- $0 < (A(x, z_1) - A(x, z_2)) \cdot (z_1 - z_2)$ for almost every $x \in \Omega$ and distinct z_1, z_2

for fixed $c_1, c_2 > 0$ and a convex generalized Orlicz function φ.

Let φ be a convex $\Phi(\Omega)$-function and define

$$h_\varphi(B(y,r)) := \int_{B(y,r)} \varphi\left(x, \frac{1}{r}\right) dx.$$

We get a Hausdorff measure of a set E by a standard construction

$$\mathcal{H}_\varphi(E) = \lim_{\delta \to 0} \mathcal{H}_{\varphi,\delta}(E) = \lim_{\delta \to 0} \inf_{C_E^\delta} \sum_j h_\varphi(B_j),$$

where C_E^δ is a countable collection of balls $B_j \subset \Omega$ such that they cover E and have radii less than δ.

We can also define a relative φ-capacity following the standard construction as in Baruah, Harjulehto & Hästö (2018): for a compact $K \subset \Omega$

$$\mathrm{cap}_\varphi(K, \Omega) := \inf_{v \in S_K} \int_\Omega \varphi(x, |\nabla v|)\, dx,$$

with similar test functions
$S_K := \{v \in W^{1,\varphi}(\Omega) \cap C_0(\Omega) : v \geqslant 1 \text{ in } K \text{ and } v \geqslant 0\}$.

For open sets U we define

$$\mathrm{cap}_\varphi(U, \Omega) = \sup_{\substack{K \subset U \\ K \text{ compact}}} \mathrm{cap}_\varphi(K, \Omega)$$

and for any set E

$$\mathrm{cap}_\varphi(E, \Omega) = \sup_{\substack{E \subset U \subset \Omega \\ U \text{ open}}} \mathrm{cap}_\varphi(U, \Omega).$$

Theorem

Suppose E is a relatively closed subset of Ω. Then the following are equivalent

- $W_0^{1,\varphi}(\Omega) = W_0^{1,\varphi}(\Omega \setminus E)$
- $cap_\varphi(E, \Omega) = 0$.

Since $\mathcal{H}_\varphi(E) < \infty$ implies that $\mathrm{cap}_\varphi(E, \Omega) = 0$ by De Filippis & Mingione (2020), we get the following result (without harmonicity of u):

Corollary

Let $E \subset \Omega$ be a relatively closed subset of Ω such that $\mathcal{H}_\varphi(E) < \infty$ and $u \in W^{1,\varphi}(\Omega)$ satisfying

$$\int_{\Omega \setminus E} A(x, \nabla u) \cdot \nabla w \, dx = 0$$

for all $w \in C_0^\infty(\Omega \setminus E)$. Then u is a A-harmonic on the whole Ω.

In the case of $\varphi(x, t) = t^p$, a full characterization was obtained by Kilpeläinen and Zhong (2000).

Theorem

Let $E \subset \Omega$ be closed and $s > 0$. Suppose that u is a continuous function in Ω, A-harmonic in $\Omega \setminus E$ such that

$$|u(x_0) - u(y)| \leqslant C|x_0 - y|^{(s+p-n)/(p-1)}$$

for all $y \in \Omega$ and $x_0 \in E$. If E is of s-Hausdorff measure zero, then u is A-harmonic in Ω.

Corollary

Let $0 < \alpha < 1$. A closed set E is removable for α-Hölder continuous p-harmonic functions if and only if E is of $n - p + \alpha(p-1)$ Hausdorff measure zero.

See also Carleson (1967), Hirata (2011), Ono (2013).

A variable exponent analogue of the classical case can be found in paper of Latvala, Lukkari & Toivanen (2010).

Theorem

Let $p(\cdot)$ be a $\log$-Hölder continuous and $E \subset \Omega$ be closed and let $u \in C(\Omega)$ be a weak solution to $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = 0$ in $\Omega \setminus E$, and assume that

$$|u(x_0) - u(y)| \leqslant M|x_0 - y|^\alpha$$

for all $y \in \Omega$ and $x_0 \in E$ for some $0 < \alpha < 1$. If

$$\mathcal{H}_{s(\cdot)}(E) = 0,$$

where

$$s(x) = n - p(x) + \alpha(p(x) - 1),$$

then u is a weak solution in Ω.

See also Fu & Shan (2015).

In Orlicz growth the result the following result was proven by Challal & Lyaghfouri (2011).

Theorem

Let $E \subset \Omega$ be a closed set and $s > 0$. Assume that u is a continuous function in Ω, A_{φ_0}-harmonic in $\Omega \setminus E$, and such that for some $\alpha \in (0,1)$

$$|u(x) - u(y)| \leqslant L|x-y|^{\alpha} \quad \forall y \in \Omega, \forall x \in E.$$

If E is of m-Hausdorff measure zero, with $m = \tau(\alpha)$, then u is A-harmonic in Ω.

Here $\tau(\alpha) = (\alpha - 1)\frac{a_0}{a_0+1}(1+a_1) + \left(\frac{a_0}{a_0+1} + \frac{1}{a_1+1}\right)n - 1$ and a_0 and a_1 correspond to $p-1$ and $q-1$ from $(\mathrm{aInc})_p$ and $(\mathrm{aDec})_q$.

Double phase case was settled by Chlebicka & De Filippis (2020)

Theorem

Let $\frac{q}{p} \leqslant 1 + \frac{\alpha}{n}$ and $E \subset \Omega$ be a closed subset and $u \in C(\Omega)$ be a continuous solution to $-\operatorname{div} A_H(x, Du) = 0$ in $\Omega \setminus E$ such that, for all $x_1 \in E$, $x_2 \in \Omega$,

$$|u(x_1) - u(x_2)| \leqslant C_u |x_1 - x_2|^{\beta_0}$$

for a positive, absolute constant C_u and some $\beta_0 \in (0, 1]$. If $\mathcal{H}_{H_\sigma(\cdot)}(E) = 0$, for $\sigma := 1 - \frac{\beta_0}{q}(p - 1)$ then u is a solution in Ω.

Here $H_\sigma(x, z) := |z|^{p\sigma} + a(x)^\sigma |z|^{q\sigma}$, $\frac{1}{p} < \sigma \leqslant 1$.

Corresponding result in generalized Orlicz spaces:

Theorem (Chlebicka & K)

Suppose $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, is a bounded open set and A satisfies structural conditions with a convex Φ–function $\varphi : \Omega \times [0, \infty) \to [0, \infty)$ satisfying (A0), (A1), $(\text{aInc})_p$ and $(\text{aDec})_q$ with some $1 < p \leqslant q \leqslant n$. Let $E \subset \Omega$ be a closed subset and $u \in C(\Omega) \cap W^{1,\varphi}(\Omega \setminus E)$ be a continuous solution to $-\operatorname{div}(A(x, \nabla u)) = 0$ in $\Omega \setminus E$ such that there exist some $C_u > 0$ and $\theta \in (0, 1]$

$$|u(x_1) - u(x_2)| \leqslant C_u |x_1 - x_2|^\theta \qquad \text{for all } \; x_1 \in E, \; x_2 \in \Omega.$$

If $\mathcal{H}_{\mathcal{J}_{\theta,\varphi}}(E) = 0$, with $\mathcal{J}_{\theta,\varphi}(B(y, r)) = r^{-\theta} \int_{B(y,r)} \varphi(x, r^{\theta-1})\, dx$, then u is A-harmonic in Ω.

If $\varphi(x,t) = t^p$, then

$$\mathcal{J}_{\theta,\varphi}(B(y,r)) = r^{-\theta} \int_{B(y,r)} \varphi\left(x, r^{\theta-1}\right) dx \leqslant C r^{-\theta} r^n r^{(\theta-1)p} = C r^{n-p+\theta(p-1)}$$

If $\varphi(x,t) = t^p + a(x) t^q$, then

$$\begin{aligned}\mathcal{J}_{\theta,\varphi}(B(y,r)) &= r^{-\theta} \int_{B(y,r)} r^{p(\theta-1)} + a(x) r^{q(\theta-1)}\, dx \\ &\leqslant C \int_{B(y,r)} r^{-p\left(1-\frac{\theta}{q}(p-1)\right)} + a(x)^{1-\frac{\theta}{q}(p-1)} r^{-q\left(1-\frac{\theta}{q}(p-1)\right)}\end{aligned}$$

Main steps of the proof:

- Existence and uniqueness of solutions v to obstacle problems
- Hölder continuity of v for Hölder continuous ψ
- The following estimate

$$
\begin{aligned}
-\operatorname{div} A(x, \nabla v)(B(x_0, r)) &=: \mu(B(x_0, r)) \\
&\leqslant Cr^{-\theta} \int_{B(x_0,r)} \varphi\left(x, r^{\theta-1}\right) dx,
\end{aligned}
$$

  where v is a solution to a an obstacle problem, where u is the obstacle.
- Show what v is actually A-harmonic in Ω and equals to u almost everywhere.

Normally the argument is to use Hölder's inequality

$$\mu(B_{2r}(x_0)) \leqslant \int_{B_{4r}(x_0)} \eta^q \, d\mu = q \int_{B_{4r}(x_0)} \eta^{q-1} A(x, \nabla v) \cdot \nabla \eta \, dx$$
$$\leqslant C \left\| \frac{\varphi(\cdot, |\nabla v|)}{|\nabla v|} \right\|_{L^{\varphi^*}} \|\nabla \eta\|_{L^\varphi}$$

However, the Luxemburg norms are difficult to estimate and the best we got was

$$\mu(B_{2r}(x_0)) \leqslant C \frac{1}{r\varphi^{-1}\left(x_0, \frac{1}{|B(x_0,r)|}\right)} \left(\int_{B(x_0,r)} \varphi\left(x, r^{\theta-1}\right) dx \right)^{1/p'} .$$

This is more or less equivalent to known Orlicz result and recovers classical and variable exponent results.

The way forward was to use Young's inequality instead of Hölder's inequality (here η is a standard cut-off function with $|\nabla\eta| \leqslant \frac{4}{r}$)

$$
\begin{aligned}
\mu(B(x_0,2r)) &\leqslant \int_{B(x_0,4r)} \eta^q \, d\mu = q \int_{B(x_0,4r)} \eta^{q-1} A(x,\nabla v)\cdot\nabla\eta \, dx \\
&\leqslant cqr^{-\theta} \int_{B(x_0,4r)} \frac{\varphi(x,|\nabla v|)}{|\nabla v|} r^\theta |\nabla\eta| \, dx \\
&\leqslant cqr^{-\theta} \int_{B(x_0,4r)} \varphi^*\left(\frac{\varphi(x,|\nabla v|)}{|\nabla v|}\right) + \varphi(x, r^\theta|\nabla\eta|) \, dx \\
&\leqslant cqr^{-\theta} \int_{B(x_0,4r)} \varphi(x,|\nabla v|) + \varphi(x, r^\theta|\nabla\eta|) \, dx \\
&\leqslant cqr^{-\theta} \int_{B(x_0,8r)} \varphi\left(x, \frac{\operatorname{osc} v(x)}{r}\right) + \varphi(x, r^\theta|\nabla\eta|) \, dx \\
&\leqslant cqr^{-\theta} \int_{B(x_0,4r)} \varphi(x, r^{\theta-1}) \, dx
\end{aligned}
$$

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Thank you for your attention!